ON SOUND TRANSMISSION THROUGH BOUNDARY LAYERS OF AIRCRAFT FUSELAGES AND ENGINE DUCTS

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Summary

The acoustics of shear flows is relevant to the transmission of sound through the boundary layers of aircraft fuselages, to the modes of sound in nozzles, and to sound propagation in shear layers. Yet, there is only one exact solution of the acoustic wave equation in the simplest case of an unidirectional shear flow, namely for the linear velocity profile, and for low Mach number mean flow. In the present communication we report on four new solutions: (I) a solution for the linear shear, which is distinct from, and arguably simpler than, those in the literature; (II) the solution for an exponential shear flow, representing the asymptotic suction profile of a boundary layer; (III) solution for a parabolic shear flow in a duct, including eigenfrequencies and eigenmodes; (IV) besides the preceding cases of sound in low Mach number mean flow, a solution without this restriction, viz. for the linear shear. The distinctive feature of the acoustics of shear flows, is the existence of singularities of the wave equation, corresponding to critical layers, where sound waves can be reflected or absorbed. This may explain the cases of strong attenuation of sound, e.g. in fuselage boundary layers, which approximate methods, like ray theory, are unable to account for.

§1 - Introduction

There exists an extensive literature (see [1,2] for reviews) on approximate methods of study of sound propagation in shear flows. The simplest case would be an unidirectional shear flow, and only one case of exact solution exists, viz. for a linear velocity profile. This solution has been obtained in terms of parabolic cylinder functions [3], Whittaker functions [4,5] and confluent hypergeometric functions [6,7]. By matching the linear shear(s) to uniform stream (s), it has been used to model [3-7] sound transmission through boundary layers and shear layers, and acoustic modes in a duct containing a shear flow. One important feature of the acoustics of shear flows which has not been fully recognized in the literature, is the presence of singularities in the wave equation, corresponding to critical layers, where the Doppler shifted frequency vanishes, i.e. wave propagation would be impossible in ray theory. Of course, ray theory breaks down near a critical layer, because at singularity of the wave equation acoustic energy need not be conserved. Thus, sound waves may be absorbed or reflected at a critical layer of a shear flow; this cannot be studied by numerical methods, which break-down at a singularity of the wave equation. The presence of a critical layer is often implicit, yet often goes unmentioned in the literature. For example, all solutions of the acoustic wave equation in a linear shear flow [3-7] use a series expansion about the critical layer, sometimes without mentioning its mathematical and physical significance.

In the present communication we report on four advances on the acoustics of shear flows: The first (I) concerns a new solution for sound in a low Mach number linear shear flow, which is distinct from the existing ones, in that it splits the wave field into components which are symmetric and skew-symmetric relative to the critical level. The linear shear has been matched to an uniform stream, to represent a boundary layer with a 'kink' in the velocity profile, i.e. a discontinuous vorticity. We present (II) the exact solution for sound in an exponential shear flow, for which the vorticity decays smoothly from the wall to the free stream. Another case (III), is the acoustics of a parabolic shear flow, in a parallel-sided duct; this is a boundary-value problem, which is not of the Sturm-Liouville type, leading to the conjecture that critical layers separate flow regions with distinct eigenfrequencies and eigenfunctions. All of the three preceding cases (I to III), as well as the literature, concern sound propagation in a low Mach number shear flow, and we give the first solution without this restriction, for a linear shear (IV). We precede the discussion of these four cases, by a short derivation of the acoustic wave equation in a high-speed shear flow [8,9], in a form emphasizing the role of singularities or critical layers, in the specification of solutions.

§2 - Acoustic wave equation in a high-speed shear flow

We consider an unidirectional shear flow:

$$\vec{U} = U(y)\vec{e}_x, \tag{1}$$

for which the material derivative reads:

$$d/dt \equiv \partial/\partial t+\vec{U}.\nabla = \partial/\partial t+ U(y)\partial/\partial x . \tag{2}$$

The two components of the linearized momentum equation read:

$$du/dt +U'v + \rho_0^{-1}\partial p/\partial x = 0 , \tag{3a}$$

$$dv/dt + \rho_0^{-1}\partial p/\partial y = 0 , \tag{3b}$$

where u, v are the x,y-components of the acoustic velocity perturbation, p the acoustic pressure perturbation, $\rho_0(y)$ the mean flow mass density, $U(y)$ the mean flow velocity, and prime denotes derivative with regard to the transverse coordinate, viz. $U' = dU/dy$. For isentropic mean flow, the equation of continuity reads:

$$c^{-2}dp/dt = -\rho_0(\partial u/\partial x + \partial v/\partial y), \quad (4)$$

where c is the adiabatic sound speed (5a):

$$c^2 \equiv (\partial p/\partial \rho)_s = \gamma p_0/\rho_0, \quad (5a,b)$$

given by (5b) for a perfect gas, where γ is the ratio of specific heats. Elimination between (3a,3b,4) and use of (5b), leads to the acoustic wave equation in a high-speed shear flow:

$$\frac{d}{dt}\left[\frac{d}{dt}\left(\frac{1}{c^2}\frac{dp}{dt}\right) - \nabla(\log c^2).\nabla p - \nabla^2 p\right] + 2U'\frac{\partial^2 p}{\partial x \partial y} = 0; \quad (6)$$

the sound speed c is related to the stagnation sound speed c_0 by the adiabatic relation:

$$\{c(y)\}^2 = c_0^2 - [(\gamma-1)/2]U(y)^2, \quad (7)$$

and thus is constant $c(y) = c_0$ for a low Mach number mean flow $U^2 << c^2$, but not otherwise. The wave equation (6) relates to the Lilley equation [8,9], and is best interpreted starting from the simplest cases.

§3 - An Hierarchy of acoustic wave equations in moving media

In the case of a medium at rest $U = 0$, (6) reduces to the classical wave equation [10]:

$$U = 0 = U': \quad c^{-2}\partial^2 p/\partial t^2 - \nabla^2 p = 0. \quad (8)$$

For a low-Mach number mean flow, it leads to the convected wave equation:

$$U^2 << c^2, \ U' = 0: \quad c^{-2}d^2p/dt^2 - \nabla^2 p = 0, \quad (9)$$

which holds also [11,12] for a non-uniform, steady potential mean flow. If the Mach number is unrestricted, the sound speed is not constant, and we are led [13,14] to the high-speed wave equation in a potential flow:

$$U'=0: \ \frac{d}{dt}\left(\frac{1}{c^2}\frac{dp}{dt}\right) - \frac{1}{c^2}\nabla(c^2).\nabla p - \nabla^2 p = 0. \quad (10)$$

In the case of quasi-one-dimensional propagation in a duct of varying cross-section S(x), the Laplacian is replaced by the duct operator [15,16]:

$$\nabla^2 \to \frac{1}{S}\frac{\partial}{\partial x}S\frac{\partial}{\partial x}, \ p \leftrightarrow \phi, \quad (11a,b)$$

and acoustic pressure by acoustic potential; thus we obtain the high-speed nozzle wave equation [17,18]:

$$\frac{d}{dt}\left(\frac{1}{c^2}\frac{dp}{dt}\right) - \frac{1}{2c}\frac{dc}{dx}\frac{\partial p}{\partial x} - \nabla p \frac{1}{S}\frac{\partial}{\partial x}\left(S\frac{\partial p}{\partial x}\right) = 0, \quad (12)$$

which simplifies, at low Mach number, to the low Mach number nozzle equation [19,20]:

$$U^2 << c^2: \ d^2p/dt^2 - (c^2/S)(\partial/\partial x)(S\partial p/\partial x) = 0, \quad (13)$$

and in the absence of mean flow, to the horn equation:

$$U = 0: \ \partial^2 p/\partial t^2 - (c^2/S)(\partial/\partial x)(S\partial p/\partial x) = 0. \quad (14)$$

Both these linear acoustic wave equations (8-10; 12-14) and their non-linear generalizations [21,22], are of second-order, i.e. allow acoustic waves propagating in opposite directions. The coupling to vorticity in the mean flow:

$$U' \neq 0, \ U^2 << c^2: \ \frac{d}{dt}\left(\frac{1}{c^2}\frac{d^2p}{dt^2} - \nabla^2 p\right) + 2U'\frac{\partial^2 p}{\partial x \partial y} = 0, \quad (15)$$

leads to a third-order wave equation (6), even for low Mach number mean flow (15).

§4 - Vanishing of Doppler shifted frequency and critical layers

Since the mean flow quantities depend only on the transverse coordinate y, it is convenient to use a Fourier decomposition in time t and longitudinal coordinate x:

$$p(x,y,t) = \int_{-\infty}^{+\infty}\int P(y;k,\omega)e^{i(\omega t - kx)}dk\,d\omega, \quad (16)$$

where P is the acoustic perturbation spectrum, for a wave of frequency ω and longitudinal wavenumber k, at position y. In the case of low Mach number mean flow (15), it satisfies the wave equation:

$$(\omega - kU)P'' + 2kU'P' + (\omega - kU)[(\omega - kU)/c^2 - k^2]P = 0, \quad (17)$$

which has a singularity or critical level $y = y_c$ where the Doppler shifted frequency vanishes:

$$0 = \omega_*(y_c) \equiv \omega - k\ U(y_c). \quad (18)$$

At the critical level P" cannot be determined from P, P', and thus numerical methods can fail: (i) a method with constant step, is likely to 'jump over' the singularity, and give erroneous results; (ii) if a step 'hits' the singularity, then an "overflow" $P''(y_c) = \infty$ results; (iii) a method with variable step, like the Runge-Kutta method, would reduce the step to zero as the singularity is approached, and never go beyond. Thus an analytical solution is needed, at least in the neighbourhood of the critical layer.

The wave equation (17, 18):

$$\omega_* P'' - 2\omega_*' P' + \omega_*[\omega_*^2/c^2 - k^2]P = 0, \quad (19)$$

can be written in a Sturm-Liouville like form:

$$\frac{d}{dy}\left(\frac{1}{\omega_*^2}\frac{dP}{dy}\right) + \left(\frac{1}{c^2} - \frac{k^2}{\omega_*^2}\right)P = 0, \quad (20)$$

but the eigenvalue $\lambda = -k^2$ appears in several terms (18). Thus the results of Sturmian [23] theory:

- there is an infinity of eigenvalues;
- the eigenfunctions corresponding to distinct eigen-values are orthogonal;
- the eigenfunctions are complete, i.e. any square-integrable function is a linear function of them,

may all fail. It can be shown, in the case of the parabolic shear, that there is no single set of eigenvalues and eigenfunctions, holding across critical layers.

The role of the singularities in the solution of the wave equation, and the physics of sound near critical layers, is best illustrated by a sucession of examples.

§5 - Sound in a low Mach number linear shear flow

The (case I) linear shear flow (*Figure 1*):

$$U(y) = qy, \quad (21)$$

with constant vorticity q, has a critical layer (18) at:

$$y_c = \omega/kq \quad (22)$$

The change of variable:

$$\xi \equiv \omega_*(y)/\omega = 1-(kq/\omega)y = 1-y/y_c, \qquad (23a)$$

$$\Phi(\xi) \equiv P(y;k,\omega), \qquad (23b)$$

places the critical layer at the origin ($y = y_c$ implies $\xi_c = 0$), and the wall at the point unity ($y = 0$ implies $\xi = 1$). The wave equation (17,21):

$$\xi\Phi''-2\Phi'+\alpha\xi(\beta\xi^2-1) = 0, \qquad (24)$$

involves two dimensionless parameters:

$$\alpha \equiv (\omega/q)^2, \quad \beta \equiv (\omega/kc)^2, \qquad (25a,b)$$

namely, the square of the ratio of wave frequency to flow vorticity (25a), and β (25b), which is unity $\beta = 1$ for horizontal propagation $\omega = kc$, $\beta > 1$ for oblique waves $\omega > kc$, $\beta < 1$ for evanescent waves. Since the only singularities of the wave equation (24) are the critical layer $\xi = 0$ and the point at infinity $\xi = \infty$, a series expansion about the critical layer has infinite radius of convergence. Besides, the critical layer is a regular singularity of the equation, so a power series solution exists:

$$\Phi_\sigma(\xi) = \xi^\sigma \sum_{n=0}^{\infty} a_n(\sigma)\xi^n, \qquad (26)$$

with index σ and recurrence formula for the coefficients a_n to be determined. Substitution of (26) in (24) leads to the recurrence formula for the coefficients:

$$(n+\sigma+1)(n+\sigma-2)a_{n+1}(\sigma) = \alpha a_{n-1}(\sigma)-\alpha\beta a_{n-3}(\sigma). \qquad (27)$$

If we set $n = -1$, we obtain $\sigma(\sigma-3)\,a_0 = 0$. If $a_0 = 0$, then $a_n = 0$ for all $n = 1,2,\ldots$ in (27), and a trivial solution $\Phi = 0$ results (26). Thus $a_0 \neq 0$, so $\sigma(\sigma-3) = 0$ is the indicial equation, which has roots $\sigma = 0,3$. The upper root $\sigma = 3$ leads to:

$$\Phi_3(\xi) = \sum_{n=0}^{\infty} a_n(3)\xi^{2n+3} \equiv F(\xi), \qquad (28)$$

which is an odd function of distance from the critical layer, and the lower root to an even function of distance from the critical layer:

$$\Phi_0(\xi) = \sum_{n=0}^{\infty} a_n(0)\xi^{2n} \equiv G(\xi), \qquad (29)$$

The general integral is a linear combination of both:

$$Y \equiv y/y_c: \quad P(y;k,\omega) = c_1F(1-Y)+c_2G(1-Y), \qquad (30)$$

where the constants of integration are determined from boundary conditions (§10). The plots of odd (*Figure 2*) and even (*Figure 3*) wave functions, show that the oscillations of the acoustic wave 'die out' near the critical layer $Y = 1$, so that there is small variation of acoustic pressure up to the wall $0 \leq Y \leq 1$, in contrast with the oscillations of the acoustic pressure away from the wall $Y \geq 1$.

§6 - Sound transmission accross an exponential boundary layer

We consider next (case II) an exponential shear (*Figure 4*):

$$U(y) = V(1-e^{-y/L}), \qquad (31)$$

with independent choice of free stream velocity $U(\infty) = V$, and boundary-layer 'thickness' L. The critical layer (18) is located at:

$$y_c = -L\log(1-\Omega), \quad \Omega \equiv \omega/kV, \qquad (32a,b)$$

which shows that there are three cases: (i) if the Doppler shifted frequency is positive in the free stream $\omega > k\,V$, then it is positive everywhere $\omega > k\,U(y)$, and $\Omega > 1$ in (32b) shows that y_c is complex in (32a), i.e. there is no critical layer; (ii) if the Doppler shifted frequency is negative in the free stream $\omega < k\,V$, then it is positive at the wall $\omega_*(0) = \omega$, it must vanish in the boundary layer, and $\Omega < 1$ in (32b) implies real $y_c > 0$ in (32a); (iii) the borderline case $\omega = k\,V$ concerns the critical layer in the free stream $\Omega = 1$ or $y_c = \infty$.

The change of independent variable:

$$\zeta \equiv e^{-y/L}/(\Lambda-\Omega) = e^{-(y-y_c)/L}, \qquad (33)$$

places the critical layer $y = y_c$ at position unity $\zeta_c = 1$, and the free stream $y = \infty$ at the origin $\zeta = 0$. The transverse wavenumber in the free stream is:

$$\bar{k} = \sqrt{(\omega-kU)^2/c^2-k^2} \equiv i\vartheta L, \qquad (34)$$

so that $\zeta^\vartheta \sim e^{i\bar{k}y}$, i.e. We have propagating waves $e^{\pm i\bar{k}y}$ in the free stream if $(\omega - kU)^2 > k^2c^2$, and unstable $e^{\bar{k}y}$ and surface $e^{-\bar{k}y}$ modes otherwise. The change of dependent variable:

$$P(y;k,\omega) = \zeta^\vartheta H(\zeta), \qquad (35)$$

leads from (17,31,33) to the differential equation

$$(1-\zeta)\zeta H''+[(1+2\vartheta)+\zeta(1-2\vartheta)]H'+[2\vartheta-\Lambda^2(1-\zeta)(2-\zeta)]H=0, \qquad (36)$$

involving the parameter:

$$\Lambda \equiv (\omega-kV)L/c, \qquad (37)$$

which is dimensionless like ϑ in (34):

$$\vartheta \equiv \sqrt{K^2-\Lambda^2}, \quad K \equiv kL. \qquad (38a,b)$$

The differential equation (36) has regular singularities at the free stream $\zeta = 0$, and the critical layer $\zeta = 1$, and an irregular singularity at $\zeta = \infty$, i.e.$y = -\infty$ below the wall. Thus there are (*Figure 5*) three pairs of solutions, i.e. one pair around each singularity $\zeta = 0, 1, \infty$, with radius of convergence limited by the next singularity. To cover the whole flow region, we may need analytic continuation, between pairs of solutions. For $\Lambda = 0$ we would have a Gaussian hypergeometric equation (36), where $\zeta = \infty$ would also be a regular singularity. In general $\Lambda \neq 0$, so $\zeta = \infty$ is an irregular singularity of the extended hypergeometric equation (36). The Frobenius-Fuchs method applies near the regular singularities $\zeta = 0,1$, but not near the irregular one $\zeta = \infty$.

§7 - Modes in a duct with a parabolic shear flow

We consider (Figure 6) a parallel-sided duct, with walls at $y = \pm L$, containing (case III) a parabolic shear flow

$$U(y) = U_0\,(1-y^2/L^2), \qquad (39)$$

with velocity $U_0 = U(0)$ on axis. The critical layer(s) are (18) located at:

$$y_c = \pm L\sqrt{1-\Omega}, \quad \Omega \equiv \omega/kU_0, \qquad (40a,b)$$

showing that there are four cases: (i) for sound propagation downstream $k > 0$ with $\omega = k\,U_0$, the

Doppler shifted frequency $\omega_*(y) = \omega - k\,U(y) \geq 0$ is positive except on axis $\omega_*(0) = \omega - k\,U_0 = 0$, which is the critical layer; (ii) for sound propagation downstream $k > 0$ with Doppler shifted frequency positive on axis $\omega > k\,U_0$, there is no critical layer; (iii) for sound propagation downstream $k > 0$ with Doppler shifted frequency negative on axis $\omega < k\,U_0$, there are two critical layers (40a), symmetric relative to the axis; (iv) for sound propagation upstream $k < 0$, the Doppler shifted frequency is always positive $\omega_*(y) > \omega$, so there is no critical layer in the flow region, i.e. for $\Omega < 0$ the critical layers (40a) are outside the duct $|y_c| > L$.

The change of variable:

$$\zeta = (y/L)^2/(1-\Omega), \quad P(y;k,\omega) \equiv \psi(\zeta), \tag{41a,b}$$

puts both critical layers at the point unity $\zeta = 1$, which is a regular singularity of:

$$(1-\zeta)\zeta\psi'' + (1/2+3\zeta/2)\psi' + \left[(\Omega-1)^2/M^2\right](1-\zeta) \left\{\left[K(1-M)/4\right](1-\zeta)^2 - 1\right\}\psi = 0, \tag{42}$$

where M is the Mach number on axis and K the dimensionless horizontal wavenumber:

$$M \equiv U_0/C, \qquad K \equiv kL. \tag{43a,b}$$

The axis of the duct $\zeta = 0$ is also a regular singularity, and $\zeta = \infty$ an irregular singularity, so the differential equation (42) is of the extended hypergeometric type. Using rigid wall boundary conditions, the first eigenvalue k_1 of the even mode E_1 is given in *Table 1* for fixed frequency Ω and several Mach numbers, and vice-versa in *Table 2*. The corresponding plots of the even eigenfunction, show that the acoustic pressure increases towards the wall, more so for lower Mach number (*Figure 7*) and lower frequency (*Figure 8*).

§8 - Critical layers and critical flow points in high Mach number shear flows

The preceding three solutions Cases I-III in §5-7) concerned sound in a low Mach number shear flow (15, 17), and we consider next a case (IV) without restriction on Mach number (6), leading (16) to the wave equation:

$$(\omega - kU)P'' + \left[2kU' + (\omega - kU)c'/c\right]P' + (\omega - kU)\left[(\omega - kU)^2/c^2 - k^2\right]P = 0, \tag{44}$$

Introducing the stagnation sound speed c_0 from (7) and:

$$\varepsilon = \sqrt{(\gamma-1)/2}, \tag{45}$$

we obtain the wave equation (44):

$$(\omega - kU)\left(c_0^2 - \varepsilon^2 U^2\right)P'' + \left[k\left(c_0 - \varepsilon^2 U^2\right) - \varepsilon^2 U(\omega - kU)\right] 2U'P' + (\omega - kU)\left[(\omega - kU)^2 - k^2\left(c_0^2 - \varepsilon^2 U^2\right)\right]P = 0; \tag{46}$$

which has up to five singularities: (i) the critical layer for sound (18); (ii) two critical flow points, where the sound speed vanishes:

$$c(y_\pm) = 0: \quad U(y_\pm) = \pm c_0/\varepsilon = \pm c_0\sqrt{2/(\gamma-1)}; \tag{47}$$

(iii) the points at infinity $y = \pm\infty$ may also be singularities.

We consider again a linear shear flow (21), this time without restriction on Mach number, and make the change of variable:

$$\zeta \equiv y/y_c = kqy/\omega, \quad P(y;k,\omega) = T(\zeta); \tag{48a,b}$$

which places the critical layer at the point unity, leading to:

$$(1-\mu^2\zeta^2)(1-\zeta)T'' + 2(1-\mu^2\zeta)T' + \alpha(1-\zeta)\left[1-\mu^2\zeta^2 - \beta(1-\zeta)^2\right]T = 0, \tag{49}$$

which involves the same two dimensionless parameters (25a, b) as in the low Mach number case, and, in addition:

$$\mu \equiv \varepsilon\omega/kc_0 = \sqrt{\beta(\gamma-1)}/2. \tag{50}$$

The differential equation (49) has four singularities, viz. three regular singularities, at the critical layer $\zeta=1$, and critical flow points $\zeta = \pm 1/\mu$, and an irregular singularity at infinity $\zeta = \infty$. A differential equation with four regular singularities is reducible to Lamé or Heun type, and since in the present case one of the singularities is irregular, we have an extension of this type. Although there are four pairs of solutions, we never need more than two, to cover the flow region, which lies between the wall and the critical flow points. There is one case (*Figure 9*), when the critical flow point $y_+ > 2y_c$ is farther from the wall than twice the distance of the critical layer, when the solution around the critical layer covers the whole flow region.

§9 - Solutions which are finite or have logarithmic singularity at the critical layer

In the case of the low Mach number linear shear flow (Case I; §5), the sound field was finite at the critical layer, for the linear shear, but in all other cases it has a logarithmic singularity there. We illustrate this with the case (IV; §8) of linear shear flow with unrestricted Mach number (49). The change of variable

$$\eta = (\zeta-1)/(1/\mu - 1), \quad T(\zeta) \equiv R(\eta), \tag{51a,b}$$

Places the critical layer $\zeta=1$ at the origin $\eta = 0$, one critical flow point $\zeta = 1/\mu$ at point unity $\eta = 1$, and the other at:

$$\zeta = -1/\mu, \quad \eta = (\mu+1)/(\mu-1) \equiv j, \tag{52a,b}$$

so that the differential equation:

$$\eta(\eta-1)(\eta-j)R'' - 2\left[j - \mu\eta/(\mu-1)\right]R' - \alpha(1-1/\mu)^2\eta\left[(\eta-1)(\eta-j) + \beta\eta^2/\mu^2\right]R = 0, \tag{53}$$

has regular singularities at $\eta=0, 1, j$. The Frobenius-Fuchs expansion about the critical layer:

$$R_0(\eta) = \sum_{n=0}^{\infty} b_n(\sigma)\,\eta^{n+\sigma}, \tag{54}$$

Leads to the recurrence formula for the coefficients:

$$j(n+\sigma+1)(n+\sigma-2)b_{n+1} = 2\left[j/(j-1)\right](n+\sigma)(n+\sigma-2)b_n + \left[\alpha j(1-1/\mu)^2 - (n+\sigma-1)(n+\sigma-2)\right]b_{n-1} - \alpha(1-1/\mu)^2\left[(1+j)b_{n-2} - (1+\beta/j^2)b_{n-3}\right], \tag{55}$$

Setting $n = -1$, leads to the indicial equation $\sigma(\sigma-3) = 0$, which has roots $\sigma = 0, 3$. The upper root $\sigma = 3$, leads to a solution:

$$R_3(\eta) = \sum_{n=0}^{\infty} b_n(3)\eta^{n+3} \sim 0(\eta^3), \quad (56)$$

which vanishes at the critical layer. The lower root corresponds to a solution $R_0(\eta)$ which is a constant multiple of (56), so that a new, linearly independent solution, is specified [23, 24] by:

$$\bar{R}_0(\eta) = \lim_{\sigma\to 0} \frac{\partial}{\partial\sigma}\left[\sigma R_\sigma(\eta)\right]; \quad (57)$$

The latter has a logarithmic singularity:

$$\bar{R}_0(\eta) = R_3(\eta)\log\eta + \bar{\bar{R}}_0(\eta), \quad (58)$$

which is dominated by (56), so that the sound field is finite at the critical layer:

$$\bar{\bar{R}}_0(\eta) = \sum_{n=0}^{\infty} d_n(0)\,\eta^n, \quad d_n(0) \equiv \lim_{\sigma\to 0}\frac{\partial}{\partial\sigma}\left[\sigma b_n(0)\right]. \quad (59)$$

The logarithmic singularity:

$$\log\eta \sim \log(\zeta - 1) \sim \log(y - y_c), \quad (60)$$

involves a phase jump of $i\pi$, as the critical layer is crossed, viz. $y - y_c$ changes sign, and $\pm i\pi$ is added to the logarithm. The sign of the phase jump is determined [25] from:

$$\log\eta \sim \log(\zeta - 1) \sim \log(Kqy / \omega), \quad (61)$$

by giving the frequency a small negative imaginary part $\omega = \bar{\omega} - i\delta$, corresponding to a perturbation growing slowly with time:

$$\exp(i\omega t) = \exp(i\bar{\omega}t)\exp(t\delta); \quad (62)$$

Substitution in (55) gives:

$$\log\eta \sim \log(kqy / \bar{\omega} - 1 + iqky\delta / \bar{\omega}), \quad (63)$$

so that the phase is 0 for $y > y_c$ and $-i\pi$ for $y < y_c$, viz. a phase jump of $-i\pi$. We plot in *Figures 10* and *11*, respectively the modulus and phase of the acoustic pressure, versus dimensionless distance from the critical layer

§10 - Choice of surface wave or radiation conditions, and wave reflection and absorption at the critical layer

In Figures 10 and 11, the two constants of integration c_1, c_2 in a general solution of the type

$$P(y;k,\omega) = R(\eta) = c_1\, R_3(\eta) + c_2\, \bar{R}_0(\eta), \quad (64)$$

were determined by requiring the acoustic pressure to vanish at the critical flow point

$$P(y_+;k,\omega) = 0, \quad (65)$$

and normalizing the acoustic pressure to the valve at the wall:

$$Q(Y) \equiv P(y;k,\omega) / P(0;k,\omega). \quad (66)$$

There are other possible types of boundary condition, which we illustrate in the case of the exponential shear flow. The solution about the free stream:

$$P(y;k,\omega) = C_+\, P_+(y;k,\omega) + C_-\, P_-(y;k,\omega), \quad (67)$$

consists of outward P_+ and inward P_- propagating waves for $\bar{k}$ real in (34), and unstable P_+ and surface P_- modes for imaginary $\bar{k}$. Thus in the case $|\omega - kV| > kc$ of real $\bar{k}$, we select an outward propagating wave by setting $C_- = 0$, and an inward propagating wave by setting $C_+ = 0$; in the case $|\omega - kV| < kc$ of imaginary $\bar{k}$, a finite sound field in the free stream requires $C_+ = 0$. In both cases one constant of integration is determined.

The solution (61) has radius of convergence limited by the critical layer. In the neighrourhood of the latter it is replaced by:

$$P(y;k,\omega) = c_1\, P_1(y;k,\omega) + c_2\, P_2(y;k,\omega), \quad (68)$$

where P_1 is finite at the critical layer,and P_2 has a lograrithmic singularity. The two 'arbitrary' constants c_1, c_2 in (62) are related to $C_\pm$ in (61) by analytic continuation, i.e. the coefficients D in:

$$\begin{bmatrix} C_+ \\ C_- \end{bmatrix} = \begin{bmatrix} D_{+1} & D_{+2} \\ D_{-1} & D_{-2} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \quad (69)$$

are fixed, because we can have only two boundary conditions. For example, the absence of logarithmic singularity at the critical layer $C_2 = 0$, would generally imply $C_\pm \neq 0$ both inward and outward propagating waves in the free stream, i.e. the critical layer reflects waves. If we want to have only inward propagating waves in the free stream ($C_+ = 0$; no reflection), then $C_1 \neq 0 \neq C_2$ there is generally a lograithmic singularity at the critical layer, which absords sound.

§12 - Sound reflection, absorption or emission by rigid, impedance or moving walls

The second condition could be a boundary condition at a wall. The spectrum of the transverse acoustic velocity

$$v(x,y,t) = \int\int_{-\infty}^{+\infty} V(y;k,\omega)\, e^{i\omega t - kx}\, dk\, d\omega, \quad (70)$$

is related to that of the pressure (16) by (3b)

$$P'(y;k,\omega) = -i\rho_0\left[\omega - k\,U(y)\right] V(y;k,\omega). \quad (71)$$

Since the flow velocity is zero at the wall, this simplifies to:

$$P'(0;k,\omega) = -i\;\rho_0\;\omega\;V(0;k,\omega). \quad (72)$$

In the particular case of sound reflection from a rigid wall, the normal velocity is zero, and thus the normal derivative of the acoustic pressure vanishes:

$$P'(0;k,\omega) = 0. \quad (73)$$

Another case is a moving wall v(x,0,t), with velocity spectrum:

$$V(0;k,\omega) = \frac{1}{4M^2}\int\int_{-\infty}^{+\infty} v(x,0,t)\, e^{-i(kx-\omega t)} dx\, dt, \quad (74)$$

which causes an acoustic pressure gradient (72). Another case, of partial reflection and absorption, is a wall with impedance Z:

$$P(0;k,\omega) = Z\;V(0;k,\omega); \quad (75)$$

this leads to a boundary condition:

$$L\,P'(0;k,\omega) = -(i/z)\;\;P(0;k,\omega), \quad (76)$$

where:

$$z \equiv Z / \rho_0\,\omega L \quad (77)$$

is the dimenssionless impedance.

We illustrate in *Figure 12* the logarithm acoustic pressure, normalized to the value at Y = y/L = 10, versus dimensionless distance from a rigid wall. The case of an impedance wall (*Figure 13*) differs mainly in a downard instead of upward inflexion of the acoustic pressure near the wall. The preceding two cases were surface waves, which decay monotonically towards the wall. In the case of propagating waves:

$$Q(Y) \equiv \log\{P(y;k,\omega) / P(10L;k,\omega)\}, \qquad (78)$$

the logarithm of amplitudes:

$$\mathrm{Re}(Q) = \log|P(y;k,\omega) / P(10L;k,\omega)|, \qquad (79)$$

shows amplitude oscillations (*Figure 14*), with phase jumps:

$$\mathrm{Im}(Q) = \arg\{P(y;k,\omega)\} - \arg\{P(10L;k,\omega)\}, \qquad (80)$$

at the nodes of the mode shape function (*Figure 15*).

§13 - Discussion

The existence of critical layers for sound in a shear flow, has been sometimes mentioned in the literature as 'turning points' [9,26]. The role of critical layers in the acoustics of vortical flows, is as important as for internal waves in atmospheres [27], inertial waves in rotating fluids [28], or waves in viscous [29,30] and ionized [31,32] media, subject to magnetic fields [33,34] or Hall currents [35,36]. Although all these subjects sound different, they have in common [37,38] singularities of the wave equation, where wave absorption, reflection or transformation are possible. The figures concerning sound in shear flows have exhibited a number of phenomena which could not be accounted for in potential flows, and even less by ray theory.

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LIST OF TABLES

Table I

M	k_1
0.1	0.468195
0.3	0.41293
0.5	0.367105

($\Omega = 0.5$)

Table II

Ω	k_1
0.6	0.561843
0.8	0.749152
1.0	0.936485

($M = 0.1$)

Legends for the Figures

Figure 1 - Sound propagation in a linear shear flow.

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Figure 3 - Waveforms symmetric relative to the critical layer.

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Figure 8 - First even eigenfunction, for fixed low Mach number, and several dimensionless frequencies.

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Figure 11 - Phase shift of acoustic pressure, relative to wall value.

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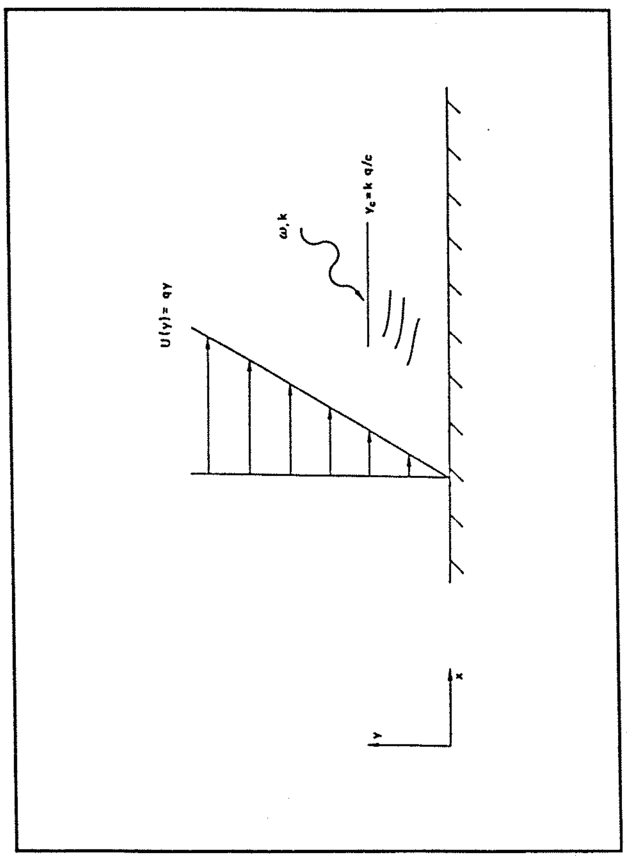
Figure 13 - As *Figure 12*, for impedance wall.

Figure 14 - As *Figure 13*, for logarithm of ratio of amplitudes of propagating wave.

Figure 15 - As *Figure 13*, for phase shift measured, from ten boundary thicknesses from the wall.



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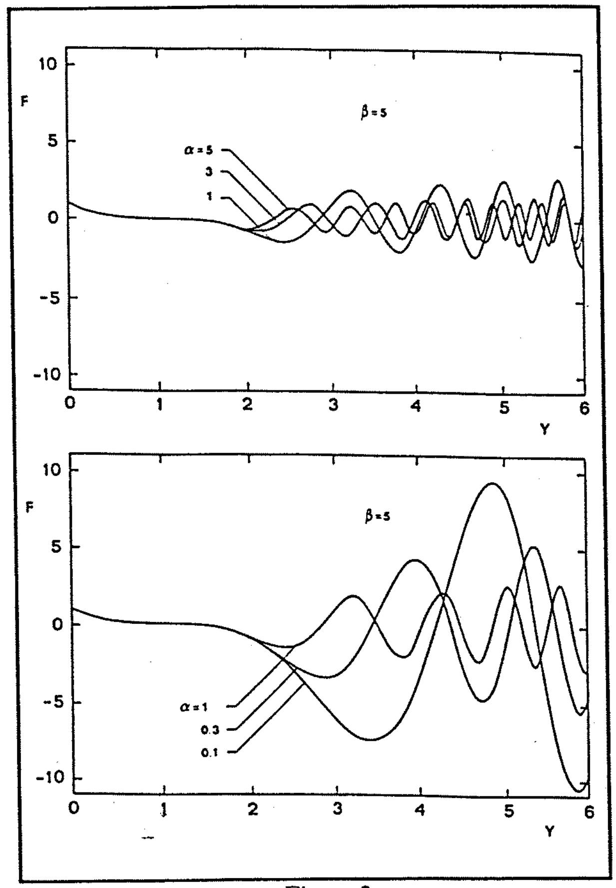

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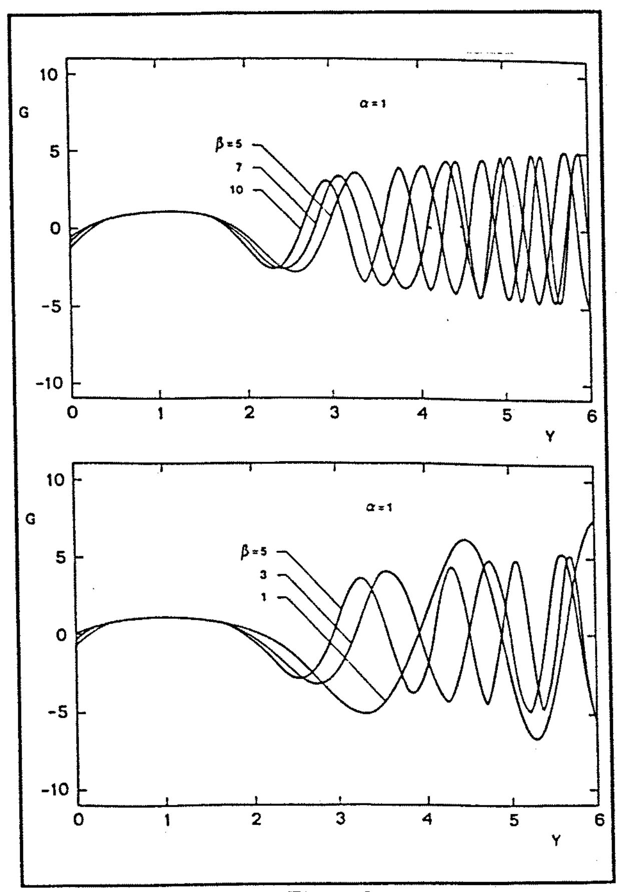

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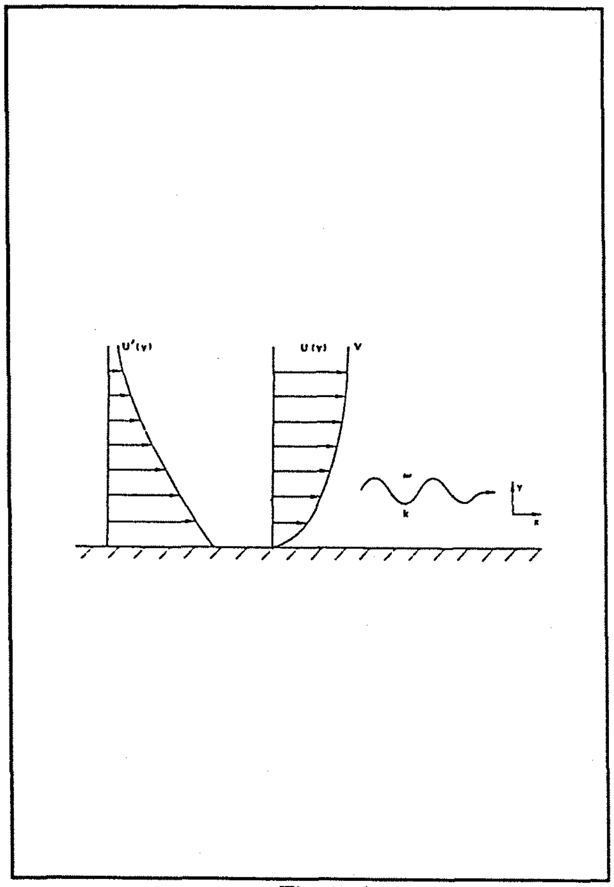

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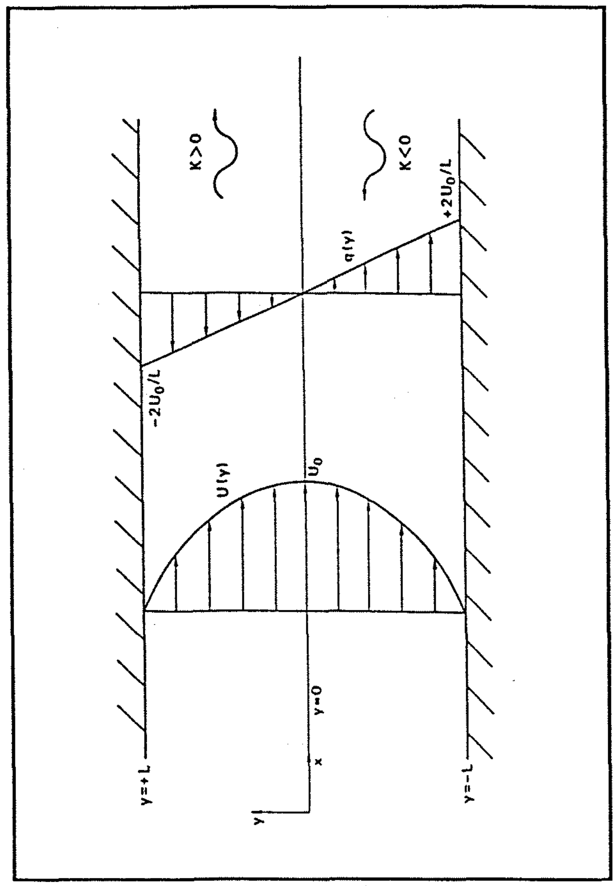

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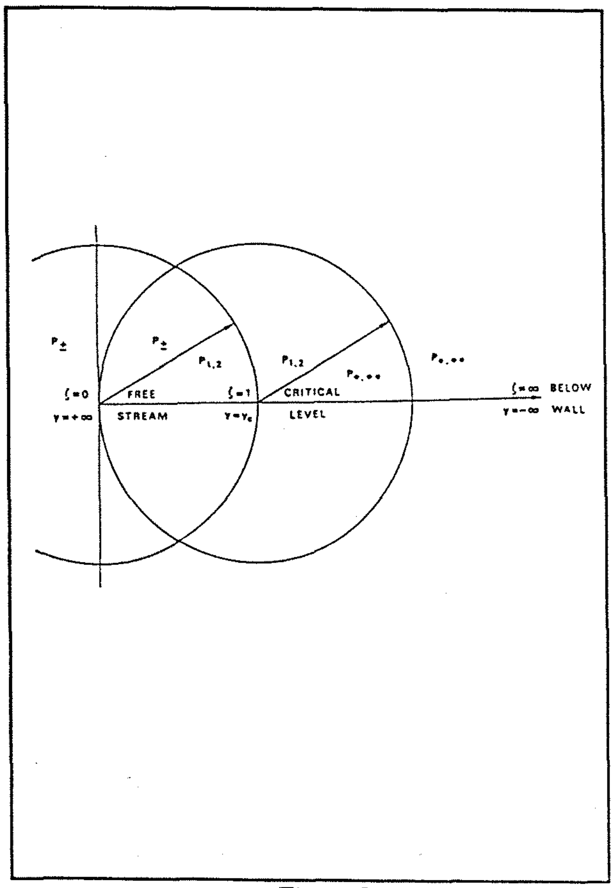

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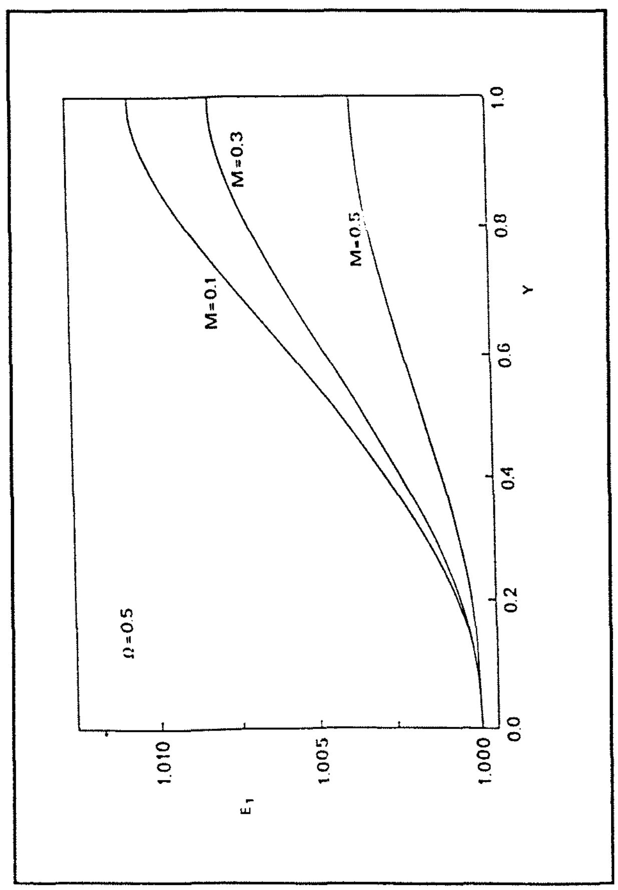

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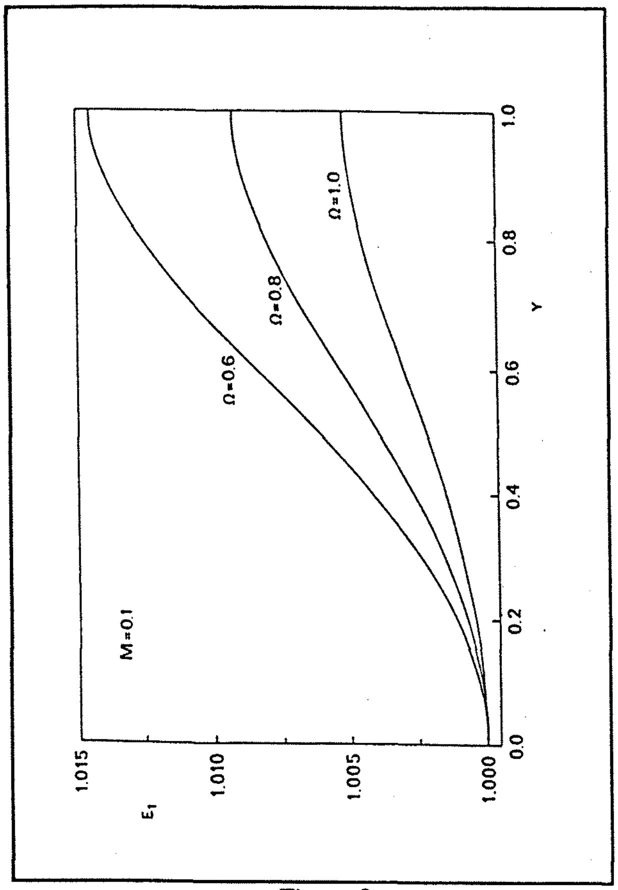

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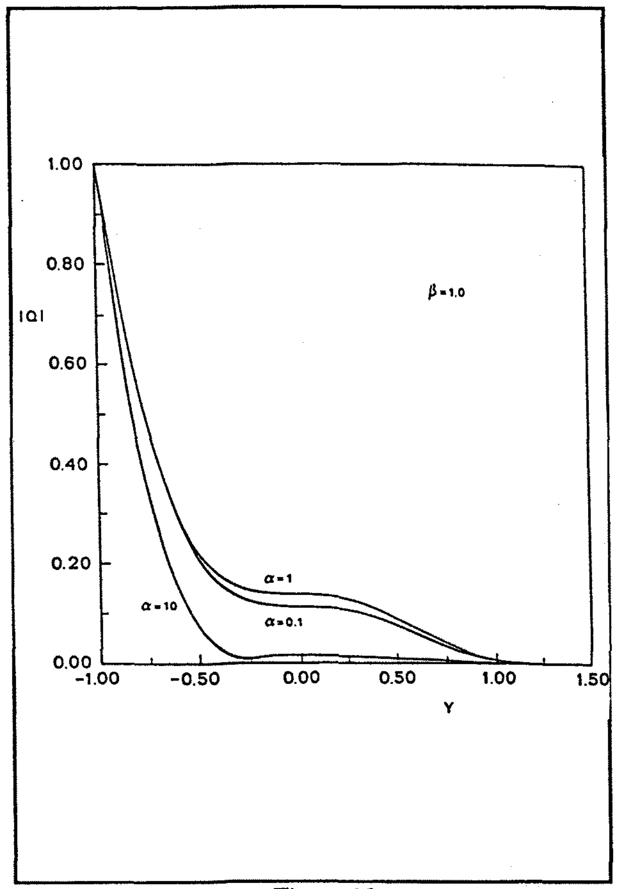

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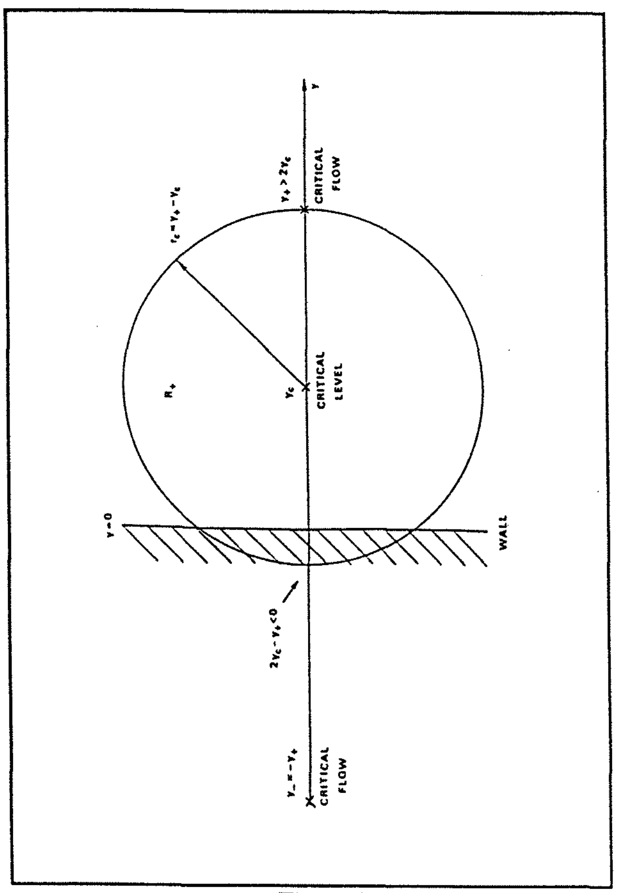

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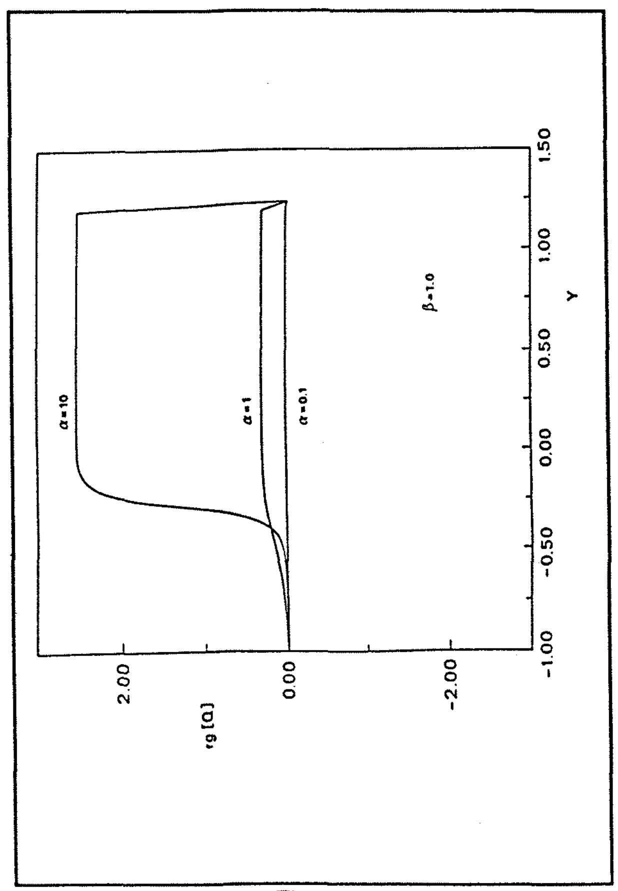


Figure 11

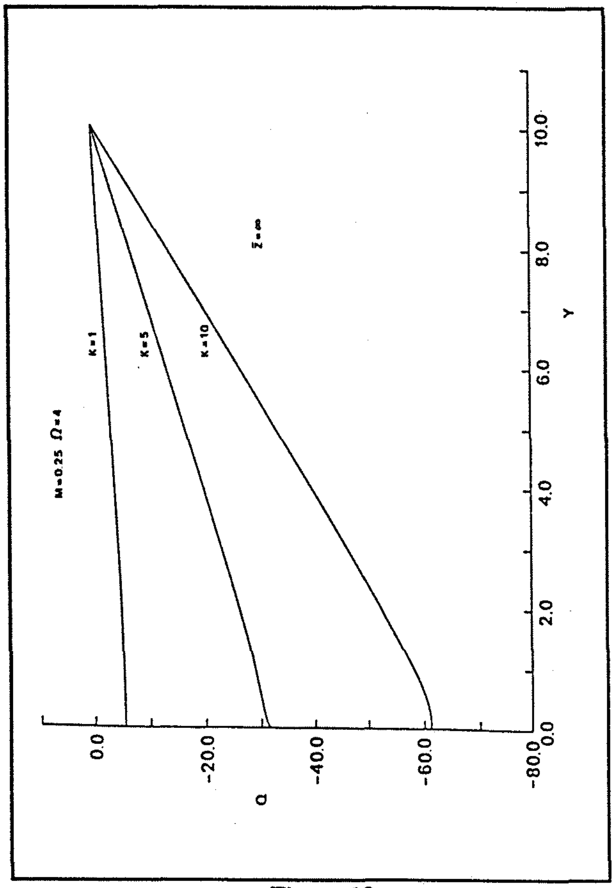
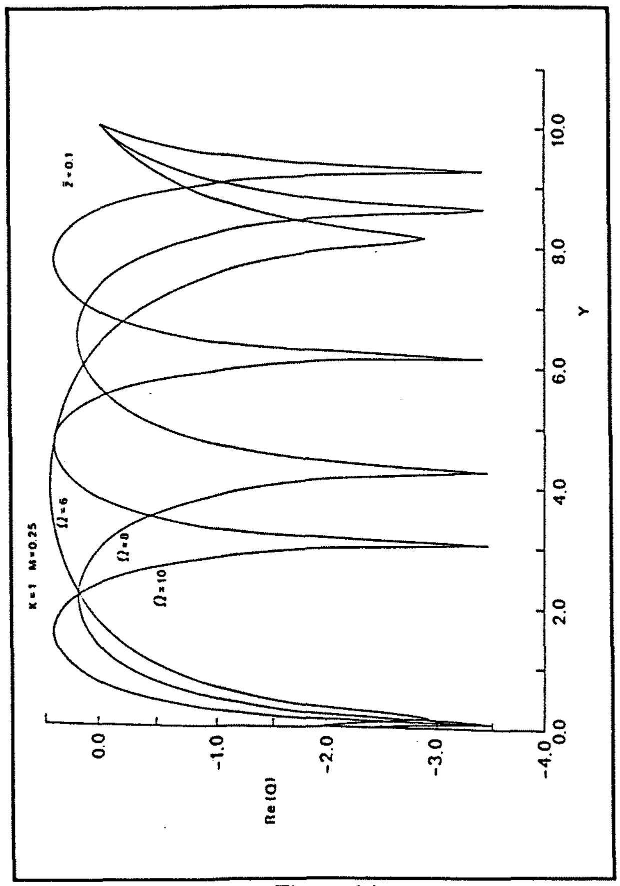


Figure 12



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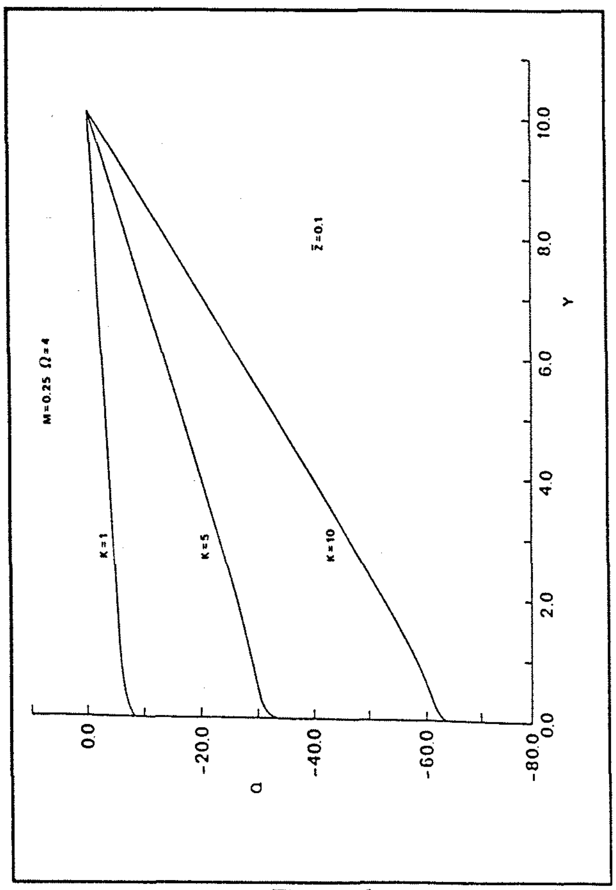

Figure 13

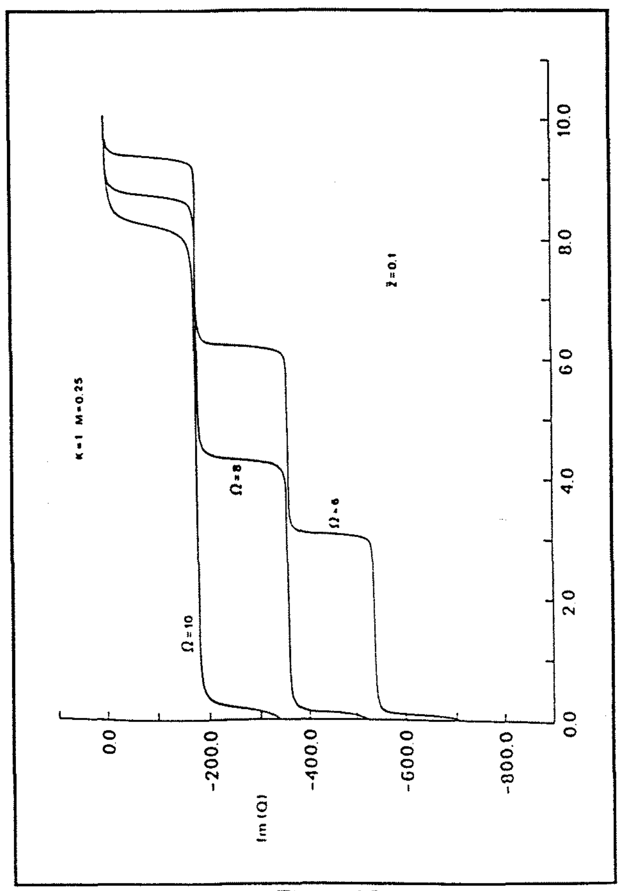

Figure 15