DEVELOPMENT OF A PLATE ELEMENT FOR THE STUDY OF A COMPOSITE PLATE INCLUDING DELAMINATIONS.

F. ROUDOLFF - Y. OUSSET

ONERA - BP 72 - 92322 Chatillon cedex - FRANCE

INTRODUCTION

The use of composite laminates has become increasingly widespread in industry in recent years. The main reason is the weight gain that can be expected at constant stiffness. However, these materials present a relatively poor resistance to localized impact loading. Experimental ultrasonic studies [1] showed that one of the main modes of degradation is composed of multiple delaminations that can be impossible to detect to the naked eye and can cause significant reductions in the stiffness of the material.

Since a detailed three-dimensional finite elements analysis is difficult to apply without overcoming locking, we retained a Love-Kirchhoff plate model. The Mindlin model [2] also presenting locking problems, we decided to use a mixed formulation of the Love-Kirchhoff plate model developed by Destuynder, Nevers [3] and Salaun [4]. This formulation is based on a decomposition of the transverse shearing force into the sum of a gradient and a rotational. It can also be written for the thin shell model of W. T. Koiter [5]. After having developed the mixed formulation for the shell model of Koiter, we give a numerical approximation and expose the problems due to the application to plates and delamination. We then give the expression of the energy release rate and finally numerical examples to validate the model.

MIXED MODEL OF KOITER

a) Shell model of Koiter

Let us consider a thin elastic shell which mid-surface is ω. We suppose that the shell admits a tangent plane and a normal N. The projection operator π on the tangent plane is defined by :

$$I = \pi + N.\overline{N}$$

where I is the identity in R^3.
Let us note $u_t = \pi.u$ and u_3 the normal composant of the displacement. The Koiter's shell model is then written :

* rotation :

The Kirchhoff-Love hypothesis is made on the displacements :

$$\theta = \frac{\partial N}{\partial m} \, . \, u_t - \frac{\overline{\partial u_3}}{\partial m}$$

* plane strains :

$$\gamma_t(u) = \frac{1}{2}\left(\pi \, . \, \frac{\partial u_t}{\partial m} + \frac{\overline{\partial u_t}}{\partial m} \, . \, \pi\right) + u_3 \frac{\partial N}{\partial m}$$

* curvature change :

$$\rho(u) = \frac{1}{2}\left(\pi \, . \, \frac{\partial \theta}{\partial m} + \frac{\overline{\partial \theta}}{\partial m} \, . \, \pi\right)$$
$$- \frac{1}{2}\left(\pi \, . \, \frac{\partial u_t}{\partial m} \, . \, \frac{\partial N}{\partial m} + \frac{\partial N}{\partial m} \, . \, \frac{\overline{\partial u_t}}{\partial m} \, . \, \pi\right)$$

* behavior law :

$$\sigma_t = S_t^{-1} \, . \, \gamma_t(u) + x_3 \, S_t^{-1} \, . \, \rho(u)$$

where σ_t is the stress tensor projection on the tangent plane and S_t the projection of the

compliance tensor.
The displacement field is then solution of the following variational system :

$$a(u,v) = l(v) \quad \forall\ v \in V$$

where $l(v)$ is the loading fonctional,

$$a(u,v) = \int_\omega Tr(n\ .\ \gamma_t(v)) + \int_\omega Tr(m\ .\ \rho(v))$$

with $n = \int_{h^-}^{h^+} \sigma_t(u)\ dx_3$

and $m = \int_{h^-}^{h^+} x_3\ \sigma_t(u)\ dx_3$

and V = { v kinematically admissible }

b) Mixed formulation

In the formulation above, u_3 must be C^1. To avoid this condition, let us introduce the shear force Q that we split into the sum of a gradient ϕ and a rotational ψ :

$$Q = \nabla\phi + rot\,\psi$$

The Koiter mixed model is then written :

$$\begin{cases} a(u,v) + b(\Lambda,v) = l(v) & \forall\ v \in V \\ b(\tilde{\Lambda},u) = 0 & \forall\ \tilde{\Lambda} \end{cases}$$

with $\Lambda = (\phi,\psi,C_i)$ where C_i are constants (there is one for each non connex free edge), and

$$b(\Lambda,u) = \int_\omega \nabla\phi\ .\ (\theta - \frac{\partial N}{\partial m}\ .\ u_t + \frac{\partial u_3}{\partial m}) - \int_\omega \psi\ curl\ (\frac{\partial N}{\partial m}\ .\ u_t - \theta) - \sum_{i=1}^{n_2} C_i \int_{\hat{\gamma}_2^i} (\theta - \frac{\partial N}{\partial m}\ .\ u_t)\ .\ \tau$$

N_2 represents the number of connex composantes of the free edge $\hat{\gamma}_2$ and $curl\,\mu = \mu_{2,1} - \mu_{1,2}$.

NUMERICAL APPROXIMATION

a) Resolution

The geometry is approached by triangles. The shape functions are linear for u_3, ϕ and ψ and linear plus bubble function for u_t and θ . The bubble function is $27\ \lambda_1\ \lambda_2\ \lambda_3$ if λ_i are the barycentrical coordinates. After discretization of the bilinear forms a and b leads to the following linear system :

$$\begin{cases} A\ X + B^t\ \phi + C^t\ \psi + D^t\ K = F \\ B\ X = 0 \\ C\ X = 0 \\ D\ X = 0 \end{cases}$$

where the matrix A corresponds to the discretization of $a(.,.)$ defined above when B, C and D correspond to the discretization of the part of the functional b that makes intervene ϕ for B, ψ for C and the constants associated to the free edges for D (K is the vector of constants associated to the free edges). A is symetric when B, C and D are rectangular. X is the vector of nodal values of generalized displacements (u and θ). To solve the linear system, two ways can be adopted ;

i) the non-zero terms are only stocked for A, B, C and D and the global system is solved with a conjugate gradient. That is the "Morse" stocking.

ii) X is eliminated as follow :

$$X = A^{-1}\ [F - B^t\ \phi - C^t\ \psi - D^t\ K\ ]$$

so that we obtain the following system in ϕ, ψ and K :

$$\begin{cases} (B\ A^{-1}B^t)\ \phi + (B\ A^{-1}C^t)\ \psi + (B\ A^{-1}D^t)\ K = B\ A^{-1}\ F \\ (C\ A^{-1}B^t)\ \phi + (C\ A^{-1}C^t)\ \psi + (C\ A^{-1}D^t)\ K = C\ A^{-1}\ F \\ (D\ A^{-1}B^t)\ \phi + (D\ A^{-1}C^t)\ \psi + (D\ A^{-1}D^t)\ K = D\ A^{-1}\ F \end{cases}$$

This system is solved with a conjugate gradient, possibly preconditioned. Each iteration only needs one resolution of a system where intervenes the matrix A that is factorized under the shape $A = L.D.L^t$, where L is an inferior triangular matrix with a unit diagonal. B, C and D are always stocked "Morse" when A is stocked "profile" here.

b) Application to plates

When the model is applied to plates, this is to say when the shell curvature becomes 0 (or small), A becomes singular. A can be regularized by adding either :

i) a matrix K_S of transverse shear stiffness given by the Mindlin model.

ii) or a linear arrangement of matrixes B and C : $\alpha B^t . B + \beta C^t . C$

The first solution has only been tested at the beginning.

c) Application to delamination

The delamination plate is modelled as an assembly of sub-plates where continuity of u, θ and ϕ is assumed through the delamination front :

$$\begin{cases} [u_t] = 0 \\ [u_3] = 0 \\ [\theta] = 0 \\ [\phi] = 0 \end{cases}$$

where [.] represents the jump of the considered function.

On the other hand, the bending/membrane problem must be taken into account.

d) Numerical example for the stocking

Because of the connecting conditions the stocking becomes expensive when A is stocked "profile" and it would be good of having a minimization algorithm for the profile of the matrix. In order to evaluate the costs (memory place and time), a square, clamped plate under a negative force in its center is calculated. Results are written in table 1 for two different meshes. The plate is not delaminated. The renumbering algorithm used here is the one of Gibbs-Poole-Stockmeyer.

Table 1

Stocking mode	461 d.o.f.		2225 d.o.f.	
	place	time	place	time
P. without r.	85116	89 s	1785160	3252 s
P. with r.	26653	35 s	263427	139 s
Morse	7695	40 s	37936	492 s

where :

P. without r. = Profile without renumbering

P. with r. = Profile with renumbering

With "Morse" stocking, memory place needed by A is minimum. Nevertheless, the bad conditioning of the matrix has an influence on the convergence of the method. In order to obtain a correct precision, the number of iterations of the conjugate gradient must be forced to several times the dimension of x. The second way of penalizing A has then been tested. Because of $B.X = 0$ and $C.X = 0$, we have $A.X = (A + \alpha. (B^t . B + C^t . C) X$ and the system to solve is not changed. This method leads to better results than the previous one by decreasing the number of iterations but it presents the disadvantage of increasing the memory place. The memory place as a function of the number of degrees of freedom for the two different ways of stocking is shown on figure 1 with :

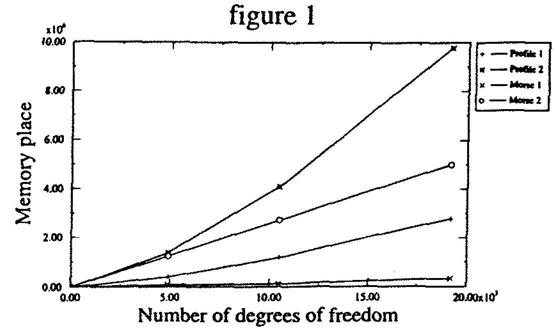
Profile 1 = Profile with the transverse shear matrix for penalizing matrix.

Profile 2 = Profile with $\alpha. (B^t . B + C^t . C)$ for penalizing matrix.

Morse 1 = Morse with the transverse shear matrix for penalizing matrix.

Morse 2 = Morse with $\alpha. (B^t . B + C^t . C)$ for penalizing matrix.

figure 1



The renumbering algorithm of Gibbs-Poole-Stockmeyer has also been used to decrease the memory place when A is stocked "profile". However, the continuity conditions do not allow an as important gain as with only one plate, especially for coarse meshes.

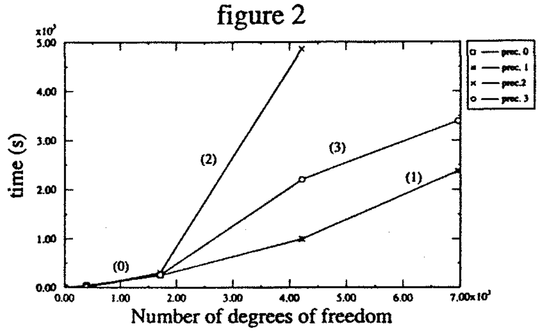
e) Preconditioning of A

Two kinds of preconditioning have been tested in order to decrease the number of iterations when the system is directly solved by a conjugate gradient. Results presented on the figure 2 are for a double Cantilever beam. Different meshes have been calculated. When there is no preconditioning, computations have not been done for more than 1700 degrees of freedom because of the time that becomes very important compared to the one with preconditioning.

On the figure 2, the different preconditionings are the following :

preconditioning 0 : no preconditioning.
preconditioning 1 : preconditioning with incomplete Crout factorization.
preconditioning 2 : preconditioning with the diagonale of A.
preconditioning 3 : preconditioning SSOR with a unit constant.

figure 2



Let us suppose that the delamination propagates of δl at the interface of two layers of the composite plate. The tool used to study the propagation of this delamination is the energy release rate. This one epresents the thermodynamic force G so that $G\ \delta l$ is the potential energy that is disponible in the structure in order to have a progression of the delamination front of a length δl.
The delamination front is well described by a velocity field w such that :

$$w = \begin{Bmatrix} w_1(x_1, x_2) \\ w_2(x_1, x_2) \\ 0 \end{Bmatrix}$$

The support of w is confined to a small neighborhood of γ_f, γ_f being the delamination front.
Destuynder [6] gave the following expression of the energy release rate if the loading support does not intersect w support (if the neighborhood of γ_f is unloaded) :

$$G^{\varepsilon}(w) = -\frac{1}{2}\int_{\Omega^{\varepsilon}} \sigma_{ij}\ u_{j,i}\ div\ w + \int_{\Omega^{\varepsilon}} \sigma_{ik}\ u_{k,\lambda}\ w_{\lambda,i}$$

When ε vanishes $G^{\varepsilon}(w)$ becomes [7] :

$$G^{0}(w) = -\frac{1}{2}\int_{\Omega} S_{\alpha\beta\lambda\mu}\ \sigma^{0}_{\alpha\beta}\ \sigma^{0}_{\lambda\mu}\ \frac{\partial w_{\gamma}}{\partial x_{\gamma}} + \int_{\Omega} \sigma^{0}_{\alpha\beta}\ \frac{\partial u^{0}_{\beta}}{\partial x_{\lambda}}\ \frac{\partial w_{\lambda}}{\partial x_{\alpha}} + \int_{\Omega} \sigma^{0}_{\alpha\beta}\ \frac{\partial}{\partial x_{\beta}}\ (u^{0}_{\lambda}\ \frac{\partial w_{\lambda}}{\partial x_{\alpha}})$$

Let us note :

$$\begin{cases} n_{\alpha\beta} = R^{M}_{\alpha\beta\lambda\mu}\ \gamma_{\lambda\mu} \\ q_{\alpha\beta} = R^{C}_{\alpha\beta\lambda\mu}\ \gamma_{\lambda\mu} \\ p_{\alpha\beta} = -\ R^{C}_{\alpha\beta\lambda\mu}\ K_{\lambda\mu} \\ m_{\alpha\beta} = -\ R^{F}_{\alpha\beta\lambda\mu}\ K_{\lambda\mu} \end{cases}$$

with $K_{\lambda\mu}(u_3) = -\,u_{3,\lambda\mu}$

and using the mixed formulation of the shell model, the energy release rate expression becomes :

$$G^0(w) = -\frac{1}{2}\sum_{i=1}^{3}\int_{\omega_i}\{\,(n^i_{\alpha\beta}(u) + p^i_{\alpha\beta}(u_3))\,\gamma^i_{\alpha\beta}(u)$$

$$+\,(m^i_{\alpha\beta}(u_3) + q^i_{\alpha\beta}(u))\,\gamma^i_{\alpha\beta}(\theta)\,\}\ div\ w$$

$$+\sum_{i=1}^{3}\int_{\omega_i}\{\,(n^i_{\alpha\beta}(u) + p^i_{\alpha\beta}(u_3))\,u_{\beta,\lambda}$$

$$-\,(m^i_{\alpha\beta}(u_3) + q^i_{\alpha\beta}(u))\,u_{3,\lambda\beta}\,\}\,\frac{\partial w_\lambda}{\partial x_\alpha}$$

$$-\sum_{i=1}^{3}\int_{\omega_i}(rot\ \psi^i + \nabla\,\phi^i)_\alpha\,\theta^i_\alpha\,\frac{\partial w_\lambda}{\partial x_\alpha}$$

$$+\sum_{i=1}^{3}\int_{\partial\omega_i}\{m^i_{\alpha\beta}(u_3) + q^i_{\alpha\beta}(u)\}\,n^i_\beta\,\theta^i_\lambda\,\frac{\partial w_\lambda}{\partial x_\alpha}$$

NUMERICAL EXAMPLES

a) First of all, the model has been validated by comparison with an analytical computation.

Let us consider a square plate which side is 1m, thickness 10 mm, simply supported on its sides and under a negative force of - 1000 N in its center. The characteristics of the material are as follows :

$E = 150000\ MPa$ and $\nu = 0.35$.

An analytical Kirchhoff-Love computation gives $u_3 = -\,8.15E{-}04\ m$. The same plate calculated with the mixed finite elements gives the same result whatever the resolution method used.

b) A Cantilever beam has been meshed by introducing first of all one plate and on the other hand two plates connected together with the continuity conditions. The two meshes were exactly the same. Except for ψ that does not need to be continue at the interface of the plates, the two methods give the same results.

c) Let us consider a double Cantilever beam (figure 3). The results obtained after convergence are presented for u_3 and ϕ in the middle of the beam on figures 4 and 5. From the expressions of ϕ and ψ we obtain the shear force.

figure 3

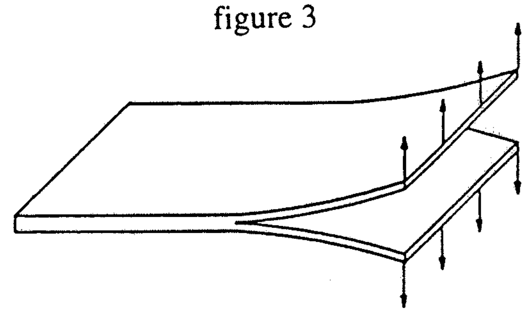


figure 4

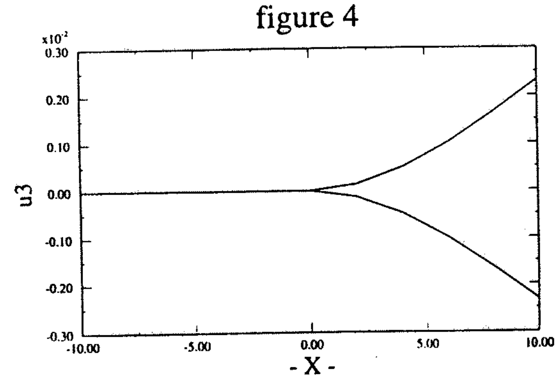
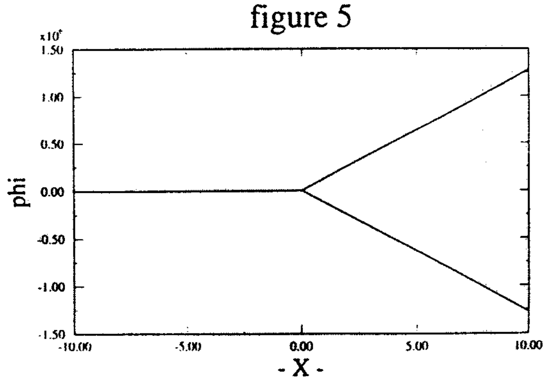


figure 5



CONCLUSION

An excellent approximation of the shear force as well as the generalized plane stress tensor and the bending moments is obtained by using the mixed formulation. The model has been validated by a comparison with an analytical calculation. The mixed elements present the advantage of not locking with bending. Moreover, the model leads to an expression of the energy release rate as a function of the only first derivatives of the unknown values of the problem. Let us finally note that the determination of the derivatives is done with the initial mesh and that we do not need a second mesh of the structure with a progression of the delamination in order to compute the energy release rate.
However, the resolution of the system with a conjugate gradient can be long when using a factorization of A needs an important memory place. Three kinds of preconditioning have been tested to decrease the number of iterations. It would however be interesting to find a compromise between time and memory place. A better renumbering algorithm that takes into account the continuity conditions between plates would be a good way of decreasing the memory place when A is factorized.

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