BOUNDARY ELEMENT ANALYSIS OF FATIGUE AND DAMAGE TOLERANCE

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Abstract

This paper describes an application of the boundary element method to the analysis of fatigue crack growth and damage tolerance in linear elastic fracture mechanics.

Fatigue crack growth is simulated with an incremental analysis of the crack extension, based on the maximum principal stress criterion; for each increment of the crack extension, the boundary element method is applied to perform a stress analysis of the cracked structure and the J-integral technique is used to compute the stress intensity factors.

Since the crack extension is modelled with new boundary elements, remeshing is not required in the analysis. This represents a practical advantage of the present formulation over the finite element method, which needs a continuous remeshing to follow the crack extension.

Damage tolerance analysis is carried out as a post-processing procedure on results of the crack extension. Results of this strategy of analysis are presented for several cracked structures.

Introduction

Cracks, present to some extent in all engineering structures, either as a result of manufacturing fabrication defects or localized damage in service, may grow slowly during normal service conditions by fatigue due to cyclic loading. The crack growth leads to a decrease in the structural strength. As a consequence, when the service loading cannot be sustained by the current residual strength, fracture occurs leading to the failure of the structure. In this context, damage tolerance defines whether a crack can be sustained safely during the projected service life of the structure, generating the following information, upon which fracture control decisions can be made:

- The effect of a crack on the structural residual strength, leading to the evaluation of its maximum permissible size.
- The crack growth as a function of time, leading to the evaluation of the time that the crack spends to reach its maximum permissible size, from which the safe operational life of the structure is defined.

Linear elastic fracture mechanics can be used in the analysis of damage tolerance to describe the behaviour of cracks, which is determined solely by the values of the stress intensity factors.

Fatigue crack growth is simulated with an incremental crack-extension analysis, in which the crack path is computed by a criterion defined in terms of the stress intensity factors. For each increment of the crack extension, a stress analysis is carried out and the stress intensity factors are evaluated.

Numerical methods must be used for the evaluation of the stress intensity factors in engineering structures. The finite element method, with a long and well documented history in fracture mechanics applications (see Gallagher [1] and Rice *et al.* [2]), has been applied to study crack-growth processes, see [3] to [7]. An intrinsic feature of the finite element method is the need for continuous remeshing to follow the crack extension; this is a practical disadvantage of the method.

The boundary element method, well established in fracture mechanics, has also been applied in an incremental analysis of crack extension problems by Ingraffea *et al.* [8]. However, the solution of general crack problems cannot be achieved with the direct application of the method, in a single-region analysis, because the coincidence of the crack surfaces gives rise to a singular system of algebraic equations. The equations for a point located at one of the surfaces of the crack are identical to those equations for the point on the opposite surface, with the same coordinates, because the same integral equation is applied with the same integration path, at both coincident points.

Among the techniques devised to overcome this difficulty, the most general are the subregions method, Blandford *et al.* [9] and the dual boundary element method, introduced by Portela *et al.* [10]. The subregions method introduces artificial boundaries into the structure, which connect the cracks to the boundary in such a way that the domain is divided into subregions without cracks. The main drawback of this method is that, in an incremental crack extension analysis, these artificial boundaries must be repeatedly introduced for each increment of the crack extension. In addition, the method generates a larger system of algebraic equations than is strictly required.

The boundary element method proposed by Portela *et al.* [10] (BEM) incorporates two independent boundary integral equations, with the displacement equation applied for collocation on one of the crack surfaces and the traction equation on the other. Although in a single-region analysis the integration path is still the same for coincident points on the crack surfaces, the respective boundary integral equations are now distinct. Furthermore, this formulation can eliminate the need for remeshing in crack-extension problems that is typical of the finite element and multi-region boundary element methods.

This paper is concerned with the application of the BEM to the analysis of fatigue crack growth and damage tolerance in linear elastic fracture mechanics. The boundary integral equations are presented, the crack modelling strategy defined and the evaluation of the stress intensity factors explained. An incremental crack-extension analysis is performed to determine the crack path. For each increment of the analysis, in which the crack extension is modelled with new boundary elements, the BEM is applied for the stress analysis and the J-integral technique is used for the stress intensity factors evaluation. The incremental analysis is based on a prediction-correction technique to define the direction of the crack-extension increment. The maximum principal stress criterion is applied to predict the tangent direction of the crack path and then a correction is introduced to determine the actual direction of the increment of crack-extension. Fatigue crack growth is caused by cyclic loading; in the simplest case, the loading cycle has a constant amplitude and is described by a static load level with a stress amplitude ratio. Results of a fatigue crack-growth analysis, the residual-strength and fatigue-life diagrams, are also presented for a simple problem.

The Boundary Integral Equations

The boundary integral equations, on which the BEM is based, are the displacement and the traction boundary integral equations. The displacement boundary integral equation can be derived with the classical work theorem, Portela [11]. In the absence of body forces and assuming continuity of the displacements at a boundary point $\mathbf{x}'$, the boundary integral representation of the displacement components u_i is given by

$$c_{ij}(\mathbf{x}')u_i(\mathbf{x}') + \int_\Gamma T_{ij}(\mathbf{x}',\mathbf{x})\, u_j(\mathbf{x})\, d\Gamma(\mathbf{x}) = \int_\Gamma U_{ij}(\mathbf{x}',\mathbf{x})\, t_j(\mathbf{x})\, d\Gamma(\mathbf{x}), \qquad (1)$$

where i and j denote cartesian components; $T_{ij}(\mathbf{x}',\mathbf{x})$ and $U_{ij}(\mathbf{x}',\mathbf{x})$ represent the Kelvin traction and displacement fundamental solutions, respectively, at a boundary point $\mathbf{x}$. The distance between the points $\mathbf{x}'$ and $\mathbf{x}$ is denoted by r. The integrals in equation (1) are regular, provided $r \neq 0$. As the distance r tends to zero, the fundamental solutions exhibit singularities; they are a strong singularity of order $\frac{1}{r}$ in T_{ij} and a weak singularity of order $\ln \frac{1}{r}$ in U_{ij}. The symbol $\fint$ stands for the Cauchy principal-value integral, and the coefficient $c_{ij}(\mathbf{x}')$ is given by $\delta_{ij}/2$ for a smooth boundary at the point $\mathbf{x}'$ in which δ_{ij} is the Kronecker delta.

In the absence of body forces and assuming continuity of both strains and tractions at $\mathbf{x}'$ on a smooth boundary, the stress components σ_{ij} are given by

$$\frac{1}{2}\sigma_{ij}(\mathbf{x}') + \oint_\Gamma S_{ijk}(\mathbf{x}',\mathbf{x})\, u_k(\mathbf{x})\, d\Gamma(\mathbf{x}) = \\ = \fint_\Gamma D_{ijk}(\mathbf{x}',\mathbf{x})\, t_k(\mathbf{x})\, d\Gamma(\mathbf{x}). \qquad (2)$$

In this equation, $S_{ijk}(\mathbf{x}',\mathbf{x})$ and $D_{ijk}(\mathbf{x}',\mathbf{x})$ contain derivatives of $T_{ij}(\mathbf{x}',\mathbf{x})$ and $U_{ij}(\mathbf{x}',\mathbf{x})$, respectively. The integrals in equation (2) are regular, provided $r \neq 0$. As the distance r tends to zero, S_{ijk} exhibits a hypersingularity of the order $\frac{1}{r^2}$, while D_{ijk} exhibits a strong singularity of the order $\frac{1}{r}$. The symbol $\oint$ stands for the Hadamard principal-value integral. On a smooth boundary, the traction components t_j are given by

$$\frac{1}{2}t_j(\mathbf{x}') + n_i(\mathbf{x}') \oint_\Gamma S_{ijk}(\mathbf{x}',\mathbf{x}) u_k(\mathbf{x}) d\Gamma(\mathbf{x}) = \\ = n_i(\mathbf{x}') \fint_\Gamma D_{ijk}(\mathbf{x}',\mathbf{x}) t_k(\mathbf{x}) d\Gamma(\mathbf{x}), \qquad (3)$$

where n_i denotes the ith component of the unit outward normal to the boundary, at the point $\mathbf{x}'$. Equations (1) and (3) constitute the basis of the BEM, as presented by Portela *et al.* [10].

Crack-Modelling Strategy

The general crack-modelling strategy, developed by Portela *et al.* [10], can be summarized as follows:

- the crack boundaries are modelled with discontinuous quadratic elements, as shown in Figure 1;
- continuous quadratic elements are used along remaining boundaries of the structure, except at an intersection between a crack and an edge, where discontinuous or semi-discontinuous elements are required on the edge in order to avoid a common node at the intersection, as shown in Figure 1.
- the displacement equation (1) is applied for collocation on one of the crack surfaces;
- the traction equation (3) is applied for collocation on the other crack surface;
- the displacement equation (1) is applied for collocation on all non-crack boundaries.

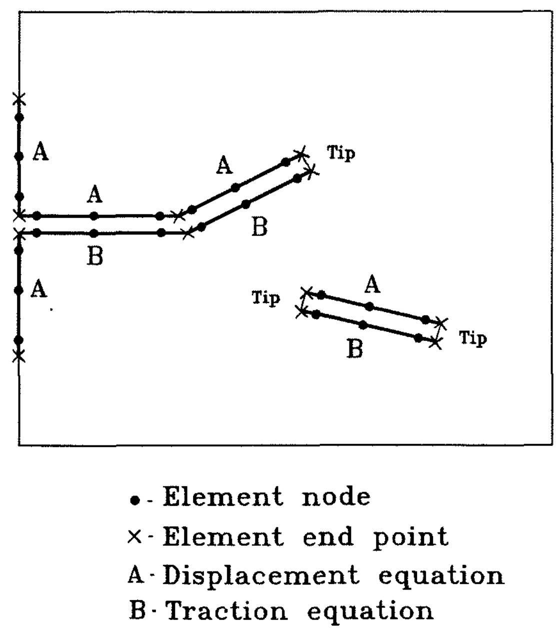


Figure 1: Crack modelling with discontinuous quadratic boundary elements.

This simple strategy is robust and allows the BEM to effectively model general edge or embedded crack problems.

In general, cracking processes can extend the cracks along curved paths. However, in practice, curved paths are usually modelled with flat increments which lead to piece-wise flat crack paths. For such cracks, all the integrals in equations (1) and (3) are most effectively carried out by direct analytic integration, as shown by Portela *et al.* [10].

When the present modelling strategy is applied in an incremental crack-extension analysis, such that each new crack-extension increment is modelled with new boundary elements, it becomes obvious that remeshing is not required, because the new boundary elements will generate new equations and up-date the ones already existing with new unknowns. In other words, the introduction of new boundary elements along a crack-extension increment will generate new rows and new columns in the matrix of the final system of equations. Assuming that the crack extension is traction-free, the right hand side of the system of equations is only extended for the positions corresponding to the new unknowns introduced. This procedure is illustrated schematically in Figure 2. If the *LU*

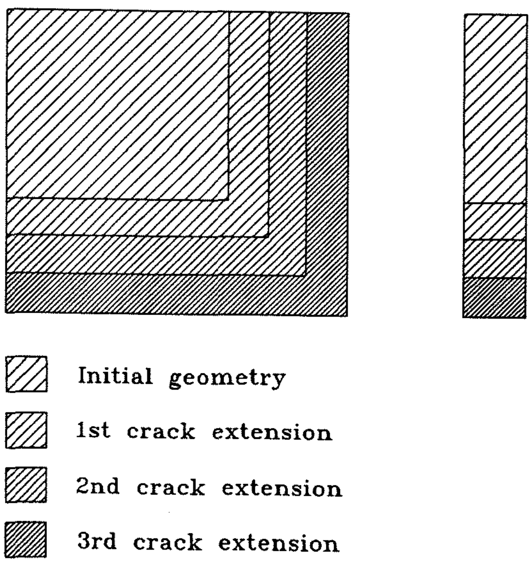


Figure 2: Schematic representation of the generation of the final system of equations.

decomposition method is adopted for the solution of the system of equations, a very efficient incremental analysis can be carried out. For each increment of the analysis, only the new rows and new columns need to be *LU*-decomposed. The existing rows and columns, already decomposed, are brought from the previous iteration into the current one.

Computation of Stress Intensity Factors

The J-integral method was chosen to obtain the stress intensity factors in this paper, because it relies on the values of the elastic field evaluated at a set of internal points. These values are provided accurately by the BEM, since the exact variation of the interior elastic field is built into the fundamental solution.

Consider a cartesian reference system with the origin at the tip of a traction-free crack, as shown in Figure 3. Rice [12] introduced the path-

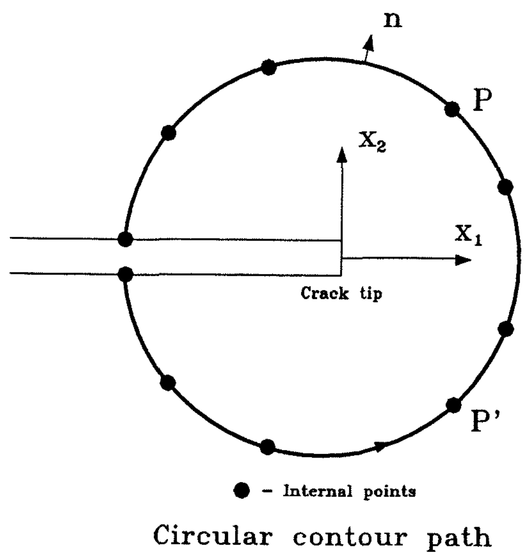


Figure 3: Coordinate reference system and contour path for J-integral.

independent J-integral which in the absence of body forces is given by

$$J = \int_S (W n_1 - t_j u_{j,1}) \, dS, \qquad (4)$$

where S represents an arbitrary contour surrounding the crack tip; W represents the strain energy density; t_j represent the traction components, given by $\sigma_{ij} n_i$, where n_i represent the components of the unit outward normal to the contour path. The relationship between the J-integral and the stress intensity factors is given by

$$J = \frac{K_I^2 + K_{II}^2}{E'}, \qquad (5)$$

where the constant E' is the elasticity modulus equal to E for plane stress conditions and $E' = E/(1 - \nu^2)$ for plane strain conditions. A simple procedure, based on the decomposition of the elastic field into symmetric and antisymmetric mode components, can be used to decouple the stress intensity factors of a mixed-mode problem, as presented by Portela *et al.* [10] for a circular contour path. The integral J is represented by the sum of two integrals as follows:

$$J = J^I + J^{II}, \qquad (6)$$

where the superscripts indicate the pertinent deformation mode, given by

$$J^M = \int_\Gamma \left(W^M n_1 - t_j^M u_{j,1}^M \right) d\Gamma, \qquad (7)$$

for $M = I$ or $M = II$. This decomposition allows the following relationships:

$$J^I = \frac{K_I^2}{E'}, \quad J^{II} = \frac{K_{II}^2}{E'}. \qquad (8)$$

The implementation of this procedure into the boundary element method is straightforward. A circular contour path, around the crack tip, is defined with a set of internal points, located at symmetrical positions relative to the crack plane of the last increment, as shown in Figure 3. The two contour points on the crack surfaces are the first and the last points of the path respectively. At these points, it is always verified that $n_1 = -1$ and $n_2 = 0$ and thus, for a traction-free crack $t_2 = 0$. The integration along the contour path can be accomplished with the trapezoidal rule. For the sake of simplicity, only circular paths containing crack nodes, were considered in this paper. The accuracy of this technique was already demonstrated with several benchmark problems by Portela *et al.* [10].

Crack-Extension Direction

The maximum principal stress criterion, which postulates that the growth of the crack will occur in a direction perpendicular to the maximum principal stress, was chosen to describe the local direction of mixed-mode crack growth. Thus, the local crack-growth direction θ_t is determined by the condition that the local shear stress is zero, that is

$$K_I \sin\theta_t + K_{II}(3\cos\theta_t - 1) = 0, \qquad (9)$$

where θ_t is an angular coordinate centered at the crack tip and measured from the crack axis ahead of the tip. As a continuous criterion, the maximum principal stress does not take account of the discreteness of the crack extension modelling procedure. Therefore, in an incremental analysis, the tangent direction of the crack-path, predicted by equation (9), must be corrected to give the direction of the actual crack-extension increment.

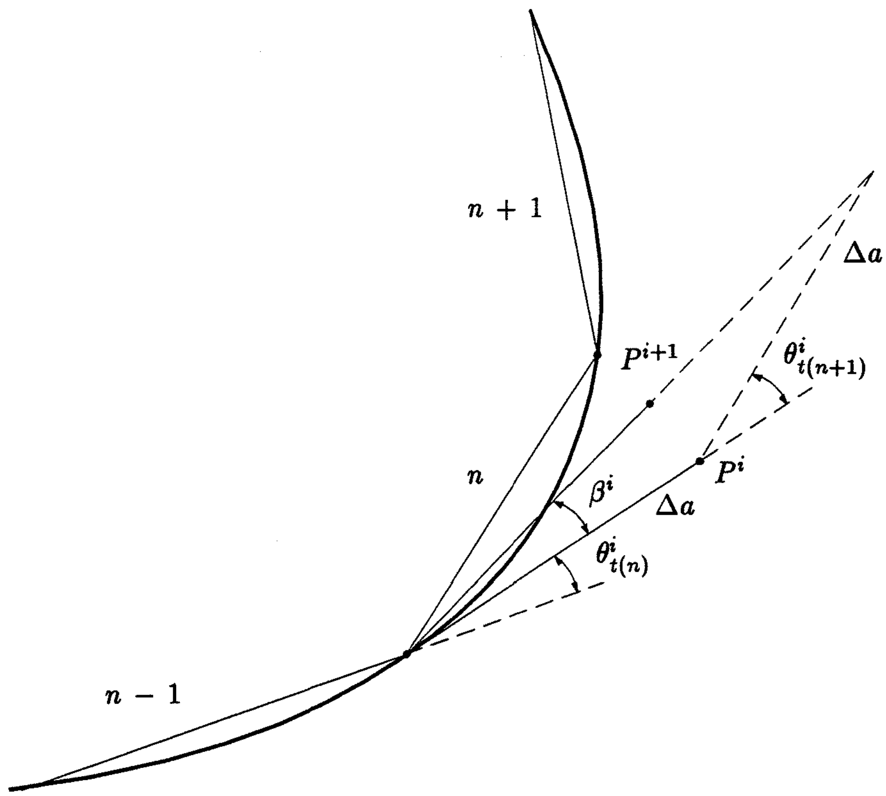
The procedure applied to define the direction of the n-th crack-extension increment introduces a correction angle β to the tangent direction $\theta_{t(n)}$ predicted by the maximum principal stress criterion, as shown in Figure 4. Using geometric relationships, this correction angle is given by $\beta = \theta_{t(n+1)}/2$, in which $\theta_{t(n+1)}$ is the direction of the next crack-extension increment, also computed with the maximum principal stress criterion. For the current nth crack-extension increment, the ith iteration of this predictor-corrector procedure can be summarized as follows:

- for the first iteration only, evaluate the crack-path tangent direction $\theta_{t(n)}^i$ with the maximum principal stress criterion, equation (9);
- along the direction computed in the previous step, extend the crack one increment Δa to P^i and evaluate the new stress intensity factors;
- with the new stress intensity factors and the maximum principal stress criterion, equation (9), evaluate the next crack-path direction $\theta_{t(n+1)}^i$;
- define the correction angle $\beta^i = \theta_{t(n+1)}^i/2$, measured from the increment defined in the second step;
- correct the crack-extension increment, defined in the second step, to its new direction given by $\theta_{t(n)}^{i+1} = \theta_{t(n)}^i + \beta^i$;
- starting from the second step, repeat the above steps sequentially while $|\beta^{i+1}| < |\beta^i|$.

When the size of the crack-extension increment, Δa tends to zero, the angle $\theta_{t(n+1)}$ also tends to zero and so does the correction angle, β. This means that in the limit, the direction of the incremental crack-extension tends to the direction of the tangent of the continuous crack path.

Incremental Analysis

The incremental analysis of crack extension assumes a piece-wise linear discretization of the unknown crack path. For each increment of the crack extension, the BEM is applied to carry out a



n – current crack-extension increment
Δa – increment size
i – iteration number
θ_t – increment direction computed with the tangent criterion
β – correction angle ($\beta = \theta_{t(n+1)}/2$)
P – crack tip location

Figure 4: Incremental crack-extension direction.

stress analysis of the structure and the J-integral is used for the evaluation of the stress-intensity factors. The steps of this basic computational cycle, repeatedly executed for any number of crack-extension increments, are summarized as follows:

- carry out a BEM stress analysis of the structure;
- at the crack tip compute the stress intensity factors with the J-integral technique;
- compute the direction of the crack-extension increment;
- extend the crack one increment along the direction computed in the previous step;
- repeat all the above steps sequentially until a specified number of crack-extension increments is reached.

For the sake of simplicity, the increment of the crack extension is discretized with a fixed number of new boundary elements. The length of the crack extension increment may be defined as the result of a compromise between accuracy and computational cost; the smaller the crack increment the more accurate and expensive is the analysis.

The results obtained from an incremental analysis of crack-extension are a crack-path diagram and diagrams of the stress intensity factor variation along the crack path.

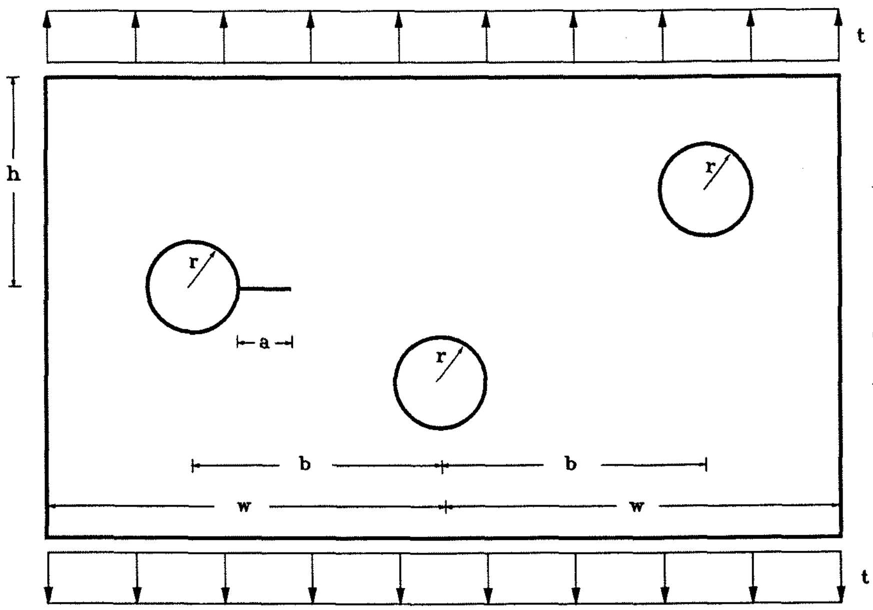


Figure 5: Crack growing from a hole in a rectangular plate.

Crack Extension Applications

A single application of the crack-extension incremental analysis will be presented in this section. Consider a crack growing from a fastener hole in a three-hole rectangular plate with the characteristic dimensions $w = 1.5\ m$, $h = 1\ m$,$b = 0.6\ m$ and $d = 0.5\ m$, schematically represented in Figure 5. The initial crack was considered with the length $a = 0.1\ m$ and the holes with radius $r = 0.2\ m$. A uniform normal traction $t = 10\ MN/m^2$ is applied at the ends of the plate in the direction perpendicular to the axis of the initial crack. The plate was assumed with the elastic constants $E = 200000\ MN/m^2$ and $\nu = 0.25$. An incremental analysis of the crack extension was carried out. Results, obtained with the maximum principal stress criterion and a crack increment equal to three times the length of the initial crack-tip element, are presented in Figures 6 and 7, in terms of stress intensity factors and crack path, respectively.

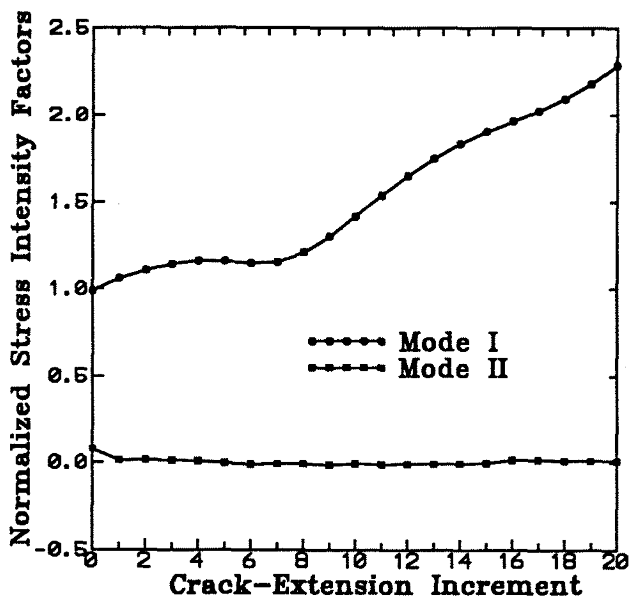


Figure 6: Stress intensity factors for the cracked plate problem.

Fatigue Crack Growth and Damage Tolerance

In general, fatigue crack-growth is driven by variable amplitude loading. In the simplest case,

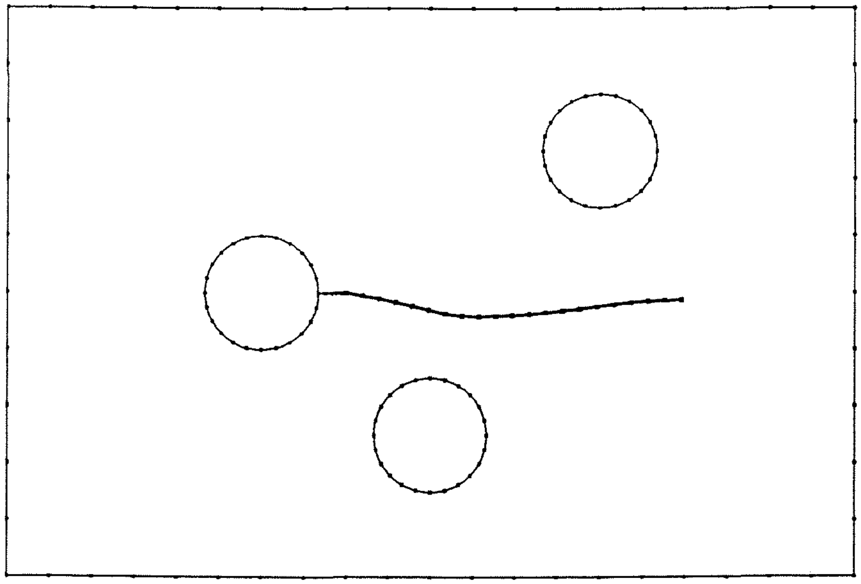


Figure 7: Crack path for the cracked plate problem.

the one considered here, the loading cycles have a constant amplitude and may be described by a constant amplitude load with a constant amplitude stress ratio. The aim of the damage tolerance analysis is to obtain the residual-strength and fatigue-life diagrams associated with the fatigue crack growth. From these diagrams, both the maximum permissible size of each crack and the safe operational life of the structure can be evaluated. In this paper, the analysis of fatigue crack growth is introduced as a post-processing procedure on the results of the crack-extension analysis.

The residual-strength diagram shows the variation in the maximum load that the cracked structure can sustain, that is, the load that causes fracture instability, as the crack length varies. Thus, for a given crack length a, the residual strength, σ_c is given by the critical load which is defined as

$$\sigma_c = \frac{K_{Ic}}{Y\sqrt{\pi a}}, \qquad (10)$$

where Y is the geometry factor. For the same crack length, the geometry factor is constant and hence, can be defined in terms of the applied reference stress, σ_r and the corresponding mode I equivalent stress intensity factor, K_{Ieq} as

$$Y = \frac{K_{Ic}}{\sigma_c\sqrt{\pi a}} = \frac{K_{Ieq}}{\sigma_r\sqrt{\pi a}}. \qquad (11)$$

In a mixed-mode analysis the equivalent mode I stress intensity factor can be defined, for the maximum principal stress criterion, as

$$K_{Ieq} = K_I \cos^3\frac{\theta_t}{2} - 3K_{II}\cos^2\frac{\theta_t}{2}\sin\frac{\theta_t}{2}. \qquad (12)$$

From equation 11, the following relationship between the residual strength and the applied reference stress holds:

$$\sigma_c = \frac{K_{Ic}}{K_{Ieq}}\sigma_r. \qquad (13)$$

At each step of an incremental analysis, the residual strength is represented conveniently in a normalized form, derived from equation (13) as

$$\frac{\sigma_c}{\sigma_{c*}} = \frac{K_{Ieq*}}{K_{Ieq}} \qquad (14)$$

where σ_{c*} and K_{Ieq*} represent the residual strength and the corresponding mode I equivalent stress intensity factor, respectively, computed at the initial crack length.

The fatigue-life diagram shows the variation in the number of loading cycles, required to extend the crack, as a function of the crack length. It was computed from the generalized Paris model defined as

$$\frac{da}{dN} = C(\Delta K_{eff})^m, \qquad (15)$$

where a is the crack length, N is the number of load cycles, C and m are material dependent constants and $\Delta K_{eff} = \Delta K_{Ieq}$ is the range of the effective stress intensity factor. The stress intensity factor range of the individual modes is given by $\Delta K = K_{max} - K_{min} = K_{max}(1 - R)$, in which $R = K_{min}/K_{max} = \sigma_{min}/\sigma_{max}$ is the stress amplitude ratio of the loading cycle. The number of loading cycles required to extend the crack a given increment is evaluated by integration of equation (15) with the trapezoidal rule, applied for each increment of the analysis.

For the plate with a cracked fastener hole represented in Figure 5, a fatigue cracking process was generated by a constant amplitude cyclic loading with a stress ratio $R = 2/3$. The constants used in Paris law were defined as $C = 4.624E - 12$ and $m = 3.3$. Results of fatigue crack growth and damage tolerance analysis were obtained for this problem. The residual strength diagram is presented in Figure 8. The fatigue-life diagram, obtained

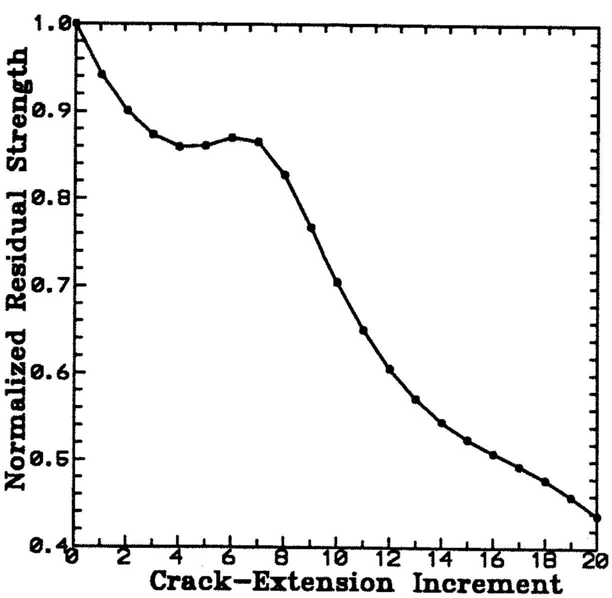


Figure 8: Residual-strength diagram for the cracked plate problem.

for the case in which the stress ratio=0.66666, C =4.624E-12 and m =3.3, is presented in Figure 9. With these diagrams and a knowledge of the fracture toughness of the material, a complete damage tolerance analysis can be carried out.

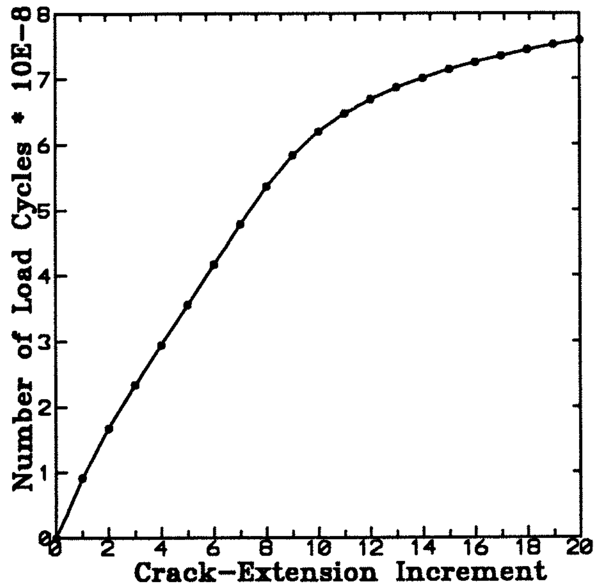


Figure 9: Fatigue-life diagram for the cracked plate problem.

Conclusions

In this paper, the boundary element method is applied to the incremental analysis of fatigue crack growth and damage tolerance using linear elastic fracture mechanics. For each increment of the crack extension, a stress analysis of the structure is carried out and the stress intensity factors are evaluated with the J–integral technique. This basic computational cycle is repeated for an arbitrary number of crack extension increments.

The boundary element method incorporates two independent equations: one is the displacement boundary integral equation and the other is the traction boundary integral equation. When the displacement equation is applied for collocation on one of the crack surfaces and the traction equation is applied for collocation on the other, general mixed-mode crack problems can be solved in a single-region formulation. This feature constitutes a practical advantage of the dual boundary element method over the finite element and multi-region boundary element methods, because remeshing, extensively used in these methods, is no longer required when the crack extension increment is modelled with new elements.

The new boundary elements, introduced for the discretization of each crack extension increment, generate new algebraic equations and introduce new unknowns in the equations already existing. An *LU* decomposition method is used to solve the system of algebraic equations. The *LU* decomposition is carried out incrementally over the new rows and new columns of the system matrix. The remaining part of the system matrix is carried over, already decomposed, from the previous iteration of the analysis.

The reliability of the whole incremental analysis process lies in the accuracy of the stress intensity factors evaluation, as well as on the accuracy of the prediction of the crack-extension increment direction. In this paper, the stress intensity factors are evaluated by the J-integral technique, applied with a circular contour around each crack tip. Since the exact variation of the interior elastic field is built into the fundamental solution of the boundary element method, this technique is very effective. In contrast, in the finite element method, the field is approximated by a polynomial variation in each element.

A simple prediction-correction procedure was adopted for the definition of the direction of each crack-extension increment. It predicts the tangent direction of the crack path using the maximum principal stress criterion and then introduces a correction to this direction with information derived from one step ahead of the current crack-extension increment.

Results of an incremental crack-extension analysis are presented. A fatigue crack-growth and damage tolerance analysis is also presented for the simplest case of constant amplitude loading cycles.

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