SINGULAR PERTURBATION THEORY APPLIED TO THE FLIGHT CONTROL SYSTEM DESIGN

Wang Zhongjun, Xu Ruijuan
Department of Aircraft Engineering
Jiang Yunxiang
Department of Automatic Control
Northwestern Polytechnical University
Xian, Shaanxi Province, 710072 P.R.China

Abstract

A class of singular perturbation method developed by B.Porter and A.Bradshaw is one of the most practical method to design flight tracking control system . If the rank of matrix *[CB]* is deficient , it must introduce measurement matrix *M* to complete the system design , but there is arbitraty in selecting matrix *M* . In general , the high-gain control law can make closed loop system robust. The main contributions of this paper are followings:

(1) The eigenstructure assignment method has been applied to select measurement matrix *M* and it is calculated according to partial eigenvalues and eigenvectors of closed loop system (slow mode) .

(2) In the computation of null space of matrix, it uses singular value decomposition in complex domain rather than real domain to select some eigenvectors of closed loop system . This simplify the computation complexity.

(3) It analyzes the robustness of designed flight control system quantitatively by calculating the minimum singular values of invers-difference matrix.

In the last part of this paper, an example of direct lift control is presented to verify proposed method. It points out that not only the robustness of closed loop system designed by using above method is very good , but the system's response is excellent also.

1. Introduction

There are many methods to design flight control system. Eigenstructure assignment method (*EAM*) has the properties of less computationlly and assigned eigenvalue/ eigenvectors of closed loop system to decouple modes directly, but it is necessary to verify the robustness of closed loop system. Control laws designed by singular perturbation method (*SPM*) in corporating high-gain, error-actuated controller make system have good robustness. If rank of matrix *[CB]*is deficient , it is necessary to introduce measurement matrx *M*, there is arbitrary in selecting matrix *M*. Reference[1] suggests that matrix *M*should choose as sparse as possible , if and only if martix *[FB]* is full rank. Reference[2] utilizes pole assignment to compute matrix *M*and suggests that eigenvectors should be considered in computiug matrix *M*. The objective of this paper is to design flight control system by taking the advantges of *SPM*and *EAM*. The designed controllers not only make closed loop system heve good response (modes decouple), but have good robustness also [7] . At last , it gives the robustness comparisions between this two methods.

2. Singular Perturbation Methods in Design of Tracking System

Through linear translation, any completely controllable and observable linear time-invariant system may be represented by following state equations.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} u \tag{1}$$

$$y = [C_1 \quad C_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2}$$

To design tracking system which makes output $y\ (t)$ track input command $v\ (t)$, extra measurement matrix Mmust be introduced if $[CB]$ is rank deficient [3], that is

$$\begin{aligned} w(t) &= y + M\dot{x}_1 \\ &= [C_1 + MA_{11} \quad C_2 + MA_{12}] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= [F_1 \quad F_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned} \tag{3}$$

The control law equation is of the form

$$u(t) = g\{K_0 e(t) + K_1 z(t)\} \tag{4}$$

where

$$e(t) = v(t) - w(t) \tag{5}$$

$$z(t) = z(0) + \int_0^t e(t)dt \tag{6}$$

In the steady −state,$\dot{x}_1 \to 0$,then,

$$\lim_{t\to\infty}(w(t) - y(t)) = \lim_{t\to\infty}(M\dot{x}_1) = 0 \tag{7}$$

it requires

$$\lim_{t\to\infty} e(t) = \lim_{t\to\infty}(v(t) - w(t)) = 0 \tag{8}$$

So, $e\ (t)$ is the difference between output $y\ (t)$ and input command $v\ (t)$ in steady−state . In equation (1) ~ (6), $x_1 \in R^{n-l}$,$x_2 \in R^l$,$u \in R^l$,$y \in R^l$,$v \in R^l$,$z \in R^l$,$g \in R^+$,$w \in R^l$, A_{11}, A_{12}, A_{21}, A_{22}, B_2, C_1, C_2, F_1, F_2, M, K_0, K_1 are the matrices of appropriate dimensions and rank $[C_2B_2] < l$, rank $[F_2B_2] = l$. After substituting equations (3) ~ (6) into (1), (2), closed loop system may be written as follows :

$$\begin{bmatrix} \dot{z} \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -F_1 & -F_2 \\ 0 & A_{11} & A_{12} \\ gB_2K_1 & A_{21} - gB_2K_0F_1 & A_{22} - gB_2K_0F_2 \end{bmatrix} \begin{bmatrix} z \\ x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} I_l \\ 0 \\ gB_2K_0 \end{bmatrix} v(t) \tag{9}$$

$$y = [0 \quad C_1 \quad C_2] \begin{bmatrix} z \\ x_1 \\ x_2 \end{bmatrix} \tag{10}$$

When g→∞, poles of closed loop system are consisted of $Z_1 U Z_2 U Z_3$, the system can be asymptotic described by slow mode and fast mode.

Poles of slow mode are $Z_1 U Z_2$

$$Z_1 = \{\lambda \in C: |\lambda K_0 + K_1| = 0\} \tag{11}$$

$$Z_2 = \{\lambda \in C: |\lambda I_{n-l} - A_{11} + A_{12}F_2^{-1}F_1| = 0\} \tag{12}$$

Poles of fast mode are Z_3

$$Z_3 = \{\lambda \in C: |\lambda I_l + gF_2B_2K_0| = 0\} \tag{13}$$

From reference[3], there are l assignable poles in Z_1 set, Z_2 set contains all transmission zeros of system. When g→∞, l poles become asymptotic uncontrollable in Z_1 set, (n− l) poles in Z_2 set become asymptotic unobservable. The locations of transmission zeroes can be changed by selecting output matrix C or measurement matrix M. In order to make system work stable , it requires

$$Z_1 U Z_2 U Z_3 \subset C^- \tag{14}$$

letting

$$F_2 B_2 K_0 = diag\{\sigma_1, \sigma_2, \cdots \sigma_l\} \tag{15}$$

the matrix K_0 can be calculated from equation (15) , if $\sigma_i > 0$,i= 1, 2, ⋯ l, poles of fast mode in Z_3 set are $-g\sigma_i$ when g→∞.

If condition (14) may be satisfied and selected matrix Mmake $[F_2B_2]$full rank , the closed loop system (9) will work stable.

3. Computation of Measurement Matrix M

Taking

$$S = F_2^{-1} F_1, \quad S \in R^{l \times (n-l)} \tag{16}$$

then, the transmission zeroes of system (1), (2) are goverened by

$$|\lambda I_{n-l} - A_{11} + A_{12} S| = 0 \tag{17}$$

$$(\lambda_i I_{n-l} - A_{11} + A_{12} S)\xi_i = 0 \tag{18}$$

Letting

$$S \cdot \xi_i = \omega_i \tag{19}$$

and substitute (19) into (18), it has

$$(\lambda_i I_{n-l} - A_{11})\xi_i + A_{12}\omega_i = 0 \tag{20}$$

or

$$[\lambda_i I_{n-l} - A_{11} \quad A_{12}]\begin{bmatrix} \xi_i \\ \omega_i \end{bmatrix} = 0 \qquad (21)$$

where $\xi_i \in R^{n-l}$, $\omega_i \in R^l$, ξ_i is the eigenvectors corresponding to transmission zeroes , $\begin{bmatrix} \xi_i \\ \omega_i \end{bmatrix}$ lie in the null space of $[\lambda_i I_{n-l} - A_{11} \quad A_{12}]$, and equations (*18*) ~ (*21*) are satisfied for all transmission zeroes , so

$$S[\xi_1, \xi_2, \cdots \xi_{n-l}] = [\omega_1, \omega_2, \cdots \omega_{n-l}]$$

$$S = [\omega_1, \omega_2, \cdots \omega_{n-l}][\xi_1, \xi_2, \cdots \xi_{n-l}]^{-1} \qquad (22)$$

The null space vectors of $[\lambda_i I_{n-l} - A_{11} \quad A_{12}]$ can be calculated by using singular value decomposition [8] (i= 1, 2, ···n−l) . It is possible to select $\begin{bmatrix} \xi_i \\ \omega_i \end{bmatrix}$ from above null space to satisfy equation (22) and have some physical meanings (for example, mode decouple) [6,9] .

If the selected transmission zeroes are complex, the null space vectors of $[\lambda_i I_{n-l} - A_{11} \quad A_{12}]$can be calculated by using singular vulue decomposition in complex domain.This Simplify the computation complexity comparision with reference[4,5], that is ,

$$[\lambda_i I_{n-l} - A_{11} \quad A_{12}] = U\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}V^* \qquad (23)$$

and $U = [U_1 \ U_2]$, $V = [V_1 \ V_2]$, V^* is the conjugate transpose matrix of V, where $\Sigma = diag\{\sigma_1, \sigma_2, \cdots, \sigma_{n-l}\}$, $\sigma_i = \sqrt{\mu_i}, i = 1,2\cdots,n-l$,

$U \in R^{(n-l)\times n}$, $V \in R^{n\times n}$, $\Sigma \in R^{(n-l)\times(n-l)}$, $V_1 \in R^{n\times(n-l)}$, $V_2 \in R^{n\times l}$.

μ_i are the eigenvalues of matix $[\lambda_i I_{n-l} - A_{11} \quad A_{12}]^* \ [\lambda_i I_{n-l} - A_{11} \ A_{12}]$, V_2 are the eigenvectors corresponding to zero eigenvalaes and lie in the null space of $[\lambda_i I_{n-l} - A_{11} \quad A_{12}]$.

After computing matrix S, solving equation (*6*) and (*16*) simultaneously, the matrix M is obtained by

$$M = (C_2 S - C_1)(A_{11} - A_{12} S)^{-1} \qquad (24)$$

4. Robustness Analysis

The block diagram of closed loop system (*9*) , (*10*) is presented in Fig.1, where the controller transfer function is

$$H(s) = g\{K_0 + K_1 / s\} \qquad (25)$$

the general plant transfer function is

$$G(s) = F(sI - A)^{-1} B \qquad (26)$$

The matrix

$$L(s) = H(s)G(s) \qquad (27)$$

is called U node return ratio. The matrix

$$S(s) = I + L(s) \qquad (28)$$

is cacled U node return difference matrix, and the matrix

$$T^{-1}(s) = I + L^{-1}(s) \qquad (29)$$

is called U node inverse−diffrerence matrix.

The minimum singular value of inverse− difference matrix have relations with system robustness, minimum gain margins and minimum phase margins of system can be calcaluted from minimun sigular value of inverse−difference matrix as following [10] .

Suppose the changed loop transfer function (return ratio) due to system parameters variation and work enviroment change has the form of

$$L'(s) = (I + \Delta L(s)) \cdot L(s) \qquad (30)$$

The system will be stable in frequency range $[\omega_1 \ \omega_2]$, if the inverse−difference matrix $I+L'(s)$ has nonzero−determinant over this frequency range and the maximum allowable perturbation matrix $\Delta L(s)$ is

$$||\Delta L(s)|| < \sigma_{min}(T^{-1}(j\omega)), \ \omega \in [\omega_1 \ \ \omega_2] \qquad (31)$$

where $\sigma_{min}$is the minimum singular value of inverse−difference matrix.

The minimum gain margins and phase margins in each of the plant's control input channel over this frequency range is described as

$$GM = min(20 log_{10}(1 + \sigma_{min}(T^{-1}(j\omega)))) \qquad (32)$$

and

$$PM = min(2arcsin(0.5\sigma_{min}(T^{-1}(j\omega)))) \qquad (33)$$

So, the larger the minimum singular value of inverse–difference matrix, the better the system robustness.

5. Design Example

The short period dynamic motion equations including actuators dynamics of an aircrft which flies at a height of 5000m and a Mach number of 0.5, are described by state equations (1), where [9]

$$A=\begin{bmatrix}0 & 0 & 1.7577 & 0.1670 & 0.2655\\ 0 & -1.5466 & -24.144 & -31.3147 & -15.4163\\ 0 & 1.0 & -1.7577 & -0.1670 & -0.2655\\ 0 & 0 & 0 & -20 & 0\\ 0 & 0 & 0 & 0 & -20\end{bmatrix}$$

$$B=\begin{bmatrix}0 & 0\\ 0 & 0\\ 0 & 0\\ 20 & 0\\ 0 & 20\end{bmatrix}$$

$$x=[\gamma\ q\ \alpha\ \delta_e\ \delta_f]^T$$

$$u=[\delta_{ec}\ \delta_{fc}]^T$$

γ, q, α, δ_e, δ_f are flight path angle, pitch rate, angle of attack, elevator deflection angle and flaperon deflection angle derived from trim state respectively. δ_{ec}, δ_{fc} are input command of elevator, flaperon actuators respectively, the angle unit is radian, the pitch rate unit is radian / sec.

The control objective is to design a control law of the form of equation (4), which makes the aircraft realize fuselage pointing mode ($\alpha_1$mode), i.e $\gamma=0$, $\theta=\alpha$, and take

$$C=\begin{bmatrix}1 & 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0 & 0\end{bmatrix},\quad y=\begin{bmatrix}\theta\\ \gamma\end{bmatrix}$$

In this example, $n=5$, $l=2$, the desired eigenvalues of short period mode and γ mode are selected based on flight quality requirements. These eigenvalues are determined by equation (12), i.e.

$$\lambda^d_{3,4}=-2.5\pm j2.5 \qquad (\xi=0.707, \omega_n=3.54)$$

$$\lambda^d_5=-2.0 \qquad (\gamma\ \ mode)$$

If $K_1=2K_0$, and $\sigma_i=1$ ($i=1, 2$, in equation (15)), then,

$$\lambda^d_{1,2}=-2.0$$

$$\lambda^d_{6,7}=-g \qquad (g\to\infty)$$

When $g\to\infty$, the asymptotic eigenvalues of closed loop system are presented above ($\lambda^d_1\sim\lambda^d_7$)

In short period motion the main physical variables are angle of attack α and pitch rate q. In γ mode, the main variable is flight path angle γ. The desired eigenvectors corresponding to these modes must have the following forms to decouple eachother [9]

$$\overset{\lambda^d_{3,4}}{\begin{bmatrix}0 & 0\\ 1 & 1\\ x & x\\ x & x\\ x & x\end{bmatrix}}\quad \overset{\lambda^d_5}{\begin{bmatrix}1\\ 0\\ 0\\ x\\ x\end{bmatrix}}\quad \begin{bmatrix}\gamma\\ q\\ \alpha\\ \delta_e\\ \delta_f\end{bmatrix}$$

Through singular value decomposition, the vectors in the null space of $[\lambda_i I_{n-l}-A_{11}\ \ A_{12}]$ ($i=3, 4, 5$) are

$$\lambda^d_{3,4}=-2.5\pm j2.5$$

$$\begin{bmatrix}-0.1649 & 0\\ 0.3902\pm j0.7489 & -0.2205\pm j0.2569\\ 0.2367\mp j0.2278 & 0.0955\mp j0.0073\\ 0.1506\mp j0.2744 & -0.3642\mp j0.0106\\ -0.0809\mp j0.2175 & 0.8612\mp j0.0416\end{bmatrix}$$

$$\lambda^d_5=-2.0$$

$$\begin{bmatrix}0.5262 & 0\\ -0.2542 & -0.2039\\ 0.6533 & 0.102\\ 0.4774 & 0.3632\\ 0.0608 & 0.034\end{bmatrix}$$

It can choose three vectors as $\begin{bmatrix}\xi_i\\ \omega_i\end{bmatrix}$, which satisfy equation (22) and is as close as poosible to the form of desired eigenvectors, that is

$$\begin{bmatrix}\xi_1 & \xi_2 & \xi_3\\ \omega_1 & \omega_2 & \omega_3\end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0.5262 \\ -0.2205 \pm j0.2569 & -0.2542 \\ 0.0955 \mp j0.0073 & 0.6533 \\ -0.3642 \mp j0.0106 & 0.4774 \\ 0.8612 \mp j0.0416 & 0.0608 \end{bmatrix}$$

Based on the chosen eigenvalue / eigenvectors, the matrix M, K_0, K_1 is calculated.

$$M = \begin{bmatrix} 0.3913 & 0.0255 & 0.4223 \\ 0.4943 & 0.0058 & -0.0024 \end{bmatrix}$$

$$K_0 = \begin{bmatrix} -0.029 & -0.2722 \\ -0.0664 & 0.5451 \end{bmatrix}, \quad K_1 = 2K_0$$

Letting input command $v = \begin{bmatrix} 1^\circ \\ 0 \end{bmatrix}$, i.e, $\gamma_c = 0^\circ$, $\theta_c = \alpha_c = 1^\circ$, the response of closed loop system are illustrated in Fig.2 ($g = 200$). There is less difference between Fig. 2 and Fig.3 which is the result of reference[9]. When simulation time is greater than 1.2 seconds, θ arrives its steady-state value, γ is constant, the fuselage pointing is achieved without chaning flingt path.

The minimum singular values of invers-difference matrix versus frequency are presented in Fig. 4. Over the frequency $(10^{-1}, 10^2)$, the minimum gain margins, minimum phase margins and minimun singular values of the system designed in this paper (marked *SPM*) is *5.2174db*, *48.6203°* and *0.8234* respectively, the corresponding values of the system designed in reference[9] (marked *EAM*) is *0.2727db*, *1.964°* and *0.0343* respectively. So, the system designed in this paper has good robustness and good response.

6. Conclusions

The fight tracking control system is designed by using singular perturbation method and eigenstructure assignment method in this paper. It is not only considering the system transmission zeroes, but considering the eigenvectors corresponding to them also.

The calculation results show that the designed system not only has good response (mode decouple), but good robustness also.

7. References

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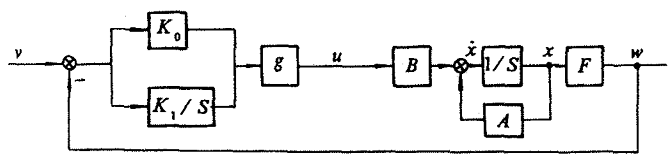

Fig. 1 tracking control system block diagram

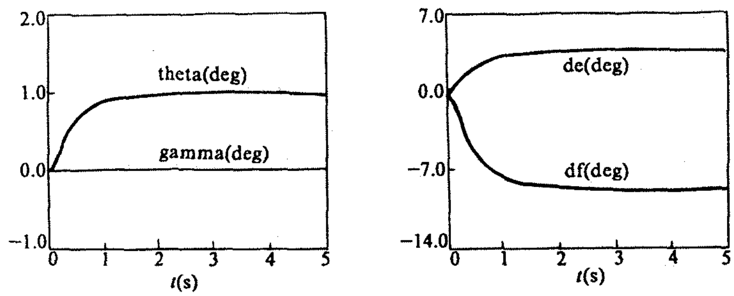

Fig.2 $\alpha_1$mode

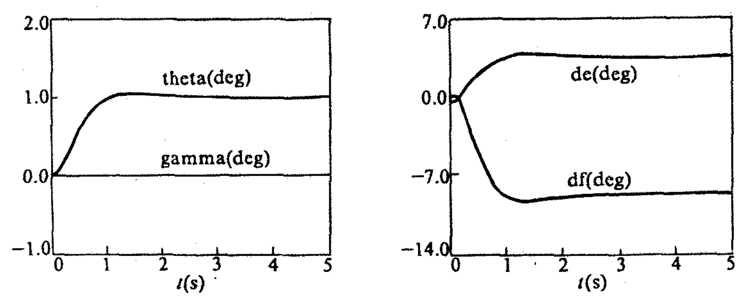

Fig.3 $\alpha_1$mode (c.f. reference[9])

Fig. 4 minimum singular value versus frequency