ESTIMATION OF THE PROBABILITY OF LARGE FLIGHT PARAMETERS DEVIATIONS

V.P.Kuzmin, V.A.Yaroshevsky
TsAGI, Zhukovsky, Russia

Abstract

Problems of the flight the success of which is estimated by meeting constraints for the trajectory parameters at a fixed moment or throughout the whole flight time are examined. Results of studies on the methodology development of the aircraft successful flight probability estimation are given. Numerical solutions of various problems are discussed. Results obtained by the above methodology are compared with other analytical and numerical estimations for model examples. Solutions of applied problems are presented.

Introduction

Problems concerning the estimation of the probability of large flight parameters deviations from nominal values have become extremely vital during the last decade due to the introduction of probability norms into the practice of aeronautical science and technology.

Two types of problems are examined. A typical example of the first type problem ("local" problem) is the analysis of automatic landing process. The major requirement imposed by the existing norms on the landing control system of a civil airplane is that the probability of a grave air accident at automatic landing may not exceed $10^{-6} \div 10^{-8}$. The possibility of an air accident is first of all a function of values of flight parameters (vertical speed V_y , range L, lateral displacement Z from the runway axis etc.) at the airplane's touchdown moment on landing ($H = 0$). Therefore it is vital to estimate the probability of falling of these parameters outside the tolerable limits, for example $P[V_y(H=0) < V_{y\ min}]$.

To the second type (problems of "overshootings") are referred problems on the probability estimation of falling of the flight kinematic parameters outside the tolerable limits at a finite time interval. For instance for the airplane flying in the zone of turbulent atmosphere at a time interval $[t_0, t_f]$ it is necessary to estimate the probability of the angle-of-attack falling outside the safe limits, i.e. to estimate

$$P[\alpha(\tau) > \alpha_{max}], \qquad \tau \in [t_0, t_f]$$

Both types of these problems are characterized by small values of probabilities sought which hinders the use of the widely-spread Monte Carlo technique for mathematical simulation: the number of the required realizations N turns out to be extremely large reaching some millions end even milliards.

Linear estimations based on the hypotheses of the Gaussian distribution of output parameters are not always well-grounded. In reality under extreme conditions, for instance under the impact of strong wind perturbations, the deviations of airplane's control surface or the rates of the deviations reach the allowable limits and besides some perturbations have a non-Gaussian distribution especially in the domain of large deviations from averaged values.

At the same time the small value of the sought-for probabilities simplifies the use of asymptotic estimations for the calculation of extreme deviations of flight parameters.

Along with probabilistic approaches to the solution of the posed problems "guaranteed" approaches are also used in practice whenever for the available random perturbations (for example given as a the whole complex of random parameters) limiting values are specified and as predicted cases most unfavourable combinations of limiting perturbations are considered [1]. Without denying in principle the "guaranteed" approach the authors nevertheless believe that the probabilistic approach is more naturally occurring.

Approximate solution of the local problem

While formulating the first type problem for a finite-dimensional case it is assumed that the set of random perturbations is reduced to a finite number of independent Gaussian normalized random parameters $c_i \quad (i = 1, n)$:

$$p(c) = (2\pi)^{-\frac{n}{2}} \exp\left(-\frac{1}{2}\sum_{i=1}^{n} c_i^2\right)$$

The output variable y is a continuous differentiable function of c_i ; the excess probability of variable y of value y_* is defined by integral

$$P[y > y_*] = \int_{\Omega} p(c)\, dc_1 \dots dc_n \qquad (1)$$

taken around domain Ω which is located behind the surface of level $y = y_*$. If this probability is too small and surface $y = y_*$ is noticeably far from the origin of coordinates than the primary contribution to the integral value is made by the vicinity of surface point C_*, the nearest to the origin of coordinates, since in domain Ω function $p(c)$ diminishes rapidly as it moves away from point C_*. Then it is naturally to suppose that in substituting the surface of level $y = y_*$ by a plane which is a tangent to the surface in point C_* (Fig.1) one may roughly estimate the probability sought [2]:

$$P[y > y_*] \approx P_1 = \overline{\Phi}(R) = \frac{1}{\sqrt{2\pi}} \int_R^\infty \exp\left(-\frac{u^2}{2}\right) du \quad (2)$$

where $R^2 = \sum_{i=1}^{n} c_{i*}^2$

Hence for the probability estimation in a first approach (P_1) it is sufficient to solve a constrained optimization problem:

$$R^2 = \min_{c_i} \sum_{i=1}^{n} c_i^2 \quad (3)$$

under condition $y(c) = y_*$ or an equivalent problem $y_* = \max y(c)$ under condition $\sum_{i=1}^{n} c_i^2 = R^2$ where R is a function of the specified probability $P : \overline{\Phi}(R) = P$.

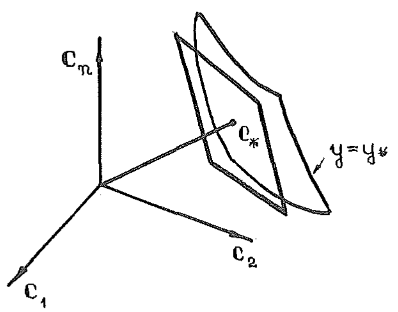

Figure 1: Location of the level surface in the space of random parameters

Solution of such problem may be based on a combination of stochastic and gradient techniques of optimization. The input labour augments noticeably with the increase of the dimensionality of the space of random parameters.

In order to improve on this estimation for non-linear function $y(c)$ one should take into account the surface level curvature, and moreover the most important here is the form of the surface level in the vicinity of point C_*. By rotating the system of axes $C_1, \ldots, C_n$ so that the axis $\tilde{C}_1$ is directed towards point C_* we shall write the surface level equation in the form:

$$\tilde{C}_1 = R + \frac{1}{2R} \sum_{i=2}^{n} \sum_{k=2}^{n} a_{ik} \tilde{C}_i \tilde{C}_k + \ldots,$$

where $a_{ii} > -1$.

Then the improved estimation of the probability sought is defined by the formula:

$$P[y > y_*] \approx P_2 = P_1 \left[\det\,(E + A)\right]^{-1/2}, \quad (4)$$

where E is the unit matrix, A is the matrix composed of elements a_{ik}.

The second approach estimation (4) may cause noticeable error if certain values of a_{ii} are close to -1 or equal to -1 (singular case).

Another estimation improvement technique (2), acceptable also for singular cases, is associated with the use of a special modification of the Monte–Carlo technique i.e. a substantial sampling technique allowing with the use of spherical coordinates to determine approximately the multi-dimensional integral (1) [3,4].

For the infinite-dimensional case a typical problem of the first type is reduced to the analysis of solutions of a stochastic equation examined in the sense of Stratonovitch [5].

$$\frac{dx}{dt} = f(x,t) + G(x,t) \cdot \xi(t), \quad (5)$$

where $\xi(t)$ is the vector of independent white noises, $f(x,t)$ and $G(x,t)$ the vector and matrix function, $x(t)$ the vector of phase coordinates.

If the initial value $x(t_0)$ is fixed and one has to evaluate the probability $P[x_1(T) < x_*]$, where $x_1(t)$ is the first component of vector $x(t)$ then in a first approach by analogy with the finite-dimensional case the problem is reduced to an equivalent variational problem [6,7] concerning determination of vector $\xi(t)$ giving minimum to value $R^2 = \sum_{i=1}^{m} \xi_i^2(t)$ under condition $x_1(T) = x_*$, where m is the dimensional representation of the white noise vector.

Then

$$P[x_1(T) < x_*] \approx P_1 = \frac{1}{\sqrt{2\pi}} \int_R^\infty \exp\left(-\frac{u^2}{2}\right) du \quad (6)$$

If the initial conditions for vector x are not fixed but have a Gaussian distribution then in the equation for R^2 a quadratic form of values $\Delta x_i(t_0)$ is added, characterizing deviations of components of vector x_0 from their averaged values.

In some cases one succeeds in solving an equivalent variational problem analytically for example for scalar equation

$$\frac{dx}{dt} + f(x) = g \cdot \xi(t) \quad (7)$$

at $t_0 = -\infty$, $f(0) = 0$, $x_* > 0$ it appears that

$g \cdot \xi = 2f(x)$ at $f(x) > 0$,

$g \cdot \xi = 0$ at $f(x) < 0$;

for equation

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + f(x) = g \cdot \xi(t) \quad (8)$$

at $t_0 = -\infty$, $k > 0$, $f(0) = 0$, $\frac{df}{dx}(x) > 0$ it appears that

$$g \cdot \xi = 4k\frac{dx}{dt}$$

From analogy with a finite-dimensional case one may improve the estimation of the probability sought if one considers the second variation of the functional and uses the neighbouring extremals technique [8] , which is equivalent to formula (4) for the finite-dimensional case:

$$P_2 = P_1 \left[\det\left(E + A\right)\right]^{-1/2}$$

where matrix A is defined as a solution of ordinary differential equations after solution of the variational problem [7].

Automatic landing

The use of the technique described for the calculation of the automatic landing process of a passenger airplane using a complete model of perturbations was performed in the following way.

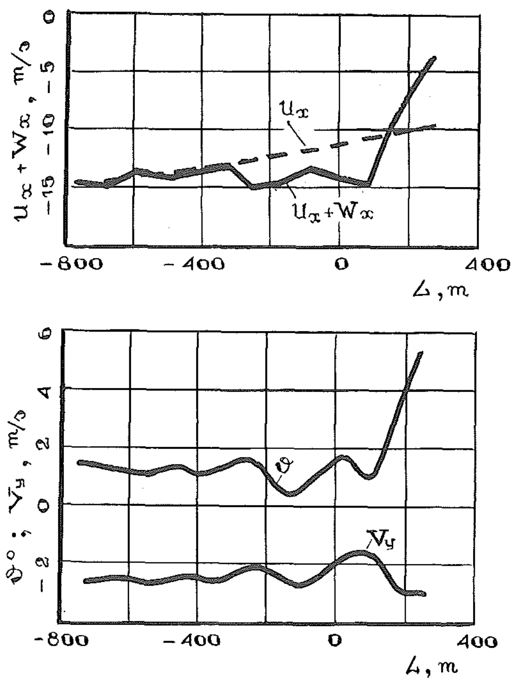

Figure 2: Automatic landing. Variation of wind $U_x + W_x$, pitch angle θ and vertical speed V_y with range L. Minimum vertical speed at touchdown.

As perturbations two components of a horizontally-directed systematic wind U_x and U_z were considered as well as the atmospheric turbulence related to them (longitudinal and lateral gusts W_x and W_z). In order to describe random functions defining the atmospheric turbulence by a finite number of random parameters these functions were replaced by canonical series [9], for instance

$$W_x(t) = \sum_{k=1}^{n} a_k \varphi_k(t),$$

where a_k are normalized independent Gaussian random parameters, $\varphi_k(t)$ are deterministic coordinate functions.

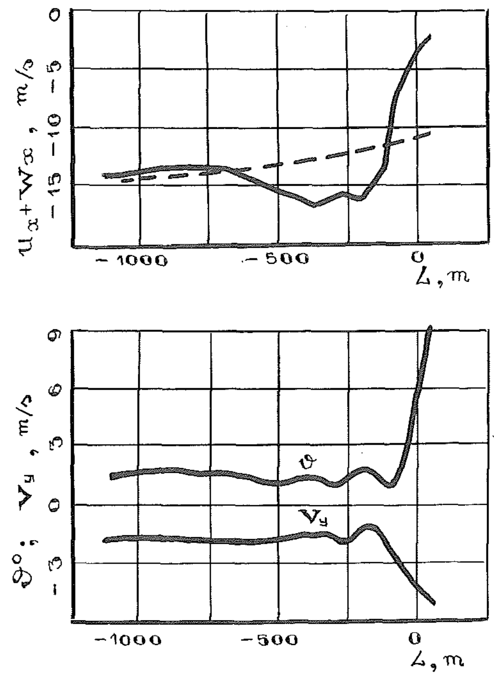

Figure 3: Automatic landing, minimum range.

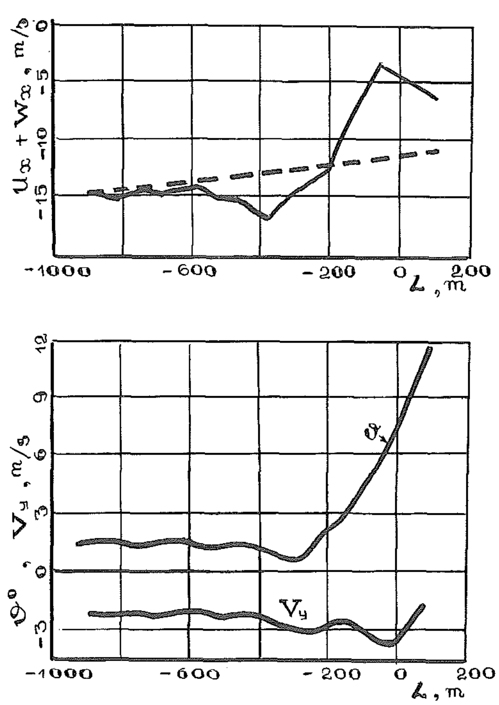

Figure 4: Automatic landing, maximum pitch angle.

This case is close to a singular one since the strongest impact on the landing accuracy is attributed to the atmospheric turbulence with root-mean-square values of gusts

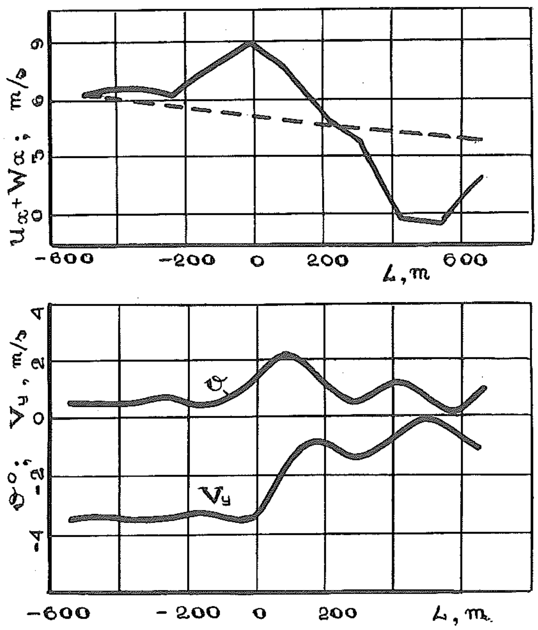

Figure 5: Automatic landing, minimum pitch angle.

σ_W proportional to the modulus of systematic wind $U = \sqrt{U_x^2 + U_z^2}$. Therefore an unconditional probability of violating specified constraints imposed on kinematic parameter values at the touchdown was calculated after calculating of conditional probabilities corresponding to various values of U_x and U_z . Fig. 2-6 reveal typical "critical" realization of the atmospheric turbulence for which extremal values of different parameters are achieved. The structure of these realization is different and is dependent first of all on transfer functions for the closed-loop system "airplane-control system".

Methodology of solution of "overshooting" problems in a simplified formulation

The problems of second type associated with the estimation of probability of the threshold level crossing by random functions at a definite time interval appear to be considerably more complicated than the problems of the first type. Here one succeeded in getting the necessary estimations only for the cases when the dynamic system excited by a white noise is similar to the system of the first or of the second order.

For a dynamic system of the first order of type (7) the probability sought $P[x(\tau) > x_*]$, $\tau \in [0,t]$ is defined in terms of solution of the corresponding Fokker-Plank-Kolmogorov equation [5] by the Fourier technique for which it is necessary to determine the spectrum of eigen values μ_k

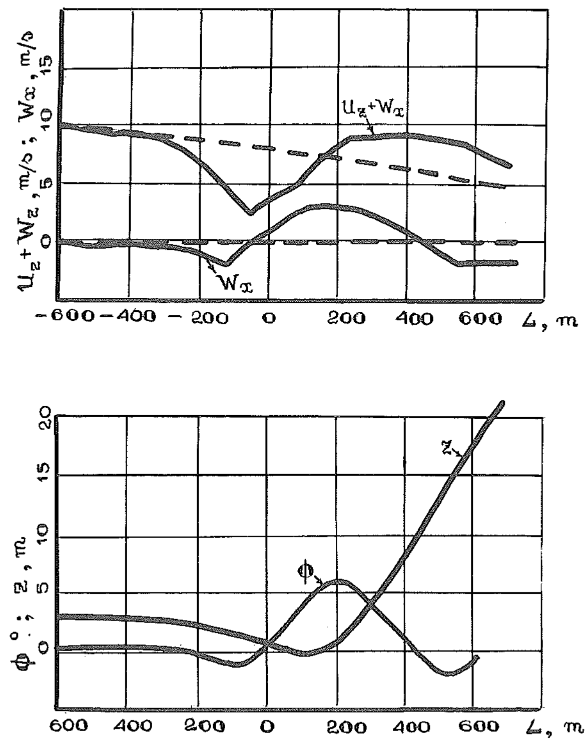

Figure 6: Variation of longitudinal and lateral wind, roll angle Φ and lateral displacement Z with range L. Maximum lateral displacement.

of a one-dimensional boundary problem:

$$P[x(\tau) > x_*] = 1 - \sum_{k=0}^{\infty} b_k e^{-\mu_k t}$$

For sufficiently large values of x_* and t this probability is determined by a simplest formula:

$$P[x(\tau) > x_*] \approx 1 - e^{-\mu_0 t} \qquad (9)$$

practically regardless of the initial distribution of variable x, and value μ_0 is defined by relation [5] :

$$\mu_0 \approx g^2 \left[\int_0^{x_*} \frac{dx}{p_{st}(x)} \right]^{-1} \qquad (10)$$

where $p_{st}(x)$ is the stationary distribution of variable x. This result relates to a simplest example of a non-differentiable random function $x(t)$ (the dispersion of derivative $\dot{x}$ is infinite, $\sigma_{\dot{x}} = \infty$).

For the case of a differentiable random function $x(t)$ ($\sigma_{\dot{x}} < \infty$) which is typical for the systems of higher order excited by a white noise at sufficiently large x_* and t the probability sought is as before described by an approximated formula (9) where value μ_0 is the least eigen value of a multi-dimensional boundary problem for which one cannot succeed in obtaining simple analytical expressions. For

Gaussian processes it is proved [10] that at $x_* \to \infty$ the value moves near $\overline{N}$-average number of intersections of function $x(t)$ of level x_* from inside per a unit of time

$$\mu_0 \approx \overline{N} = \int_0^\infty \dot{x} p_{st}(x, \dot{x})\, d\dot{x} \tag{11}$$

However in the case of large but finite values of x_* the estimation (11) may turn out to be essentially overstated since it is equivalent to the assumption that the subsequent level crossings are independent of the previous ones.

The above is illustrated by numerical predictions performed for a dynamic system [11] :

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + |x|^n \cdot \mathrm{sign}\, x = 2\sqrt{k}\xi(t) \tag{12}$$

for different $k > 0$, $n \geq 0$. The probability of overshooting

$$P[|x(\tau)| > x_*], \qquad \tau \in [0, t]$$

was defined at essential intervals of time t.

At $n = 1$ (a linear system, $\sigma_x = 1$, $\sigma_{\dot{x}} = 1$) the largest number of overshootings is observed for values of k somewhat less than 1, and for large and small values of k the rate of overshootings is reduced (Fig.7) though the estimation in this case gives the result independent on k:

$$\mu_0 \approx \overline{N} = \frac{1}{\pi}e^{-\frac{R^2}{2}} \tag{13}$$

where $R = x_*$ (in this case the estimation (11) should be doubled because intersections of two levels $\pm R$ is being fixed).

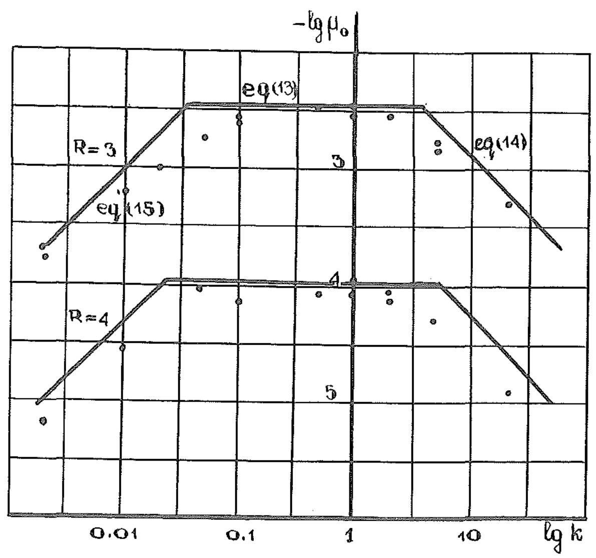

Figure 7: Comparison of analytical estimations of parameter μ_0 with numerical experiment.

The reason is in the fact that for large k variable x is the output of the chain of two aperiodic links with highly different constants of time ($2k$ and $\frac{1}{2k}$), excited by the white noise. Then the role of the link with a small constant of time appears to be of minor importance, variable $x(t)$ becomes "close to non-differentiable", intersections by function $x(t)$ of any level occur in a "flickering" manner i.e. by series. For the other limiting case at very small k, intersections of levels by function $x(t)$ also occur by series since process $x(t)$ is oscillatory and weakly damped and variable $x(t)$ having once reached level x, in many cases repeats these intersections.

As a result of solution of equation (12) at $n = 1$ for the case of large k and small k one may make approximation using solutions of equations of the first order: for the first case it is sufficient to neglect term $\frac{d^2x}{dt^2}$ in the equation (12), and for the second case to pass over to an equivalent equation to change the amplitude of oscillations [5,12]. Then using the formula similar to (10) one may obtain estimations

$$\mu_0 \approx \frac{R}{k\sqrt{2\pi}}e^{-\frac{R^2}{2}} \tag{14}$$

- at large k,

$$\mu_0 \approx kR^2e^{-\frac{R^2}{2}} \tag{15}$$

- at small k.

Results of calculations using these formulae are also given in Fig.7. The set formulae (14), (15) and (13) represents with a sufficient completeness "exact" results obtained during numerical predictions. As R increases the limits of using formula (13) are extended.

Similar numerical predictions were carried out for cases when $n = 0$ and $n = 3$ in equation (12). Correspondingly, for these cases formulae similar to those of (14), (15) and (13) were obtained [11].

The results obtained, permit to estimate the probability of overshootings of angles-of-attack for the airplane with non-linear characteristics flying in a turbulent atmosphere if the transfer function of the closed-loop system "airplane-control system" is near the transfer function of a dynamic system of the first or the second order.

Conclusions

Estimation of a small probability of violating the preset constraints for flight parameters ($10^{-6} \div 10^{-8}$) is an extremely tedious problem of mathematical simulation. For problems of the first type when parameters are defined at a fixed moment of time the proposed methodology permits to confine oneself to calculations of several hundreds realizations regardless of the value of the probability. For problem of the second type when parameters are defined at a certain interval of time a methodology is developed that can be used only in those cases when the system "airplane-control system" is similar to a dynamic system of the first or of the second order.

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