OPTIMALITY-BASED CONTROL LAWS FOR REAL-TIME AIRCRAFT CONTROL VIA PARAMETER OPTIMIZATION

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Abstract

The potential for real-time aircraft control via the control law obtained by the application of the "optimality-condition" is investigated. This optimality-based control law possesses simplicity and yet preserves the original structure of the highly complex, nonlinear, optimal control. The technique is applied to aircraft trajectory optimization, in particular, the minimum-time-to-climb and the minimum-time-to-turn problems. Robustness is also studied via simulations with respect to various initial state disturbances and plant/aerodynamic modelling uncertainties. The results indicate that the optimality-based control law exhibits excellent natural robustness in terms of meeting the final flight conditions when a certain "correction" term is included. The open-loop solutions are also given for comparison purposes.

Introduction

With the advancement in high-speed computers, interest has grown considerably in obtaining a whole range of aircraft optimal control laws for real-time aircraft control. The optimal control problem is usually solved by applying the first-order necessary conditions to the variational Hamiltonian. However, the difficulty inherent in solving the resulting nonlinear two-point boundary-value problem (TPBVP) has prevented the extensive use of this technique as design tool to acquire and implement such optimal flight control laws. To circumvent these mathematical and computational difficulties, several reduced-order techniques have been proposed. Two well-known techniques are the energy-state approximation[1-9] and singular perturbations.[10-19]

In the energy-state approximation, energy replaces the velocity and altitude as the state. By assuming small flight path angle and angle of attack, additional simplification in modelling complexity is achieved. Bryson, et al,[1] have applied this technique successfully to the classical minimum-time-to-climb problem. Other authors have had similar success applying this technique to aircraft turns[2-4] and minimum-fuel[5-9] problems. Because of its reduced order, this approach is suitable for quick aircraft performance estimation and is attractive for on-board real-time display of trajectory information. In many cases, closed-form solutions are possible. The main disadvantage to this technique is that in allowing the kinetic and the potential energies to trade back and forth in zero time at a given energy level, discontinuities in velocity and altitude histories occur. An alternate approach is the use of singular perturbation techniques introduced by Kelley.[10-12] In these approaches, the system dynamics are separated into slow and fast modes, and the solution of a higher-order problem is approximated by the solution of a series of lower-order problems. These methods have been shown by many authors to be effective and efficient in the computation of aircraft optimal trajectories. For example, Weston, et al,[13] applied it to the climb-dash intercept mission in two dimensions, while Calise[14], Visser, et al,[15] and Sheu, et al,[16] have had equal success in three-dimensional interception problems. Ardema[18-19] used it to treat aircraft pursuit-evasion and maneuverability problems.

Recently, another technique has been proposed by Lu.[20] He suggested applying the maximum principle and the information yielded by the necessary conditions to aid in parameterizing optimal feedback control laws. Here, the costate variables were parameterized instead of the control function. He successfully tested this approach to the problems of robot arm and aeroglider control. In this paper, we extend his idea to obtain optimal control laws for several aircraft maneuvers. We investigate the application of the "optimality condition" to develop a real-time control law for minimum-time-to-climb and minimum-time-to-turn problems. This optimality-based control law has the property of being simple and yet preserves the state structure information of the original optimal control. In general, the proposed control law is not linear. A brief background on the development of the optimality-based control law is given, and its application to the above problems are presented. For comparison purposes, we have included the open-loop solutions.

Optimality-Based Control Law Development

Consider the following class of free-end-time optimal control problems. The goal is to determine $u(t) \in U$ which minimizes the performance index

$$J = \phi[x(t_f)] + \int_{t_o}^{t_f} L(x,u,t)dt \tag{1}$$

subject to

$$\dot{x} = f(x,u,t) \tag{2}$$

with $x(t_o)$ given, as well as some specified $x_i(t_f)$ where $i = 1, \ldots, q$. Here, $x \in R^n$ is the state; $u \in R^m$ is the control; U is the class of piecewise continuous controls; t is the time; $L,\ f \in C^2$ are known functions of their arguments, and t_f is the terminal time.

The optimal control problem is solved by using the variational Hamiltonian,[22]

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$$H(x,u,\lambda,t) = L(x,u,t) + \lambda^T f(x,u,t) \quad (3)$$

where the costate variables $\lambda \in R^n$ are continuously differentiable functions of time. Application of the first-order necessary conditions results in

State Equations and Specified Initial and Final States

$$\dot{x} = H_\lambda^T = f(x,u,t) \quad (4)$$

$$x(t_o) = x^o \quad (5)$$

$$x_i(t_f) = x_{f_i} \quad (i = 1, ..., q) \quad (6)$$

Costate Equations and Transversality Conditions

$$\dot{\lambda} = -H_x^T \quad (7)$$

$$\lambda(t_f) = \begin{cases} \nu_i, & i = 1, ..., q \\ \phi_{x_j}|_{t=t_f}, & j = 1+q, ..., n \end{cases} \quad (8)$$

Optimality Condition

$$H_u^T = 0 \quad (9)$$

Terminal Time Condition

$$(\phi_x^T + H)|_{t=t_f} = 0 \quad (10)$$

where $\nu \in R^q$ is the undetermined multiplier vector associated with the specified terminal states. Eqns. (4)-(10) constitute a two-point boundary value problem. To obtain an approximate optimal control *law*, Lu[20] suggests solving the optimality conditions for the control u^* as a function of state and costate variables, and then instead of parameterizing the control $u(t)$, he now parameterizes the costate variables. This approach has been tested successfully in two examples. In this control law development, we extend the technique a step further and parameterize the coefficients of the states instead of the costates alone. As a hypothetical example, suppose after applying the optimality condition, we obtain

$$u^*(x,\lambda,t) = \lambda_1(t)x_1^2(t) + \frac{\lambda_1(t)}{\lambda_2(t)}x_2 + \lambda_3(t)x_3(t) \quad (11)$$

Now instead of the costates, the coefficients of the states are parameterized; that is, we can propose a nonlinear control *law* of the form

$$\bar{u}^*(x,a,t) = a_1(t)x_1^2(t) + a_2(t)x_2(t) + a_3(t)x_3(t) \quad (12)$$

The $a_i(t)$'s are now the new control functions to be determined via parameter optimization techniques. Often times, the optimality condition results in a highly complex structure for the control which involves not only the costate variables but also the atmospheric properties and aerodynamic characteristics of the aircraft. The advantage of this approach is that all these terms can be implicitly included in the $a_i(t)$'s. It is not difficult to make such engineering judgements since the structure of the optimal control obtained via the optimality condition would serve as a guide. The resulting optimality-based control law (OBCL) possesses simplicity and yet retains the original state structure of the optimality condition which may or may not be linear in nature. It is expected that this OBCL would exhibit robustness with regard to initial state disturbances and plant/aerodynamic modelling uncertainties when compared to the open-loop solution. If it does not, a further bonus is that an intuitive term could be added to the OBCL to enhance its robustness; for example, we can add $k(t)f(x)$ to Eqn. (12) to obtain

$$\bar{u}^*(x,a,t) = a_1(t)x_1^2(t) + a_2(t)x_2(t) + a_3(t)x_3(t) + k(t)f(x) \quad (13)$$

Although there is no theory to justify this term, we will show that this is indeed practical and posssible numerically. As a note, our primary goal is to obtain the simplest feedback controls needed to achieve a specific purpose. Sequential quadratic programming (SQP)[21] is chosen to solve the resulting optimal control problem with the fixed-structure control law $\bar{u}^*$ replacing u^*. This nonlinear programming method is computationally accurate and inexpensive, and in addition, its flexibility allows for performance indices and constraint changes with little reprogramming. In this technique, interpolated control functions, $a_i(t)$, based on a fixed number of control points are determined. Optimization then takes place over this set of control points rather than the entire control history.

Choice of Initial a(t)'s

A question arises immediately about how one chooses an initial set of control points for $a_i(t)$ for the optimization process. Due to an infinite range available, many combinations are possible. Although no standard procedure of selecting the initial guess for $a(t)$ is available, it still can be done and with a careful selection the optimization process will easily converge to a solution. In our technique, the initial control points are selected based on the result of the open-loop solution. We first make an engineering judgement to weight each term in the proposed control law. The values of the control points for each term are then chosen accordingly, and given the optimal states from the open-loop solution, the closed-loop control is calculated. If the value of the closed-loop control matches approximately the optimal open-loop control, these control points will be retained and used as the starting guess for the closed-loop optimization process. The advantage of this selection process is that one has complete control of distributing the values of the control points for the $a_i(t)$.

Numerical Examples

We are applying this approach to aircraft trajectory optimization. As a preliminary study, the classical point-mass aircraft minimum-time-to-climb in a vertical plane and a constant altitude minimum-time-to-turn problem are considered. Data for the aircraft maximum thrust, T(V,h), and the aerodynamic coefficients are taken from an early representation of the F-4 fighter aircraft.[1] We are using analytical representations of these data prepared by Ong.[23] In Appendices I and II, the development of the "optimality-based" control laws are given. In this section, numerical simulations based on the proposed control laws obtained in the appendices for specific flight conditions are studied. We discuss first the results of the minimum-time-to-climb problem, and in the minimum-time-to-turn

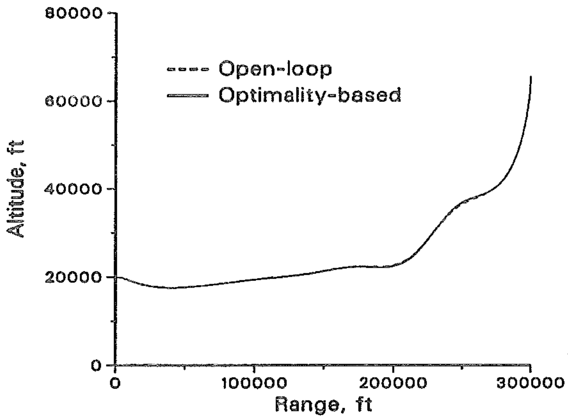

Figure 1: Open-loop and optimality-based optimal flight profile comparisons

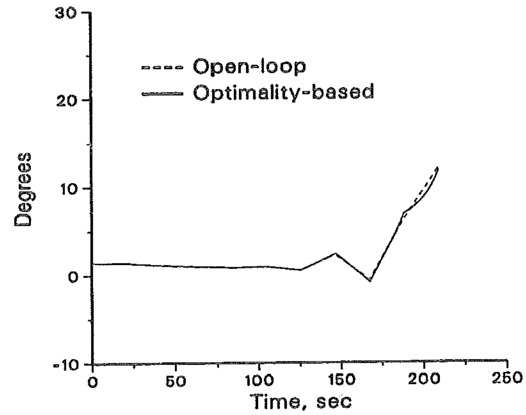

Figure 2: Open-loop and optimality-based optimal angle of attack history comparisons

problem we show that by adding a "correction" term in the OBCL the robustness can be enhanced significantly.

Minimum-Time-To-Climb Solutions

In this problem, we want the aircraft to fly from an initial state of $V(0)$ = 830 ft/s (M = 0.8), $\gamma(0) = 0°$, $h(0)$ = 20000 ft, $x(0)$ = 0, and $m(0)$ = 1088 slugs to a final state of $V(t_f)$ = 968.1 ft/s (M = 1), $h(t_f)$ = 65600 ft, and $x(t_f)$ = 300000 ft in minimum time; $\gamma(t_f)$ and $m(t_f)$ are free. Eleven equally-spaced control points were used for both $a(t)$ in the OBCL solution and $u(t)$ in the open-loop solution.

In Fig. 1, the optimal climb trajectories obtained using the proposed optimality-based control law $\bar{\alpha} = a(t)/V$ are compared to the optimal open-loop solution. Note that these two optimal trajectories are virtually identical. The aircraft first flies at almost a constant altitude to gain speed, and near the end of the flight it executes a zoom-climb maneuver. The optimal flight time for the open-loop solution is t_{ol} = 209.1 s, while that for the optimality-based control law is t_{obcl} = 209.5 s. This is not surprising because parameterizing the control function $a(t)$ to obtain $\alpha(t)$ is equivalent to parameterizing the $\alpha(t)$ directly. However, the advantage here is that the proposed control law incorporates velocity feedback while the other does not. We expect that both the angle of attack histories for the open-loop and the optimality-based control law should also be nearly identical. This is the case as shown in Fig. 2. We note that the high angle of attack history near the end of the flight is a result of the zoom-climb maneuver. We anticipate the proposed control law to be more robust than the open-loop solution in the presence of external disturbances. This robustness will be investigated in the presence of two initial state disturbances: i) +5% in initial velocity, and ii) +5° in initial flight path angle. Robustness in the presence of plant uncertainties is not investigated, but will be studied in the minimum-time-to-turn problem.

In Fig. 3, the robustness of our optimality-based control law compared to the open-loop solution in the presence of the initial velocity perturbation is given. It is encouraging to see that the perturbed solutions for both the closed-loop and open-loop do not deviate greatly from the nominal solutions, but compared to the open-loop perturbed trajectory, the closed-loop deviations is smaller. This can be explained by virtue of the fact that the angle of attack control in the optimality-based control law has velocity state feedback. Thus, any perturbation in the velocity will be accounted for in the angle of attack.

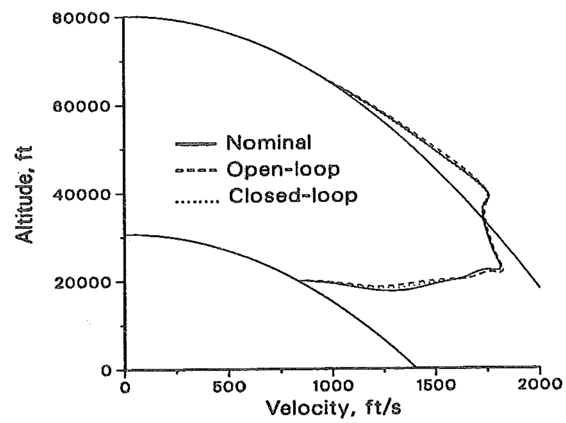

Figure 3: Energy diagram with initial velocity perturbation

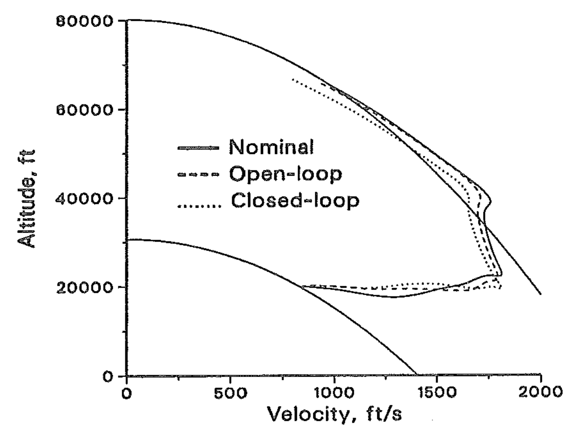

Figure 4: Energy diagram with initial flight path angle perturbation

In Fig. 4, the robustness of both the open-loop and the optimality-based control law solutions in the presence of initial flight path angle perturbation is given. Unlike the initial velocity perturbation case, both solutions deviate

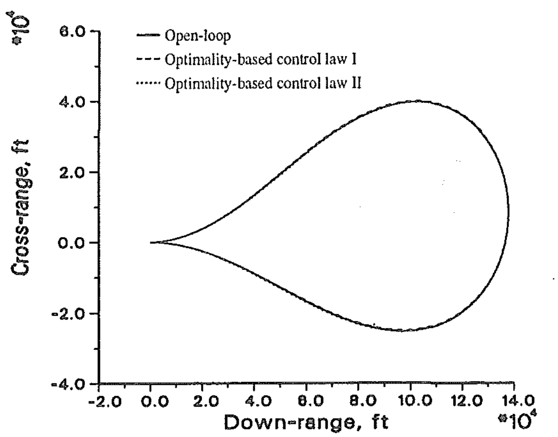

Figure 5: Optimal turn trajectories for open-loop and optimality-based control law I and II solutions

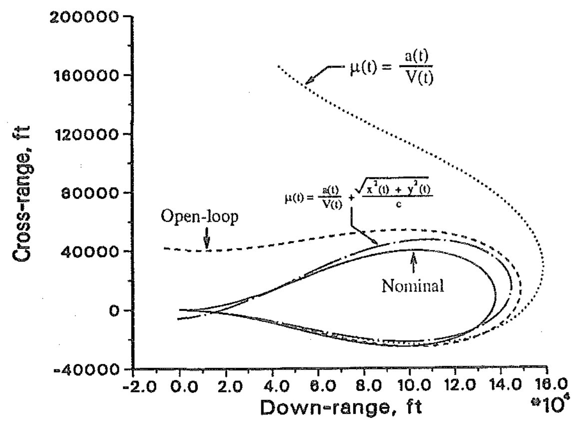

Figure 6: Trajectory comparisons with +5% initial velocity perturbation

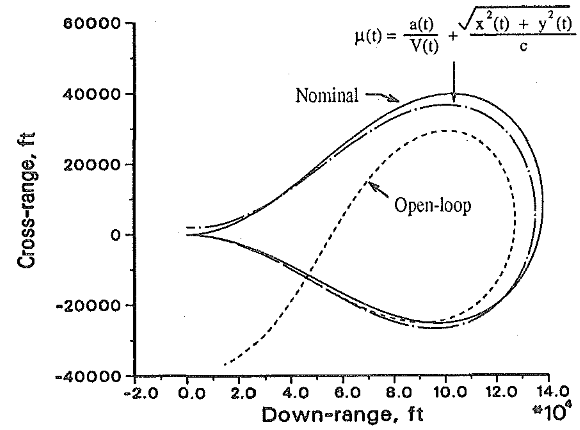

Figure 7: Trajectory comparisons with -5% initial velocity perturbation

much more from the nominal solution. In fact, the closed-loop solution surprisingly deviates more from the nominal solution than those for the open-loop solution. This can be explained by the fact that the flight path angle does not appear in the proposed control law, and because of this any perturbation in the flight path angle will not be accounted for in the control law. In an ad hoc way, if we include the flight path angle in the proposed control law, this deviation could be reduced. This idea of including more feedback states is demonstrated in the minimum-time-to-turn problem.

Minimum-Time-To-Turn Solutions

In this problem, we would like the aircraft to make a 180° turn and return to its original position, but with a different final velocity; that is, we want to fly from an initial state of $V(0) = 1355.9$ ft/s, $\psi(0) = 0°$, $x(0) = y(0) = 0$ ft, and $m(0) = 1088$ slugs to a final state of $V(t_f) = 1210.2$ ft/s, $\psi(t_f) = 180°$, and $x(t_f) = y(t_f) = 0$ ft. The final mass is free, and the altitude for the turn is 50000 ft. The proposed two optimality-based control laws i) $\bar{\mu}^* = a_1(t)/V$ and ii) $\bar{\mu}^* = a_1(t)/V + \sqrt{x^2+y^2}/a_2$ are tested against the open-loop solutions. The reason for the additional term in the optimality-based control law II will become clear when the results are discussed.

As in the case of the minimum-time-to-climb problem, eleven equally-spaced control points are employed. Fig. 5 shows the optimal flight profiles for the open-loop and optimality-based control law I and II (OBCL-I, OBCL-II). It is apparent that the optimal trajectories are nearly identical. This comes as no surprise since the problem is the same for all three cases except for the control approximations. As a result, the optimal flight times are also identical: $t_f^* = 253.3$ sec.

Thus far, the proposed OBCL-I and II perform as well as the optimal open-loop solution assuming a perfect system and no external disturbances. Athough there is no theory to back-up our claim, we would like to show that these optimality-based control laws also exhibit excellent robustness with regard to initial state disturbances and plant/aerodynamic modelling uncertainties. The next two sections address the robustness issue.

Presence of Initial State Disturbances

In this section, we investigate the robustness of each control law in the presence of i)±5% of initial velocity, ii) ±5° of initial heading angle, and iii) ±1000 ft of initial down- and cross-range distance perturbations. To save space, only the plots for the initial velocity disturbances are given as shown in Figs. 6 and 7. Table 1 shows the final state values compared to the desired final states in the presence of the other initial state disturbances.

The advantage of the correction term in OBCL-II now becomes clear. In Fig. 6, note that the open-loop solution out performs significantly OBCL-I. However, by adding the correction term as in OBCL-II, it now performs better, i.e., its robustness is enhanced which in turn allows the perturbed trajectory to follow closely the nominal solution. This additional term involves down- and cross-range as feedback states, and it allows OBCL-II to possess natural robustness. The intuitive justification for this natural robustness is as follows. When the aircraft strays "outside" the nominal trajectory, the correction term increases in value. This increases the bank angle which aids the aircraft in returning to it nominal solution. The reverse is true as shown in Fig. 7 in the presence of −5% initial velocity perturbation. In this case, the aircraft strays "inside"

Table 1: Final states values due to initial state disturbances

Uncertainties	Optimality-based control law I, II				Open-loop			
	V_f, ft	ψ_f, deg	x_f, ft	y_f, ft	V_f, ft	ψ_f, deg	x_f, ft	y_f, ft
Nominal	1210.2	180	0	0	1210.2	180	0	0
+5% init vel	1599.6, 1306.3	124.8, 190.4	42420, -877	166086, -6285	1386.3	165.6	-8493	42386
−5% init vel	- , 1140.3	- , 168.2	- , -938	- , 5881	1108.0	197.0	14207	-36725
+5° heading	1210.2, 1210.2	185.0, 185.0	0 , 0	0 , 0	1210.2	185.0	0	0
−5° heading	1210.2, 1210.2	175.0, 175.0	0 , 0	0 , 0	1210.2	175.0	0	0
+1000 ft dwn-rnge	1210.2, 1205.7	180.0, 181.9	1000, 1067	0 , -3400	1210.2	180.0	1000	0
−1000 ft dwn-rnge	1210.2, 1215.6	180.0, 178.7	-1000, -902	0 , -3256	1210.2	180.0	-1000	0
+1000 ft crss-rnge	1210.2, 1209.1	180.0, 180.1	0 , 672	1000, -113	1210.2	180.0	0	1000
−1000 ft crss-rnge	1210.2, 1211.4	180.0, 179.3	0 , -672	-1000, 100	1210.2	180.0	0	-1000

the nominal solution leading to smaller down- and cross-range values. As a result, the correction term is smaller which leads to a shallower bank angle to allow the aircraft to return to its nominal solution. No trajectory is given for the original optimality-based control law I in the presence of −5% initial velocity perturbation because as the perturbed trajectory progresses, the velocity drops so much that constant altitude flight cannot be maintained. This again shows the significant contribution of the correction term.

In Table 1, the values of the final states as a result of the initial state disturbances are given. The perturbations in the initial velocity state appear to result in the most significant deviations from the nominal final states. One reason for this is that velocity plays a major role in the problem formulation; that is, it affects the dynamic pressure, thrust, drag, and lift, all of which are pertinent quantities in the equations of motion. The initial heading angle perturbation does not affect the velocity, down-, and cross-range final states for both the two proposed control laws and the open-loop solutions. It merely shifts the final heading angle by the amount of the initial perturbation. Similar effects can be seen in the final state of the OBCL-I and open-loop solutions for initial perturbations of the down- and cross-range, i.e., the perturbation affects its own final states but not others. This is not true for OBCL-II. Its final states are affected by these perturbations. The reason for this occurrence lies in the structure of the control law which includes down- and cross-range as feedback states. Nonetheless, the deviations are small.

With Plant/Aerodynamic Modeling Uncertainties

Thus far, robustness is enhanced in the presence of state disturbances. In this section, we investigate robustness in the presence of plant/aerodynamic modelling uncertainties. Only the OBCL-II will be tested against the open-loop solution since OBCL-I has been shown not to perform better than either of the other two. We assume the following plant/aerodynamic modelling uncertainties: i) actual thrust is 95% of maximum thrust, ii) actual specific fuel consumption is 5% more than the specified value of 1600 s, iii) actual lift is 5% less than the available lift, and iv) actual drag is 5% more than the original value.

In Fig. 8 , we show the robustness of our optimality-based control law II due to these uncertainties, and in Fig. 9, the results for the open-loop case are given. From these figures, it is clear that the proposed control law performs far better than the open-loop solution. In Table 2, we note that the average deviation of the final states of the perturbed trajectories of OBCL-II from the nominal trajectory are 78 ft/s for the velocity, 3° for heading angle, 4613 ft for down-range, and 5319 ft for cross-range. Except for the final velocity deviation, these values are significantly lower than those for the open-loop solution of 62 ft/s, 8°, 8284 ft, and 18443 ft, respectively. The uncertainty in drag appears to influence the trajectory most in both cases. It is followed by thrust, lift, and specific fuel consumption in order of importance.

Conclusions

This study demonstrates the potential of the application of the "optimality-condition" to obtain optimality-based control laws for real-time aircraft control. The proposed control laws possess simplicity and yet retain most of the information present in the original, more complex, optimality condition control laws. In general, these control laws are nonlinear. In the minimum-time-to-turn problems, it was found that by including a "correction" term in the proposed optimality-based control law, its robustness is enhanced significantly in terms of meeting the specified final state conditions in the presence of initial state disturbances or plant/aerodynamic modelling uncertainties. Without this term, the control law did not perform better than the optimal open-loop control. This technique of enhancing robustness has not been introduced previously, and is first applied here. An important observation from this study is that perturbations in the velocity produce the biggest impact in the deviations from the nominal trajectory. In the presence of plant/aerodynamics modelling uncertainties, drag perturbations contribute most.

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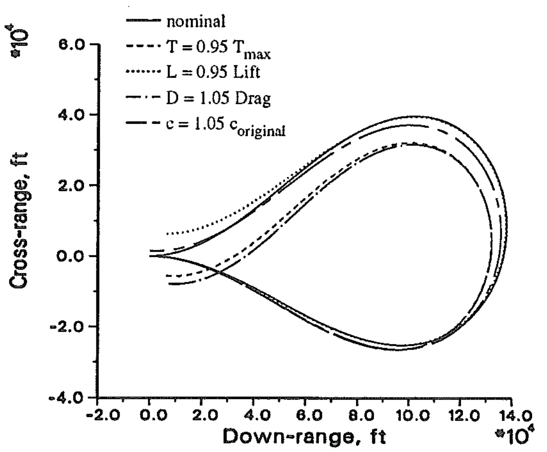

Figure 8: Optimality-based control law II solutions to plant/aerodynamic modelling uncertainties

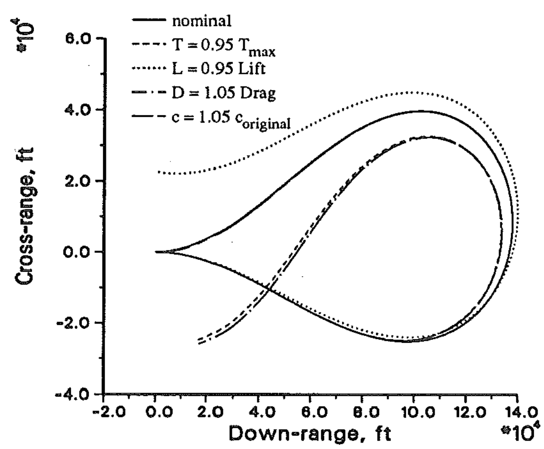

Figure 9: Open-loop solutions to plant/aerodynamic modelling uncertainties

Table 2: Final state values due to plant/aerodynamic modelling uncertainties for optimality-based contraol law II and open-loop solutions

Uncertainties	Optimality-based control law II				Open-loop			
	V_f, ft	ψ_f, deg	x_f, ft	y_f, ft	V_f, ft	ψ_f, deg	x_f, ft	y_f, ft
Nominal	1210.2	180	0	0	1210.2	180	0	0
T = 0.95 T_{max}	1087.7	174.7	5976	-5553	1088.3	191.3	15760	-25030
L = 0.95 Lift	1168.7	178.9	5081	6289	1210.2	171.2	123	22642
D = 1.05 Drag	1086.8	176.6	7291	-7893	1086.9	191.8	17054	-25881
$c = 1.05\ c_{orig}$	1184.5	176.7	102	1542	1208.0	180	197	-218
Avg. dev.	78	3	4613	5319	62	8	8284	18443

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Appendix I

Minimum-Time-To-Climb Problem

The governing equations of motion for the minimum-time-to-climb problem (see Fig. 10 for nomenclature), assuming motion over a flat nonrotating earth, are:[24]

$$m\dot{V} = T\cos\alpha - D - mg\sin\gamma \quad (14)$$
$$mV\dot{\gamma} = T\sin\alpha + L - mg\cos\gamma \quad (15)$$
$$\dot{h} = V\sin\gamma \quad (16)$$
$$\dot{x} = V\cos\gamma \quad (17)$$
$$\dot{m} = -\frac{T}{cg} \quad (18)$$

where V, γ, h, x, and m are the velocity, flight path angle,

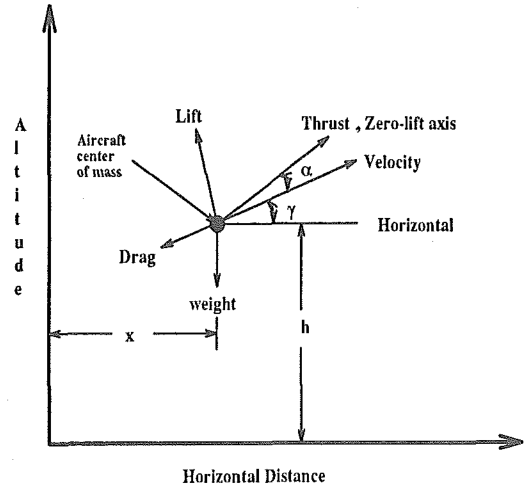

Figure 10: Geometry and nomenclature for minimum-time-to-climb problem

altitude, down-range, and aircraft mass, respectively. The angle of attack, α, is the control.

The aerodynamic forces, lift (L) and drag (D), are given by

$$L = qSC_L = qSC_{L_\alpha}\alpha \quad (19)$$
$$D = qSC_D = qS(C_{D_o} + \eta C_{L_\alpha}\alpha^2) \quad (20)$$

where $q = \frac{1}{2}\rho(h)V^2$ is the dynamic pressure, S is the aerodynamic reference area, $C_{L\alpha}$ is the lift coefficient slope, C_{Do} is the zero-lift drag coefficient, and η is the efficiency factor ($0 \le \eta \le 1$). In general $C_{L\alpha}$, C_{Do}, and η are functions of Mach number. T is the maximum thrust. It depends on speed, V, and altitude, h. A standard exponential atmosphere model is used, i.e.,[25]

$$\rho(h) = \rho_o e^{-\frac{h}{h_1}} \quad (21)$$

where $\rho_o = 2.54 \times 10^{-3}$ slug/ft^3 and $h_1 = 2.73 \times 10^4$ ft. The performance index to be minimized is $J = \int_0^t dt$. Thus, omitting the angle of attack inequality constraints and the effects of angle of attack that come through the aerodynamic drag and lift, the variational Hamiltonian is

$$H = \lambda_v \frac{T\cos\alpha - D - mg\sin\gamma}{m} + \lambda_\gamma \frac{T\sin\alpha + L - mg\cos\gamma}{mV} + \lambda_h V\sin\gamma + \lambda_x V\cos\gamma - \lambda_m \frac{T}{cg} \quad (22)$$

Applying the optimality condition, we obtain

$$H_\alpha = -\lambda_v \frac{T}{m}\sin\alpha + \lambda_\gamma \frac{T}{mV}\cos\alpha \equiv 0 \quad (23)$$

Solving (23) explicitly for α, we get

$$\alpha^*(t) = \tan^{-1}\left(\frac{\lambda_\gamma(t)}{\lambda_v(t)V(t)}\right) \quad (24)$$

From the structure of the above optimal control, we proposed the following "optimality-based" control law

$$\bar{\alpha}^*(V; a(t)) = \frac{a(t)}{V} \tag{25}$$

Note that we have modified the structure of the angle of attack control somewhat; that is, the arc tangent form in the control law is omitted to avoid numerical sensitivity since the velocity could become small in the optimization process. The $a(t)$ are then determined optimally by using the nonlinear programming method SQP.

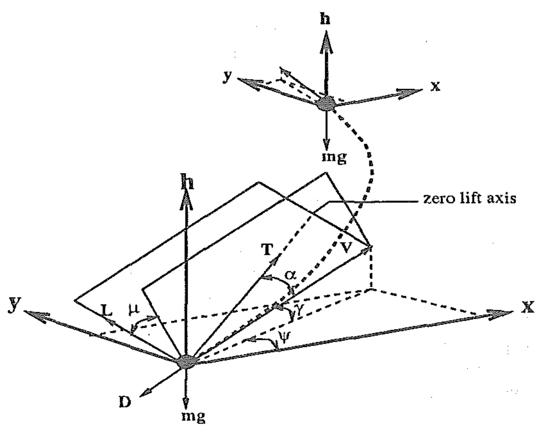

Figure 11: Geometry and nomenclature for minimum-time-to-turn problem

Appendix II

Minimum-Time-To-Turn Problem

The standard point-mass equations of motion over a flat, nonrotating earth are:[24] (see Fig. 11 for geometry and nomenclature)

$$m\dot{V} = T\cos\alpha - D \tag{26}$$
$$mV\dot{\psi} = (T\sin\alpha + L)\sin\mu \tag{27}$$
$$\dot{x} = V\cos\psi \tag{28}$$
$$\dot{y} = V\sin\psi \tag{29}$$
$$\dot{m} = -\frac{T}{cg} \tag{30}$$

where V, ψ, x, y, and m are the velocity, heading angle, down-range, cross range, and mass respectively. The bank angle, μ, is the control. The aerodynamic drag and lift are $D = qS(C_{Do} + \eta C_{L\alpha}\alpha^2)$ and $L = qSC_{L\alpha}\alpha$, respectively. If we assume small angle of attack, i.e., $\cos\alpha \approx 1,\ \sin\alpha \approx \alpha$, and $T\sin\alpha \ll L$, the angle of attack needed to maintain a constant altitude turn, determined from the vertical force balance $(T\sin\alpha + L)\cos\mu = mg$, is

$$\alpha = \frac{mg\sec\mu}{qSC_{L\alpha}} \tag{31}$$

Using these assumptions and Eqn. (31), we can reduce Eqns. (26) and (27) to

$$m\dot{V} = T - D_o - D_L\sec^2\mu \tag{32}$$
$$\dot{\psi} = \frac{g\tan\mu}{V} \tag{33}$$

where $D_o = C_{Do}qS$ and $D_L = \eta m^2g^2/C_{L\alpha}qS$. The performance index to be minimize is $J = \int_0^t dt$. Before forming the variational Hamiltonian, we would like to note that the numerical method we use allow us to provide an upper and lower limit on the bank angle. Thus, neglecting the constraints on bank angle, the variational Hamiltonian is

$$H = \lambda_v\frac{T - D_o - D_L\sec^2\mu}{m} + \lambda_\psi\frac{g\tan\mu}{V} + \lambda_x V\cos\psi + \lambda_y V\sin\psi - \lambda_m\frac{T}{cg} \tag{34}$$

Applying the optimality condition, we obtain

$$H_\mu = -\frac{2\lambda_v D_L}{m}\frac{\sin\mu}{\cos^3\mu} + \lambda_\psi\frac{g}{V}\frac{1}{\cos^2\mu} \equiv 0 \tag{35}$$

Solving Eqn. (35) explicitly for μ, we get

$$\mu^*(t) = \tan^{-1}\frac{\lambda_\psi mg}{2\lambda_v D_L V} \tag{36}$$
$$\tag{37}$$

From the structure of the above optimal control, we propose the following two "optimality-based" control laws.

Optimality-based control law I

$$\bar{\mu}^*(V; a_1(t)) = \frac{a_1(t)}{V} \tag{38}$$

Optimality-based control law II

$$\bar{\mu}^*(x, y, V; a_1(t), a_2) = \frac{a_1(t)}{V} + \frac{\sqrt{x^2(t) + y^2(t)}}{a_2} \tag{39}$$

Note that we have modified the structure of the bank angle control somewhat; that is, the arc tangent form in the original optimal control is omitted mainly to avoid numerical problem that might surface should bank angle reaches 90°. The arc tangent structure may be regarded as implicitly included in $a_1(t)$. The mass may also be assumed to be imbedded in $a_1(t)$ since we do not expect large mass variation. For the second proposed control law, we have added an additional term. This is equivalent to adding $kf(V, \psi, x, y, m)$ where $k = 1/a_2$ and $f(V, \psi, x, y, m) = \sqrt{x^2(t) + y^2(t)}$. We shall show from the numerical results that this term allows OBCL-II to possess natural robustness.