HIGH RESOLUTION UPWIND SOLUTIONS OF THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

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Abstract

A numerical scheme is presented for the solution of the steady incompressible Navier-Stokes equations. Based on the pseudo-compressibility concept, the scheme includes a conservative upwind discretization of the convective terms obtained by the flux difference splitting approach, together with a central discretization of the viscous terms. The solution is advanced in time using a linearly implicit time-marching technique. The linear systems are solved by either direct or relaxation techniques. Numerical results have been obtained for the laminar flows in the entrance of a channel and over a backward facing step. They are in close agreement with experimental data.

1. Introduction

For incompressible flows, the continuity equation reduces to a purely kinematical constraint rather than being a dynamical equation. This proves to be a major source of difficulty. Several strategies have been proposed to overcome this difficulty. One such strategy is the stream function-vorticity formulation. In two dimensions, this has the advantage of reducing the number of unknowns, as well as of eliminating the pressure and improving the coupling between the equations, but the stream function equation remains a purely kinematic equation and the approach extends with difficulty to three-dimensional configurations.

Another approach is to replace the continuity equation by a Poisson equation for pressure obtained by taking the divergence of the momentum equation and using the continuity equation. However, this approach leads to difficulties in implementing the pressure boundary conditions and the Poisson equation is also a purely spatial equation. It then has to be solved separately from the momentum equation, which affects the convergence of the process.

Finally, another approach , valid for steady flows, is to introduce an artificial time derivative term in the continuity equation. This strategy was first proposed by Chorin [1] and is known as the artificial or pseudo-compressibility method. The modified system of equations can easily be time-matched. However, if central differencing (or standard Galerkin FEM) is used, several problems appear, in particular, spurious oscillations develop at high Reynolds numbers. To overcome these problems, use can be made of staggered grids or of properly tuned artificial viscosity techniques. Another option is available. Indeed, when the pseudo-compressibility technique is used, the incompressible Navier-Stokes equations closely resemble the compressible Navier-Stokes equations. In particular, the inviscid terms in isolation constitute a hyperbolic system of conservation laws. It is therefore possible to discretize these terms using a high resolution upwind difference technique as those developed for the compressible flow equations. This approach has the advantage that the numerical dissipation is naturally built into the discretization technique without the need for developing a sophisticated artificial dissipation operator and also that when used in conjunction with implicit time stepping, it leads to well conditioned linear systems which can be solved by relaxation methods. Such an avenue was explored by a few authors [2,3,4,5]. The present paper presents a scheme based on the same philosophy. However, high resolution is achieved by a different approach, following a suggestion by Roe [6], and requires much fewer limiter function evaluations than previous schemes.

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2. Analysis

2.1 Governing equations

The 2D incompressible Navier-Stokes equations, with pseudo-compressibility included, are written, in non-dimensional form

$$\frac{\partial \mathbf{Q}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{G}}{\partial y} = \frac{1}{Re}\left(\frac{\partial \mathbf{F_V}}{\partial x} + \frac{\partial \mathbf{G_V}}{\partial y}\right) \qquad (1)$$

with

$$\begin{aligned} \mathbf{U} &= (P/\beta, u, v)^t \\ \mathbf{F} &= (u, P+u^2, uv)^t \\ \mathbf{G} &= (v, uv, P+v^2)^t \\ \mathbf{F_V} &= (0, \partial u/\partial x, \partial v/\partial x)^t \\ \mathbf{G_V} &= (0, \partial u/\partial y, \partial v/\partial y)^t \end{aligned} \qquad (2)$$

where the continuity equation has been used to simplify the viscous fluxes and P is the pressure p divided by the density ρ. β is the artificial compressibility parameter. A value of 1, which has been suggested by various authors (e.g. [2]) has been adopted in the present study. It should be noted, as pointed out by Pan and Chakravarthy [5], that although the artificial compressibility parameter β does not influence directly the steady solution since the term involving it vanishes at steady state, it does influence it indirectly because of its effect on the eigenvalues/vectors of flux jacobians and the associated conservative upwind discretization. Pan and Chakravarthy have shown that the influence, and the related errors, is strongest for large values of β, which is an additional reason to choose a value of 1 for β.

2.2 Discretization

2.2.1 Space discretization

The scheme uses a composite discretization. The well-behaved viscous terms are simply centrally differenced whereas the inviscid terms are upwind discretized using the flux difference splitting (FDS) approach pioneered by Roe [7]. Let us now describe the FDS technique on the one-dimensional inviscid equation

$$\frac{\partial \mathbf{Q}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = 0 \qquad (3)$$

The FDS discretization is based on a finite volume formulation where the variables represent cell averages. A general finite volume semi-discretization of 3 is

$$\frac{d\mathbf{Q}_i}{dt} + \frac{\mathbf{F}_{i+1/2} - \mathbf{F}_{i-1/2}}{\Delta x} = 0 \qquad (4)$$

where $\mathbf{F}_{i\pm1/2}$ are the fluxes at the cell boundaries. The FDS flux formulas are obtained by solving a linearized Riemann problem at the cell interface $i+1/2$

$$\frac{\partial \mathbf{Q}}{\partial t} + \tilde{\mathbf{A}}\frac{\partial \mathbf{Q}}{\partial x} = 0 \qquad (5)$$

where the matrix $\tilde{\mathbf{A}}$ is evaluated at some average state between $\mathbf{Q}_i$ and $\mathbf{Q}_{i+1}$. The average state is chosen such that [7]

1. for $\mathbf{Q}_{i+1} = \mathbf{Q}_i = \mathbf{Q}$, $\quad \tilde{\mathbf{A}} = \mathbf{A} = \frac{\partial \mathbf{F}}{\partial \mathbf{Q}}$
2. $\tilde{\mathbf{A}}(\mathbf{Q}_{i+1} - \mathbf{Q}_i) \equiv \mathbf{F}_{i+1} - \mathbf{F}_i$

For incompressible flows, it turns out [2,3,4,5] that the simple arithmetic average of $\mathbf{Q}_i$ and $\mathbf{Q}_{i+1}$ possesses these properties. The solution of the linearized Riemann problem then gives the first order flux formula

$$\mathbf{F}_{i+1/2} = \frac{\mathbf{F}_i + \mathbf{F}_{i+1}}{2} - \frac{1}{2}|\mathbf{A}|(\mathbf{Q}_{i+1} - \mathbf{Q}_i) \qquad (6)$$

where $|\mathbf{A}| = \mathbf{R}|\mathbf{\Lambda}|\mathbf{L}$. $\mathbf{R}$ and $\mathbf{L}$ are respectively the right and left eigenvector matrices of $\tilde{\mathbf{A}}$ and $|\mathbf{\Lambda}|$ is the diagonal matrix of its eigenvalues moduli. Their expression is given in the Appendix.

As such, the FDS flux formula is first order accurate and therefore generally produces excessive dissipation. Several strategies can be used to increase the order of accuracy. One such strategy is the MUSCL approach first introduced by van Leer [8] whereby the Riemann problem at the cell interface uses values reconstructed from neighbouring cell values. This reconstruction process may or may not be limited in order to obtain TVD properties. That approach has been used by Gorski [3], Pan and Chakravarthy [5] and, in a slightly modified form, by Hartwich and Hsu [2].

Another approach is to combine first-order flux variations across neighbouring interfaces to construct a higher order flux formula. Such an approach requires less computations since just first order flux variations are computed. It was used by Liou and van Leer [9] for the compressible Euler equations and by Athavale and Merkle [4] for the incompressible Navier-Stokes equations. It is however difficult to use limiting to obtain the TVD property with this approach.

The approach that was used in the present study follows rather the suggestion by Roe [6]. It is identical to the previous one for unlimited discretizations of linear constant coefficient systems but provides a much easier way to introduce limiting. Also, with respect to the MUSCL approach, it requires half as many limiter function evaluations. The higher order flux formula is given by the following relation

$$\mathbf{F}_{i+1/2} = \frac{\mathbf{F}_i + \mathbf{F}_{i+1}}{2} - \frac{1}{2}\mathbf{RKL}(\mathbf{Q}_{i+1} - \mathbf{Q}_i) \qquad (7)$$

where matrices $\mathbf{R}$ and $\mathbf{L}$ retain their previous meaning. With respect to the first order formula, the only modification is the replacement of the diagonal matrix $|\mathbf{\Lambda}|$ by another diagonal matrix $\mathbf{K}$ which is defined as follows.

$$\mathbf{K} = \mathrm{diag}[|\lambda^{(k)}_{i+1/2}|(1 - \frac{B(r^{(k)}_{i+1/2})}{r^{(k)}_{i+1/2}})] \qquad (8)$$

where

$$r^{(k)}_{i+1/2} = \frac{\mathbf{L}^{(k)}_{i+1/2}\,\delta\mathbf{Q}_{i+1/2}}{\mathbf{L}^{(k)}_{i+1/2-\mathrm{sgn}(\lambda^{(k)})}\,\delta\mathbf{Q}_{i+1/2-\mathrm{sgn}(\lambda^{(k)})}} \qquad (9)$$

is the ratio of k-wave intensities at two neighbouring cell interfaces and $B(r)$ is some averaging function which can be linear — $B(r) = r$ and $B(r) = 1$ correspond respectively to central and fully upwind discretizations — or non linear (limiter function). It can easily be shown that the discretization is second order accurate provided that $B(r)$ is a continuous function in the neighbourhood of $r = 1$ and $B(1) = 1$. In the present study, the min-mod limiter function was used

$$B(r) = \max(0, \min(1, r)) \tag{10}$$

Following a suggestion by Pan and Chakravarthy [5], it may be preferable to use a linear averaging function such as

$$B(r) = \frac{1}{3} + \frac{2r}{3} \tag{11}$$

which yields third order accuracy.

2.2.2 Time discretization

Once the space discretization is chosen, there remains to choose the time discretization. Since we are interested in steady solutions, we adopt an implicit technique for faster convergence. In the present study, only the backward Euler formula was used, yielding the following discretization

$$\Delta\mathbf{Q} + \delta_x \mathbf{F}^{n+1} + \delta_y \mathbf{G}^{n+1} - \frac{1}{Re}(\delta_x \mathbf{F_v}^{n+1} + \delta_y \mathbf{G_v}^{n+1}) = 0 \tag{12}$$

where δ_x represent the discretization formula in the x-direction and similarly for δ_y. For practical implementation, this discretization must be linearized. For example, we have

$$\mathbf{F}^{n+1}_{i+1/2} = \mathbf{F}^{n}_{i+1/2} + \sum_m \frac{\partial \mathbf{F}_{i+1/2}}{\partial \mathbf{Q}_m} \Delta\mathbf{Q}_m \tag{13}$$

Now, the high resolution discretization of the convective fluxes requires a 5-point stencil in each direction, leading to expensive block pentadiagonal systems. Following several authors [4,9] who have shown that the simplification does not significantly alter the convergence speed, we use for the implicit part the first order flux formula, i.e.

$$\mathbf{F}^{n+1}_{i+1/2,\mathrm{hr}} = \mathbf{F}^{n}_{i+1/2,\mathrm{hr}} + \sum_m \frac{\partial \mathbf{F}_{i+1/2,1}}{\partial \mathbf{Q}_m} \Delta\mathbf{Q}_m \tag{14}$$

where the subsripts hr and 1 denote respectively the high resolution and first order flux formulas. The implicit stencil is then reduced to 3 points in each direction but the scheme remains quite complex because of the complexity of the true flux jacobians. The first order flux formula at the cell interface $i + 1/2$ may be rewritten

$$\mathbf{F}_{i+1/2} = \mathbf{F}_i + \underbrace{\mathbf{R\Lambda^- L}}_{\mathbf{A}^-{}_{i+1/2}} (\mathbf{Q}_{i+1} - \mathbf{Q}_i)$$

The following approximate formula is then used for the flux increment

$$\sum_m \frac{\partial \mathbf{F}_{i+1/2,1}}{\partial \mathbf{Q}_m} \Delta\mathbf{Q}_m \sim \mathbf{A}_i \Delta\mathbf{Q}_i + \mathbf{A}^-{}_{i+1/2}(\Delta\mathbf{Q}_{i+1} - \Delta\mathbf{Q}_i) \tag{15}$$

The following discretization results, because of the cancellation of the $\mathbf{A}_i \Delta\mathbf{Q}_i$ terms.

$$\Big[\frac{I}{\Delta t} + \mathbf{A}^-{}_{i+1/2}\,\delta_x^+ + \mathbf{A}^+{}_{i-1/2}\,\delta_x^- - \frac{1}{Re}\delta_x \mathbf{A}_v \\ \mathbf{B}^-{}_{j+1/2}\,\delta_y^+ + \mathbf{B}^+{}_{j-1/2}\,\delta_y^- - \frac{1}{Re}\delta_y \mathbf{B}_v\Big]\Delta\mathbf{Q} = \\ \frac{1}{Re}[\delta_x \mathbf{F}^n_v + \delta_y \mathbf{G}^n_v] - \delta_x \mathbf{F}^n - \delta_y \mathbf{G}^n \tag{16}$$

where δ^+ and δ^- denote respectively the first order forward and backward difference operators.

2.3 Solution strategy

The resulting linear system of equations may be solved in various ways. For coarse grids, direct methods can be used. We used simple Gaussian elimination for the first test case, the laminar flow in the entrance of a channel. Because the system is well conditioned, relaxation methods can also be used. Other alternatives are approximate factorization (AF) or approximate LU factorization as used respectively by Gorski [3] and by Athavale and Merkle [4]. Vertical line block Gauss-Seidel relaxation was used in the present study for the second test case.

3. Results

3.1 Flow in the entrance of a channel

The flow in the entrance of a channel was computed with a Reynolds number based on the channel half-width of 5. A uniform 21 × 21 grid was used with $\Delta x = \Delta y = 0.1$. Results were obtained with both first and second order space discretizations. Gaussian elimination was used to solve the linear systems. Both calculations converged in 30 iterations, confirming the fact that the combination of first order implicit and second order explicit discretization does not significantly reduce convergence properties of the algorithm. First order velocity profiles are displayed in Fig. 1 while second order profiles are displayed in Fig. 2. The improved accuracy of the latter results is demonstrated by the capture of the off-center velocity peaks close to the channel entrance and by the more accurate value of the fully developed velocity maximum.

3.2 Flow over a backward facing step

As a more severe test-case, the laminar flow over a backward facing step was computed. This well-documented flow configuration [10] allows comparison with both experimental and other numerical results. The configuration parameters were the following : the step height was half the channel width and the Reynolds number based on channel width and mean inflow velocity was 200. A uniform 129×49 grid was used.

Converged results were obtained in less than 100 iterations for both first and second order space discretizations. The computed streamlines are displayed in Fig. 3-4. Both results are nearly identical and the separation lengths, respectively 2.83 and 2.77 channel widths for first and second order discretizations, closely agree with the experimental value of about 2.8. This clearly reveals that the present upwind discretization produces very little dissipation even in the first order case when neighbouring cells are separated by a single inviscid standing wave. This was also observed in the compressible case [11].

4. Conclusions

A numerical scheme has been developed for solving the incompressible Navier-Stokes equations. Based on the pseudo-compressibility concept, it combines central differencing of the viscous terms with upwind discretization of the convective fluxes, obtained by the flux difference splitting approach.

High resolution is obtained by a slight modification of the first order flux formula and requires half as many evaluations of the averaging (limiter) functions than the MUSCL approach. The scheme was tested for two laminar flow configurations and proved to produce accurate results in rather few iterations.

Forthcoming developments include adaptation to general curvilinear coordinate systems for the treatment of arbitrary geometries such as aerofoils, the comparative study of various solution strategies and the inclusion of acceleration techniques.

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Appendix

Expressions for the evaluation of flux formulas

The matrix $\tilde{\mathbf{A}}$ of the 1D approximate Riemann problem in the x-direction is ($\beta = 1$)

$$\tilde{\mathbf{A}} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2\bar{u} & 0 \\ 0 & \bar{v} & \bar{u} \end{pmatrix} \qquad (17)$$

where $\bar{u}$ and $\bar{v}$ denote the arithmetic average of values at neighbouring points.

The eigenvalues are $\bar{u} - \bar{a}, \bar{u}$ and $\bar{u} + \bar{a}$ with $\bar{a} = \sqrt{1 + \bar{u}^2}$. The matrix of right eigenvectors is

$$\mathbf{R} = \begin{pmatrix} \frac{1}{\bar{a}} & 0 & \frac{1}{\bar{a}} \\ \frac{\bar{u}-\bar{a}}{\bar{a}} & 0 & \frac{\bar{u}+\bar{a}}{\bar{a}} \\ \frac{\bar{v}(\bar{a}-\bar{u})}{\bar{a}^2} & 0 & \frac{\bar{v}(\bar{u}-\bar{a})}{\bar{a}^2} \end{pmatrix} \qquad (18)$$

and the matrix of left eigenvectors is

$$\mathbf{L} = \begin{pmatrix} \frac{\bar{u}+\bar{a}}{2} & -\frac{1}{2} & 0 \\ -\frac{\bar{v}}{\bar{a}^2} & -\frac{\bar{u}\bar{v}}{\bar{a}^2} & 1 \\ \frac{\bar{a}-\bar{u}}{2} & \frac{1}{2} & 0 \end{pmatrix} \qquad (19)$$

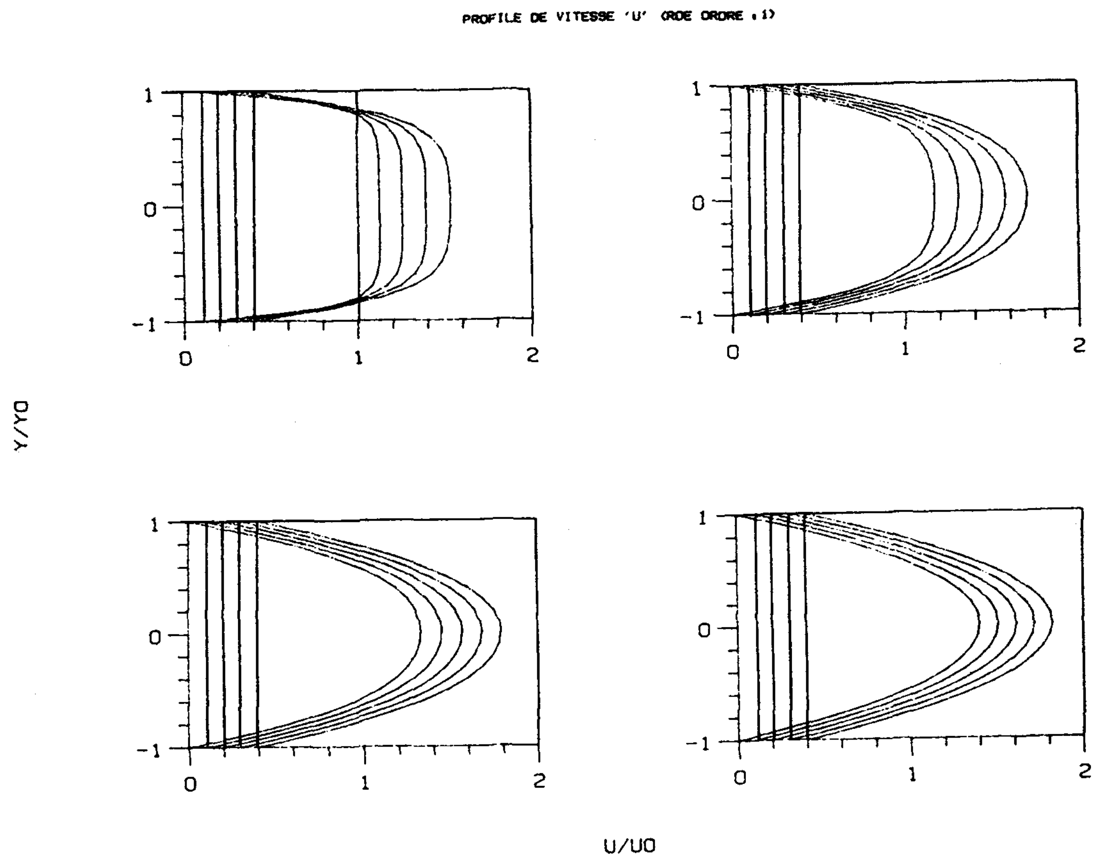

Fig.1 Flow in the entrance of a channel : first order velocity profiles

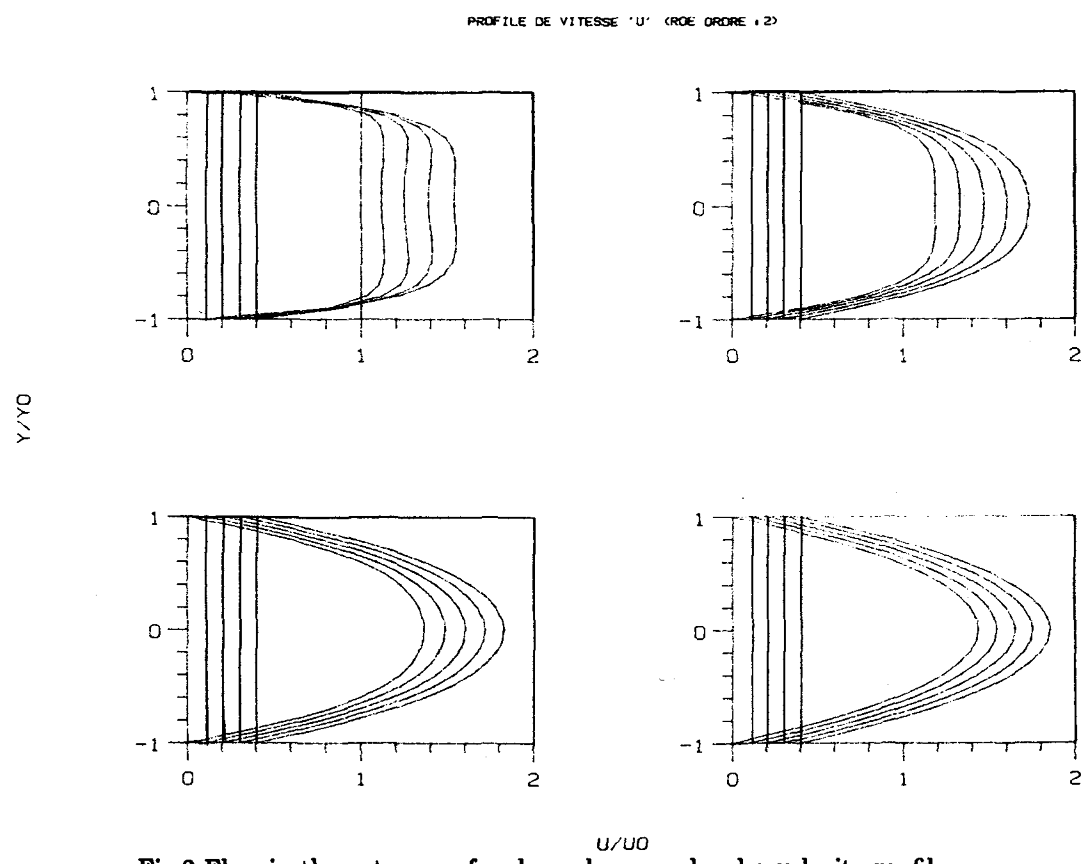

Fig.2 Flow in the entrance of a channel : second order velocity profiles

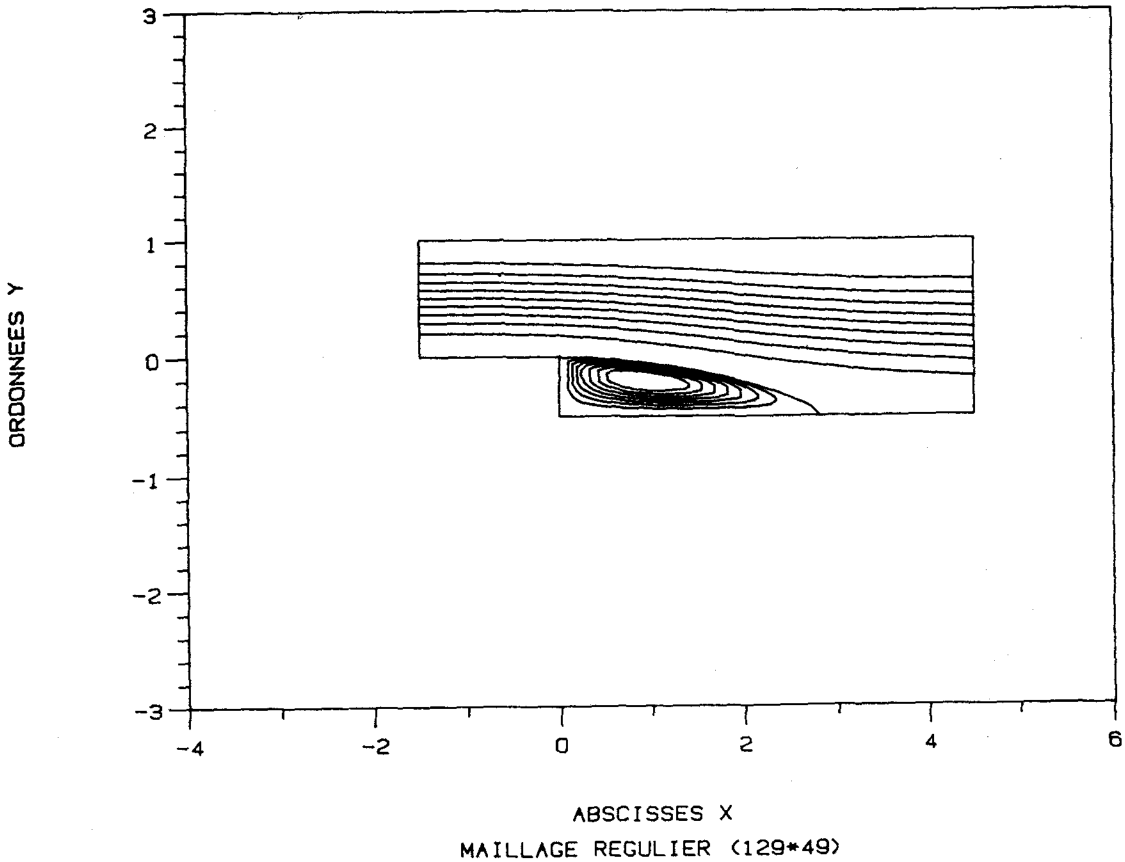

Fig.3 Flow over a backward facing step : streamlines computed with first order flux formula

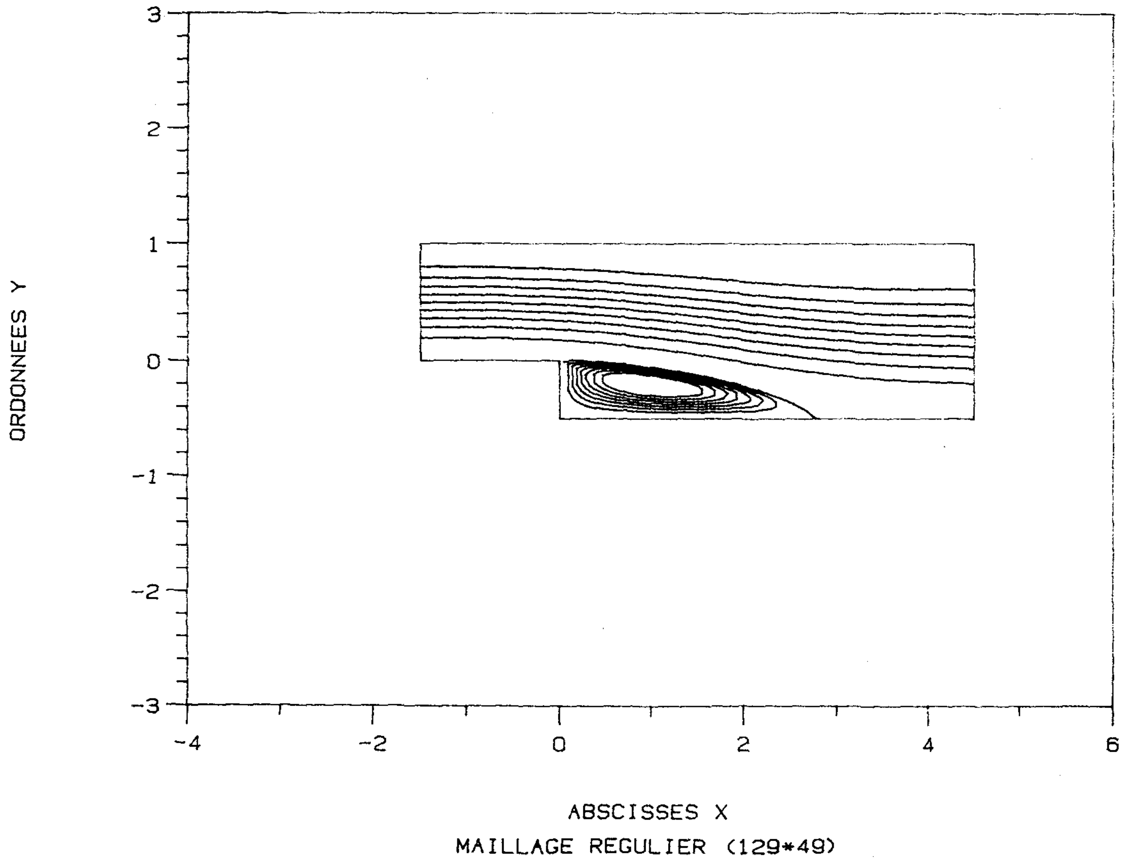

Fig.4 Flow over a backward facing step : streamlines computed with second order flux formula