BIFURCATION METHODS — A PRACTICAL METHODOLOGY FOR IMPLEMENTATION BY FLIGHT DYNAMICISTS

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Abstract

The use of bifurcation methods as a practical tool for implementation during the design and development phases of high-performance aircraft is investigated. The technique has proven itself in the research environment as an effective means of analysing the global nonlinear behaviour of manoeuvrable aircraft. The paper reports on the computer programs that have been developed to allow convenient and reliable use of the methods and on an investigation into the level of complexity required of aerodynamic models in order that the results produced by the methods are valid and useful. It is shown that it is possible for practical results to be obtained before expensive dynamic testing programmes have been undertaken; the development of a qualitative dynamic test rig for wind tunnel scale models to form part of the overall methodology is suggested in order to apply the technique with confidence on new aircraft designs.

Nomenclature

SYMBOL IN TEXT	DEFINITION
$\vec{c}$	vector of m control variables
$\vec{f}$	vector of n differentiable continuous functions
F	Jacobian matrix of $\vec{f}$
$\vec{g}$	vector of k functions
I_x, I_y, I_z	aircraft moments of inertia (kg m^2)
I_{xz}	aircraft product of inertia (kg m^2)
k	reduced state-space dimension (number of eigenvalues on the imaginary axis)
L	body-axis rolling moment (Nm)
m	aircraft mass (kg) *or* number of control variables in vector $\vec{c}$
M	body-axis pitching moment (Nm)
n	number of state variables in vector $\vec{x}$
N	body-axis yawing moment (Nm)
p	body-axis roll rate (radian s^{-1} or degree s^{-1})
q	body-axis pitch rate (radian s^{-1} or degree s^{-1})
r	body-axis yaw rate (radian s^{-1} or degree s^{-1})
t	time (s)
$\vec{u}$	vector of k transformed state variables
V_T	aircraft total translational velocity (ms^{-1})
$\vec{x}$	vector of n state variables
Y_w	side force in wind axes (due to aerodynamics, thrust and gravity) (N)
Z_w	force along z wind axis (due to aerodynamics, thrust and gravity) (N)
α	angle of attack (radian or degrees)
β	angle of sideslip (radian or degrees)
δ_a	aileron deflection (radian or degrees) (positive: left t.e. down)
δ_r	rudder deflection (radian or degrees) (positive: t.e. left)
δ_e	stabilator deflection (radian or degrees) (positive: t.e. down)
ϕ	bank angle (radian or degrees)
θ	pitch angle (radian or degrees)

SUPERSCRIPTS:

$\dot{}$	derivative with respect to t
T	transpose (of vector)

ABBREVIATIONS:

ALT.	altitude (used in plot titles) (m)
AOA	angle of attack (α)
BACTM	Bifurcation Analysis and Catastrophe Theory Methodology
t.e.	trailing edge

SYMBOLS USED ON PLOTS:

P	roll rate (p)
Q	pitch rate (q)
R	yaw rate (r)
T	time (t)
ALPHA	angle of attack (α)
BETA	sideslip angle (β)
DA	aileron angle (δ_a)
DE / DSTAB	stabilator angle (δ_e)
DR	rudder angle (δ_r)
S	all eigenvalues in left-half plane (stable)
U	1 real eigenvalue positive (unstable)
A	2 real eigenvalues positive (unstable)
B	3 real eigenvalues positive (unstable)
C	4 real eigenvalues positive (unstable)
D	5 real eigenvalues positive (unstable)
L	1 complex eigenvalue in right-half plane (possible limit cycle)
E	1 complex and 1 real eigenvalue in right-half plane (unstable)
F	1 complex and 2 real eigenvalues in right-half plane (unstable)

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SYSTEMS OF AXES:

These are defined in Figure 1.

Introduction

The use of bifurcation methods in analysing dynamics of aircraft flight motions is no longer a new concept. It originated in a project commissioned by the Office of Naval Research in the USA in 1976 undertaken by Mehra and Carroll of Scientific Systems Inc.[1,2,3] This project, which continued until about 1980, was part of a dedicated effort on the part of the Office of Naval Research to broaden the understanding of the nonlinear behaviour often experienced by combat aircraft during high angle of attack (AOA) manoeuvres.

During the 1980s various reports appeared on the implementation and development of these methods by other researchers[4,5], whilst Mehra and Carroll themselves reported on their work until about 1982.[6,7,8] The development of a bifurcation methods programme in South Africa was started by the author in 1983.[9,10,11]

The application of Catastrophe Theory and bifurcation methods to aircraft rigid-body dynamics occurred fairly early on in the development of nonlinear dynamics in the applied sciences but the achievement was by no means isolated: a new awareness of the important role of nonlinear occurrences in a wide range of events and mechanisms started to take shape in the 1970s, following on from the insight of several earlier mathematicians and scientists.[12,13,14] This relatively new field, often now dominated by the work done in chaotic behaviour, has led to a commonality between different sciences in lieu of the earlier trend towards more and more diversification and specialisation.

In line with this deeper involvement in nonlinear dynamics, use of bifurcation methods has occurred in aeronautics not only in the context of global rigid-body behaviour using the aircraft equations of motion but also in detailed analysis of certain critical motions[15], specific steady and unsteady aerodynamic stability problems[16,17] and aeroelasticity.[18]

Notwithstanding these applications of bifurcation theory in the aeronautical sciences, and the fact that the original methods of Mehra and Carroll are described in text book form[19], it is of concern that the methods appear not to be finding ready acceptance as a tool in the practical design phase of aircraft projects. This is particularly disturbing in the light of the move towards better understanding of nonlinear phenomena by researchers: as increasingly more optimum performance is demanded from new aircraft so it should be expected that designers make better use of the nonlinear conditions in which they are intended to operate. It is no longer sufficient when studying stability and handling qualities of agile aircraft to remain within the bounds of 'safe' regions in terms of departure resistance, spin susceptibility and recovery characteristics (using criteria such as Lateral Control Départure Parameter, Kalviste criteria, etc.[20,21,22]) — it is now necessary to maintain stable and easily controllable flight with relatively low pilot workload during hard manoeuvres, and any analysis technique that can assist in achieving these goals should form part of the design loop from as early in the process as possible.

The problem as perceived by engineers in the industry appears to be twofold:

1. computational difficulties — the large quantity of computation required and its sensitivity in terms of successfully finding *all* desired solutions to the nonlinear equations result in the technique appearing to be costly and time-consuming;
2. it is often perceived that bifurcation methods can only be regarded as useful once a very comprehensive mathematical model has been defined. This would require results from expensive dynamic as well as static wind tunnel tests, probably in conjunction with free-flight tests, and then full verification of the model in piloted simulation studies. Clearly, in this case, bifurcation methods would only start being useful once the aircraft were already flying.

The objective in this paper is to outline the FORTRAN computer package that has been developed as a robust, reliable and easy-to-operate core for the implementation of the bifurcation method; and to gain some insight into the level of sophistication required of an aircraft dynamic model in order for bifurcation methods to provide useful output. An opportunity for these methods to be tested on actual locally-used aircraft has not arisen and thus the work is based on a comprehensive McDonnell Douglas F-4J Phantom model.

The need for a technique such as bifurcation methods and the concepts involved in it are dealt with first. Thereafter, the computer program is described briefly together with a sample of how it is incorporated in the overall methodology. Selected results from the method for different model complexities are then presented. Finally, some conclusions are drawn in terms of the stage in aircraft design and development at which bifurcation methods become useful, the overall implications of the methods being used in the aircraft design and development environment and further work required to enhance their effectiveness.

Fundamental concepts of bifurcation methods

Ever since the advent of jet-powered combat aircraft soon after the 2^{nd} World War the trends in design of such aircraft led to configurations that were more and more susceptible to adverse handling qualities during manoeuvring flight. The quest for supersonic and turn-rate performance evolved the characteristic long-nosed swept wing designs with low wing loading and much of the inertia concentrated in the fuselage. These aircraft, whilst capable of generating the acceleration required for combat pursuit and evasion, were subject to both aerodynamic and inertial nonlinearities, and the result was often the onset of unexpected motions (such as pitch-up, wing rock, nose-slice, post-stall gyrations, etc.[23,24,25]) that limited agility to a level below the maximum manoeuvring potential. The effect was not only to impaire combat effectiveness — by hampering control and weapons aiming and tracking tasks — but also led to the loss of aircraft and lives.

The enormous effort that has been vested in overcoming these problems is evident in the superior agility of modern fighters relative to their predecessors. The majority of this work has been experimental in nature, with more and more aerodynamic analyses having entered the arena as computational fluid dynamics has developed in recent years. In the realm of flight mechanics most work has concentrated on stability criteria intended to ensure that the aircraft does not enter into dangerous motions; a heavy reliance has been placed on simulation using advanced models with comprehensive aerodynamic data gleaned from the experimental programmes.[20,26] Simulation enables not only the study of the aircraft behaviour but also the influence of control systems, piloting technique and physiological factors.

All of these techniques remain relevant and useful but none of them provides the unified global overview of the nonlinear dynamics of aircraft motion that bifurcation analysis does. It facilitates a systematic study of aircraft behaviour through its entire range of possible equilibria, for all control parameter variations, from trimmed flight through to fully developed spin and beyond into chaotic motions. As such, it provides a global understanding of the mechanics underlying the limit cycles, hysteresis effects and discontinuous motions exhibited by aircraft at high AOA.

The original technique developed by Mehra and Carroll was named BACTM, for Bifurcation Analysis and Catastrophe Theory Methodology. It is a unified approach that, given a set of continuous differentiable equations, allows a systematic analysis of the behaviour of dynamic systems.

Bifurcation analysis is the study of new solutions to a system of equations that 'bifurcate' (branch out) from some known solution as a parameter changes and is applicable to a wide range of dynamical systems.[19]

Catastrophe theory — an extension of bifurcation theory — is a mathematical language designed by René Thom[12] to describe and classify the abrupt changes in behaviour that can occur when a continuous system can have more than one stable state or path.[27] A catastrophe is the jump (bifurcation) from one state or pathway to another. The so-called 'elementary catastrophe theory' used in BACTM is limited in application to gradient-type dynamical systems. Although in general aircraft models containing nonlinear aerodynamics are not of this form, they may locally resemble a gradient system.

BACTM is founded upon 4 theorems. Mathematical statements (see ref. 1) are not given in this paper but relevant features are outlined below.

- **Centre Manifold Theorem.** This theorem, which is the key to the methodology, enables the behaviour of an autonomous dynamical system $\dot{\vec{x}} = \vec{f}(\vec{x}, \vec{c})$ (where $\vec{x}$ is an n-dimensional vector of state variables; $\vec{c}$ is an m-dimensional vector of control variables; $\dot{\vec{x}} = d\vec{x}/dt$) in a region of equilibrium where k of the n eigenvalues of the system Jacobian F $(= \partial\vec{f}/\partial\vec{x})$ simultaneously cross the imaginary axis (i.e. where a bifurcation occurs) to be described in terms of a k-dimensional system of equations $\vec{g}(\vec{u}, \vec{c})$, as well as $n - k$ linear equations. This effectively means that:
  - bifurcations are revealed through eigenvalues crossing the imaginary axis;
  - the bifurcational behaviour of a system of n equations can be studied in terms of a *reduced* system $\vec{g}(\vec{u}, \vec{c})$ of k equations;
  - results obtained from the reduced system can be generalised to higher-dimensional systems by superimposing linear systems having eigenvalues with non-zero real parts.

  The theorem also quantifies the maximum possible value of k for a system of given number of controls, m. Usually, no more than 2 or 3 eigenvalues are found to cross the imaginary axis simultaneously in an aircraft dynamical system.

- **Main Theorem of Elementary Catastrophe Theory.** This theorem classifies all bifurcational behaviour of finite dimensional *gradient* systems for up to 5 control variables. Aircraft seldom involve more than 4 controls of concern in stability analysis and under these conditions only 7 types of catastrophe are possible (for gradient systems). Global bifurcational behaviour of these 7 elementary catastrophes is described in terms of their generic functions $\vec{g}(\vec{u}, \vec{c})$ (see above) and have been extensively studied and documented.[12,19,27]

  Although aircraft dynamic models are not usually of the gradient form, results from the Centre Manifold Theorem permit information inferred from *local* catastrophes to be applied to the overall system.[19]

- **Hopf Bifurcation Theorem.** Nonlinear autonomous dynamical systems of the non-gradient type can have equilibrium solutions which are closed orbits (limit cycles). They correspond to pairs of complex eigenvalues crossing into the right half plane as a parameter (control) is varied, while the other eigenvalues remain in the left half plane. The critical point at which the eigenvalues cross the imaginary axis signifies the bifurcation from a fixed-point solution to a limit cycle. The theorem states conditions under which such Hopf bifurcations occur.

- **Global Implicit Function Theorem.** This allows conditions to be determined under which a nonlinear system of equations has a unique solution, thus signifying a lack of bifurcational behaviour. It applies only to gradient systems — where bifurcations coincide with real eigenvalues changing sign - - and will not therefore provide conditions relating to Hopf bifurcations.

In using bifurcation analysis for general aircraft systems it is the Centre Manifold and Hopf Bifurcation theorems that are of primary relevance. The central issue is the behaviour of eigenvalues of the linearised system (Jacobian matrix) as parameters are varied. The system of equations must be continuous and differentiable; it is studied under conditions of equilibrium and the parameters that are varied are usually control variables.

The first step in implementing the technique is to define the dynamical system $\dot{\vec{x}} = \vec{f}(\vec{x}, \vec{c})$. This comprises the 6 equations of motion as well as the kinematic equations for an aircraft in flight. Typically, $\vec{x} = (p, q, r, \alpha, \beta, V_T, \theta, \phi)^T$ where p, q and r are the roll, pitch, and yaw rates respectively; α, β and V_T are the angle of attack, angle of sideslip and total velocity; θ and ϕ are the pitch and roll angles of the aircraft. Typical elements of $\vec{c}$ would be $(\delta_a, \delta_e, \delta_r)^T$, i.e. aileron, stabilator and rudder control surface deflections. Thus, in this case, $n = 8$ and $m = 3$. The eight equations are the standard flight dynamics equations in which the aerodynamic forces and moments are defined in terms of stability derivatives, and these in turn are often tabulated as (nonlinear) functions of α, β and/or Mach number.

The basis of the methodology is to generate the so-called equilibrium and bifurcation surfaces.

The former is an $(n+m)$-dimensional plot of all admissible combinations of state and control variables for equilibrium of the equations of motion ($\dot{\vec{x}} = 0$). Of course, surfaces of higher dimension than 3 cannot be plotted or interpreted easily; in practice, a series of $m \times n$ 2-dimensional graphs of a state variable versus a control variable are drawn. Information on eigenvalues of F is also required in order to predict bifurcational behaviour. This is achieved by representing each possible combination of eigenvalues (i.e. left or right half plane, real or complex — see Notation pages) with a symbol which is plotted onto the equilibrium branches. Such equilibrium plots are rich in information on the dynamical system and, properly interpreted, can yield predictions of discontinuous motions, limit cycles and hysteresis over the entire flight envelope.

Bifurcation surfaces are m-dimensional plots of all admissible values of the control variables at which bifurcations occur i.e. points at which $\dot{\vec{x}} = 0$ *and* one or more eigenvalues lie on the imaginary axis. Again, these are plotted in practice as 2-dimensional graphs of one control versus another. Studied in conjunction with the equilibrium surface, the bifurcation surface provides useful information on how controls should be varied or control systems designed in order to either avoid or utilise bifurcational behaviour.

The use of equilibrium and bifurcation surfaces is best illustrated by means of a low-order mathematical example followed by a demonstration on a relatively simple aircraft model.[1,4,9,10,19]

These surfaces yield information on all possible equilibrium points as well as their linearised stability conditions (which give good insight as to the actual stability of the nonlinear system). They do not, however, describe transient behaviour as the system bifurcates from one equilibrium branch to another. To complete the picture, therefore, simulation is included in the methodology. It is important to note that the equilibrium and bifurcation surfaces indicate exactly which sets of simulation runs are necessary in order to feature relevant phenomena. Since time histories obtained from simulations are dependent on initial conditions, control sequence and duration of the run the use of simulations in isolation requires enormous numbers of runs and even then important phenomena are easily missed.

Simulation also complements the equilibrium surfaces in three other respects:

- if the computations for generating the equilibrium surfaces provide only an indication of *possible* limit cycle solutions but not their amplitude and stability properties then time histories are needed to fulfil this function;
- if the mathematical model used is a simplified one then selected simulations enable verification (or lack thereof) of the assumptions made;
- predictions of behaviour directly from the eigenvalues using the theorems may in some instances be difficult: time histories illustrate response in a practical manner.

Thus by combining simulation with the equilibrium information yielded by a suitable computer program, an in-depth and global understanding of the nonlinear dynamics of the system is realised. This approach affords the flight dynamicist a vast potential for behaviour prediction, control system design, departure and spin recovery studies, improved parameter estimation, etc.

The computer program and its implementation

Program

The program required in order to generate equilibrium surfaces is essentially a nonlinear equation solver capable of finding all possible solutions to the algebraic system $\vec{f}(\vec{x}, \vec{c}) = 0$ within a required envelope of state and control variables. The program that has been developed is written in standard double-precision FORTRAN in a modular form and with a strong emphasis on robust solution methods, versatility, user-friendliness and well-formatted output. As such it is simple to use and yet able to be applied to any system of ordinary differential equations.

The solution procedure uses the parametric continuation method based on Kubíček[28] as developed by Mehra *et al.*[2] It has been augmented by incorporating it into the automatic step-incrementation method of Bergan[29], by providing a choice of predictor-step integration methods (to provide improved performance in regions where the system becomes stiff), a more powerful corrector-step solution algorithm, and other numerical techniques to facilitate robustness and convenient solution intervals. Several routines from the IMSL maths library are utilised.

The program requires initial guesses and checks for two guesses generating the same solution branch. Input includes not only the equations of motion $\vec{f}$ but also the elements of Jacobian F. Tabular data is currently accepted for up to two dimensions.

The present version finds eigenvalues of F at each solution but does not solve for limit cycle amplitude and stability as the extra computation required to incorporate this feature would increase program run time considerably.[3]

The program also solves an augmented $(n+1)$-dimensional system of equations made up of $\vec{f}$ plus an equation in numerical form specifying the determinant of F to be zero. This generates bifurcation surfaces for jump-type bifurcations caused by a real eigenvalue of F changing sign. No simple criterion exists for determining manifolds of Hopf and other bifurcations. Since the determinant is not analytically specified the additional row of the augmented system Jacobian matrix (required by the solution algorithm) is generated numerically using Richardson extrapolation on a finite-difference method.[30]

In order to allow for some intervention and selection before plotting occurs, the output from the program is made available to a separate post-processing program and then to the graphics package. A dedicated simulation package is used for time histories. All programs and packages are run on the University's IBM mainframe computer.

This suite of software forms a highly flexible and practical implementation of the bifurcation methodology.

Implementation

In order to provide some indication of the approach used in interpreting equilibrium surfaces sample output from the program is shown in Figure 2 for a simplified F-4J model.[11] The original model[31] is a fully validated representation of the actual aircraft's behaviour in subsonic flight from trim conditions, through stall, and up to and including flat spin. It has been reduced in complexity by setting gravity terms to zero and assuming V_T to be constant. The former simplification decouples the kinematic equations from the equations of motion; it is justified by the fact that at AOA up to stall and departure aircraft dynamics are largely dictated by inertial coupling and aerodynamic nonlinearities rather than by gravity effects.[21] The latter assumption introduces little error in stability and response calculations as V_T is slow-varying relative to the other state variables.[1] Altitude is also taken as constant.

Note that these simplifications (which have been shown to be acceptable for the example being presented[11]) have been made for ease of computation (and particularly in finding initial conditions) and interpretation of results — they do not reflect any limitation of the methodology or program.

The model uses the usual dynamic and kinematic equations for a rigid body in atmospheric flight. The aircraft is taken as mirror-symmetric about the x-z plane. Aerodynamic forces and moments are represented by 28 derivatives and coefficients, most of which are tabulated as functions of α and one as a function of β as well. Thus the model incorporates the inertial coupling inherent in the equations of motion as well as extensive aerodynamic coupling.

The system as implemented, therefore, comprises the following five equations of motion ($\dot{p}, \dot{q}, \dot{r}$ in body axes; $\dot{\alpha}, \dot{\beta}$ equations in wind axes):

$$\dot{p} = \frac{(I_y - I_z)\, I_z - I_{xz}^2}{I_x I_z - I_{xz}^2}\, rq + \frac{(I_x - I_y + I_z)\, I_{xz}}{I_x I_z - I_{xz}^2}\, pq + \frac{I_z}{I_x I_z - I_{xz}^2}\, L + \frac{I_{xz}}{I_x I_z - I_{xz}^2}\, N$$

$$\dot{q} = \frac{I_z - I_x}{I_y}\, rp + \frac{I_{xz}}{I_y}\,(r^2 - p^2) + \frac{1}{I_y}\, M$$

$$\dot{r} = \frac{(I_x - I_y)\, I_x + I_{xz}^2}{I_x I_z - I_{xz}^2}\, pq + \frac{(I_y - I_z - I_x)\, I_{xz}}{I_x I_z - I_{xz}^2}\, qr + \frac{I_{xz}}{I_x I_z - I_{xz}^2}\, L + \frac{I_x}{I_x I_z - I_{xz}^2}\, N$$

$$\dot{\alpha} = q - \tan\beta\,(p\cos\alpha + r\sin\alpha) + \frac{Z_w}{m V_T \cos\beta}$$

$$\dot{\beta} = p\sin\alpha - r\cos\alpha + \frac{Y_w}{m V_T}$$

The control variables (which appear inside the aerodynamic terms of these equations) are δ_a, δ_e and δ_r. The corresponding symbols used in the computer plots are given in the Nomenclature.

Figures 2(a) and 2(b) are the roll rate (p) and AOA (α) projections of the equilibrium surface for the full F-4J stabilator range and $\delta_a = 25°$ and $\delta_r = 0$. The title 'modified model' refers to the simplifications described above and to the smoothing of tabular data. The q, r and β curves are not shown.

The 'trim' region of flight is that portion of the graphs where stability is represented by the symbol S, from $\delta_e = -7.5°$ to $-3°$. Thus roll rate p is in the vicinity of 75° per second and α about 6° (pitch rate q is about 10° per second, yaw rate r about 5° per second and β 2°. The AOA range of stability on these solution curves is 0° to 11°.

If the stabilator were to be deflected beyond $-7.5°$ in the pitch-up sense (i.e. δ_e more negative), the equilibrium surface predicts the onset of limit cycle behaviour (Hopf bifurcation). No indication is given by the current program of the nature of the limit cycle (amplitude and stability). Assuming that the dynamics remain on this branch of the curves (i.e. that oscillations are not of sufficient magnitude to bring the motion into the domain of attraction of another more stable branch) then oscillations will occur for α greater than 11° and up to 20° (β from 3° to 9°). Roll rate oscillations would start about a mean value of 60° per second and decrease to $-20°$ per second as α increases (pitch rate oscillation would average about 10° per second and yaw rate would oscillate approximately about zero: it becomes negative at $\alpha \approx 19°$, indicating adverse yaw).

Since the characteristics of the predicted limit cycle are not known in the absence of limit cycle solution techniques or simulation, it is not yet possible to define whether the oscillation represents buffet or wing rock. Certainly its onset — $\alpha \approx 11°$ — corresponds to that of buffet in the literature, but it is possible that the limit cycle grows in amplitude as α increases and becomes a wing rock motion.

At $\delta_e = -14°$ ($\alpha = 20°$) the solution branch turns back on itself and becomes unstable. The system must then jump (bifurcate) to a new solution branch, thus indicating departure. The only remaining equilibrium branch on the graphs is that denoted by possible limit cycles at an AOA of 48° (in line with the zero-gravity simplification no fully developed spin branch exists in the plots). Again, without knowledge of the oscillatory stability in this region predictions are uncertain. It appears, however, that some form of oscillation is likely: a steep spin is possible.

The magnitude of the departure in terms of the change in variables is $p \approx 100°/\text{sec}$, $q \approx 0, r \approx 110°/\text{sec}$, $\alpha \approx 28°$ and $\beta \approx 8°$. Thus the departure appears as a pitch-up motion with strong divergence in roll and yaw rates (roll departure and nose slice).

Figure 3 shows results from a simulation run intended to verify the above predictions and characterise the limit cycle phenomena. The stabilator deflection is initially zero (corresponding to oscillations about $\alpha = 5°$ according to the equilibrium curve; this should be followed by a bifurcation to the stable 'trim' behaviour as δ_e crosses $-3°$). At time $t = 30$s, δ_e is decreased and is held steady at $-6°$ from $t = 40$s to 60s. The stable behaviour predicted by the equilibrium curves is manifest. Between $t = 60$s and $t = 70$s, δ_e is decreased to $-12°$. The existence of the predicted limit cycle is confirmed (t from 70s to 120s) and the rolling oscillation is seen to dominate relative to oscillations in the remaining variables. The motion is clearly wing rock (amplitude of p about 150°/second), with fairly marked amplitude in sideslip angle (about 20°).

At $t = 120$s δ_e is decreased to $-18°$ and a very large-amplitude oscillation develops, with mean values as derived from the equilibrium surface. Phase-plane plots (not shown) reveal that this post departure motion is approximately periodic with a 'figure-8' trajectory. The motion could be described as a type of steep spin but with excessively high amplitudes in all variables. It can be expected that in practice gravity would play a significant role at these high AOA, thus reducing the severity of the oscillations. A time history (not shown) without any of the simplifications mentioned previously (i.e. gravity, velocity, and altitude terms included) yields a very close resemblance in behaviour to that of Figure 3. As expected, however, amplitudes of both the wing rock and the post-departure oscillations are reduced.

Other simulations — not reproduced here — show that transient behaviour does not significantly affect the dynamics and that when δ_e is returned from $-18°$ back to zero, the response is merely reversed, i.e. there is no hysteresis in this case. Had such hysteresis existed, a method of returning to trimmed flight would have had to be sought using other controls, and the bifurcation surface would have proved useful in revealing such control-sequence recovery techniques.

This example of the implementation of bifurcation methods by no means shows the full potential of the technique. References 1, 2, 3, 4, 10 and 19 provide more detailed studies of aircraft stability and control topics, including the use of bifurcation surfaces.

Model complexity

The output obtained from the bifurcation method technique is, of course, dependent on the system equations ($\vec{f}(\vec{x}, \vec{c})$) and their data values fed in. The equations of motion for aircraft flight are standard and available in any flight mechanics text. Ideally, the form of the equations used in applying the bifurcation method would be the same as those chosen to perform simulations of the aircraft type concerned; in other words the equations would represent the motion of the aircraft through all its flight conditions, including extremes of AOA and sideslip, spins, etc.[26]

As mentioned above for the F-4J model implementation it is often convenient to simplify the system of equations in order to achieve faster computation, ease the process of identifying initial guesses for solution branches and to produce less confusing results that enable easier interpretation. Such simplified models can only be applied in practical problems when the phenomena being investigated are caused by mechanisms that remain within the system model — for this reason the reduced order F-4J model cannot be used for the study of developed spins.

The important aspect of the overall model in terms of implementation during aircraft design and development is that of the aerodynamic data. This ranges in nature from simple semi-empirical predictions for each stability derivative, or transfer function, in the early stages of design up to extremely comprehensive data bases obtained from static and dynamic wind tunnel tests, spin tunnel tests, free-flight model tests and possibly even full-scale flight tests. The latter package of information is only likely to be available once the first aircraft have already been flying for some time.

Generally, it can be said that the nonlinearities inherent in the system via the equations are responsible for 'jump-type' phenomena whilst the various aerodynamic nonlinearities lead to oscillatory motions.[4] This would imply that the *shape* of the curves on equilibrium surfaces results from the equations themselves, whereas any oscillatory behaviour on these curves can be ascribed to the aerodynamic nonlinearities. Certainly in the simplified aircraft model with constant aerodynamic coefficients studied in Refs. 1, 4, 10 and 19 the inertial coupling within the equations manifests very interesting behaviour and produces equilibrium surfaces rich in information.

Figures 4 (a) and (b) show the roll rate and AOA equilibrium surface projections for the F-4J model, as simplified above in terms of gravity and total velocity effects, for $\delta_a = \delta_r = 0$ and a range of δ_e from $-40°$ to 40° (well beyond the actual stabilator limits). In the context of the discussion that follows this model will be referred to as the 'full model'.

Figure 5 shows the equivalent curves with the stability derivatives fixed at constant values. The correlation is disappointing: the only common curves on the p graphs are the 'central branch' and the $\sum$-shaped branch. The other two $\subset$-shaped curves have not been found. This equilibrium surface enables no conjecture in respect of the aircraft's high AOA dynamics to be made (it predicts a stable increase in α up into spin-type values with no departure whatsoever). The fact that little oscillatory motion is predicted is, however, not unexpected for constant derivative values in the light of the statements above.

This particular example shows poor results from the bifurcation method for the particular parameter values chosen. It can be deduced, therefore, that implementation of the technique at the initial stage in design when only predicted constant values of stability derivatives are available is unlikely to be feasible.

The aerodynamic data in the full model is tabulated in terms of α and β. The α tables are for $|\alpha| \leq 45°$ whilst some go up to $|\alpha| \leq 110°$. The 2-dimensional table in which β is the second independent variable covers $|\beta| \leq 30°$. A bifurcation method run was made in which it is assumed that, possibly due to less expensive wind tunnel tests having being run, data is only available for $|\alpha| \leq 30°$ and $|\beta| \leq 15°$. The p and α equilibrium plots are shown in Figure 6 and it can be seen that the shape of the curves is retained. This model 'simplification' is seen to provide the same warnings as to when nonlinear phenomena are likely to occur as the full model results. The actual behaviour at values of AOA for which instability has set in ($\alpha \approx 20°$, as in the full model), however, does not correspond to reality because the original 'steep spin' branch at $\alpha \approx 45°$ is no longer stable. In fact, no stable solution exists for α beyond about 20°. It is, of course, to be expected that, since the data is limited in α, an implementation of this model could provide indications of possible undesirable behaviour at moderate to high AOA (up to the stall region) but could not reflect motions at higher AOA.

It is also of interest to observe the results attainable from the bifurcation method when static data is available but dynamic is not. This may arise at a stage when static wind tunnel tests on a preliminary design model have been performed but dynamic tests are scheduled for later in the project. Clearly, the motions resulting from nonlinear effects are highly dynamic in nature, yet the equilibrium surface, shown in Figure 7, bears a very strong resemblance to that of the full model in terms of predicting the onset of phenomena as higher AOA is approached: the shape of the relevant curves is almost identical, although the 'steep spin' mode occurs at slightly lower AOA and the possibility of oscillatory behaviour exists from α as low as 5°. The low AOA $\sum$-shaped branch (p graph), however, takes on a new shape for $\delta_e < -2°$.

A realistic scenario could exist for which fairly comprehensive static data is available but dynamic data — which relies on a far larger investment in time and money than static tests — only takes on the form of estimated constant values. Such results for the F-4J example are shown in Figure 8. The understanding of the dynamics of the system that a flight dynamicist could infer from these equilibrium surfaces is virtually identical to that suggested by the full model. The only differences would be that no limit cycle behaviour is shown just before departure (although simulations for the full model reveal that the amplitudes of the states in this limit cycle are extremely low) and that a stable branch exists in the AOA range of 10° to 14° for $\delta_e > 5°$ (at an extremely high roll rate).

A similar run for constant values of dynamic data but for $|\alpha| \leq 30°$ and $|\beta| \leq 15°$ (not shown) — i.e. a 'combination' of Figures 6 and 8 — yields, as expected, a good prediction of the onset of unusual behaviour (except the pre-departure limit cycle). Unlike in Figure 6, however, the 'steep spin' branch *is* denoted by limit cycles, although the position of the curve is inaccurate (α too high, p too low).

The simple variations demonstrated have been selected in order to show some relationship between the value of bifurcation run output and the aerodynamic data available. Since the opportunity of applying the technique during actual aircraft development has not arisen, these levels of model complexity do not reflect stages in some specific project. The constant derivatives that have been used in place of the tables are representative of the full model at $\alpha \approx 5°$; the constants that would be available in lieu of tabular data in practise would be analytical or empirical estimates, or results from initial low-cost wind tunnel tests. Obviously, it is assumed here that these estimates are representative of the true system at some reasonable AOA.

The accuracy with which lower-complexity aerodynamic models approximate higher level models will vary from one configuration to another and from one flight condition to another. The above results have also been obtained for $\delta_a = -25°$, with similar conclusions having been drawn. The indications are that as long as reasonably good static stability derivative data is available and the dynamic derivatives can be estimated, even as constant values, then the bifurcation methods will allow effective and cheap comparison of differing design configurations and yield useful information on the nonlinear dynamics of the aircraft.

The sensitivity of the bifurcation technique to model complexity as investigated here has considered only constant versus tabular data, tabular data up to moderately high AOA versus tabular data up to developed spin incidences, and static versus dynamic data. It thus reflects broadly the level of experimental results obtained at some point in a design/development programme. The study needs to go further than this and look more closely at the sensitivity of the technique to *specific* stability derivatives so as to indicate where the most effort is required in the analytical predictions and/or testing programmes. To a large extent, of course, this is known since the relationship between high AOA phenomena and stability derivatives has been the subject of intensive study over the years.

The deductions made above about bifurcation methods generating useful results at a fairly early stage in aircraft design and development — such as when good static derivative data exists but dynamic derivative values are estimated constants — has been made with the advantage of the full model results being available for comparison. In the practical context this would not be the case. There is a need, therefore, to incorporate an experimental component into the overall methodology, and it is intended that an investigation be conducted into the design of a low-cost wind tunnel dynamic test rig to reveal qualitative information on discontinuous and limit cycle bifurcationary behaviour. If successful correlation between such scale-model motions and full-scale response is obtained then this, in conjunction with the computational aspects of bifurcation methods, would form a practical methodology enabling the flight dynamicist to make confident predictions of behaviour early on in the design phase of new aircraft.

Summary

The need for a unified global technique for studying nonlinear stability and control problems of high-performance aircraft throughout the flight envelope has been presented. Bifurcation methods are shown to effectively meet this requirement. The concepts underlying the methodology have been summarised and a brief example of implementation has been given.

It has been noted that, despite a growing need for the understanding of nonlinear behaviour in dynamic systems, some reluctance to imlement bifurcation methods exists in the aircraft design and development environment. This has been ascribed to the perceived difficulties in computing the required equilibrium and bifurcation surfaces and to the belief that highly comprehensive models are necessary.

In response to these perceptions a user-friendly suite of computer programs has been developed, incorporating a versatile and robust equilibrium solution program; and aerodynamic models of varying complexity for the F-4J Phantom have been used in the program to investigate the sensitivity of bifurcation methods to the aerodynamic model available.

The results indicate that the methods are likely to be useful provided good static stability derivative data has been obtained; estimates of some type should be available for dynamic derivatives but no need appears to exist for very comprehensive dynamic data in order to produce valid and practical output. Similar results obtained for other aircraft configurations, in several flight conditions, would serve to confirm these findings. Preferably, the methods should be implemented within an actual aircraft design and development environment.

The work done indicates further that, to account for the often counter-intuitive phenomena associated with high AOA aircraft dynamics, the overall bifurcation methods methodology would be strongly enhanced by complementing the computational procedures with a simple experimental facility. The aim would be qualitative analysis of scale-model aircraft response during bifurcationary behaviour. It is intended that further work will involve the development of such a low-cost dynamic test rig for wind tunnels and the investigation of its effectiveness and viability. The potential thus exists for bifurcation methods to evolve into a broader experimental/computational tool to be used with confidence by flight dynamicists.

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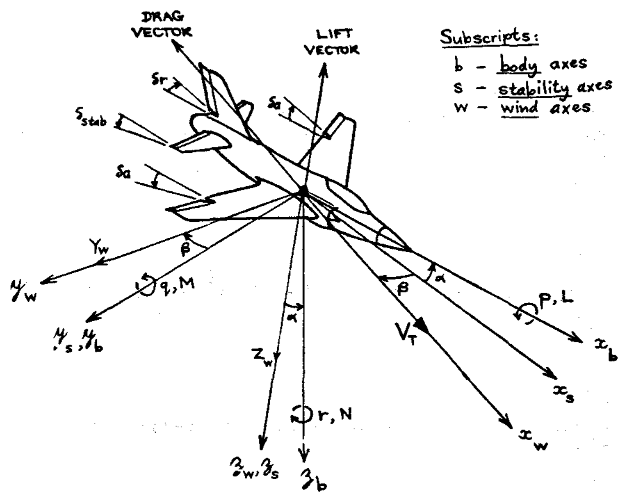

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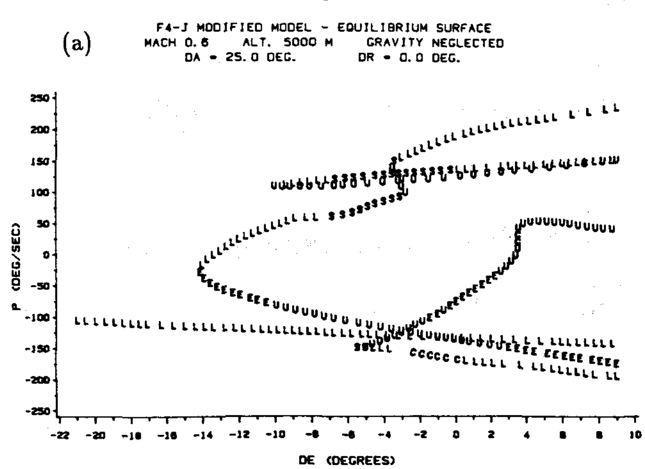

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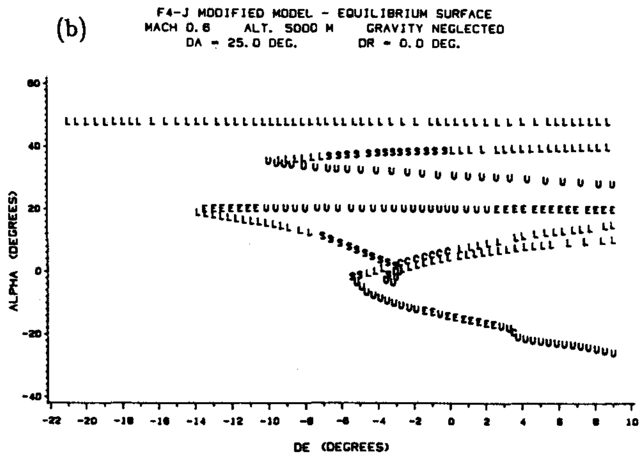

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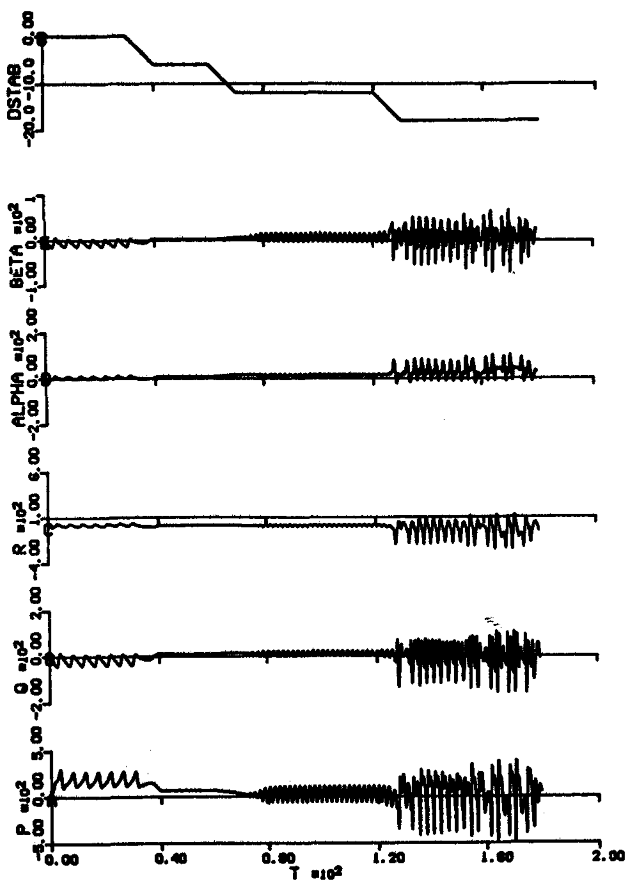

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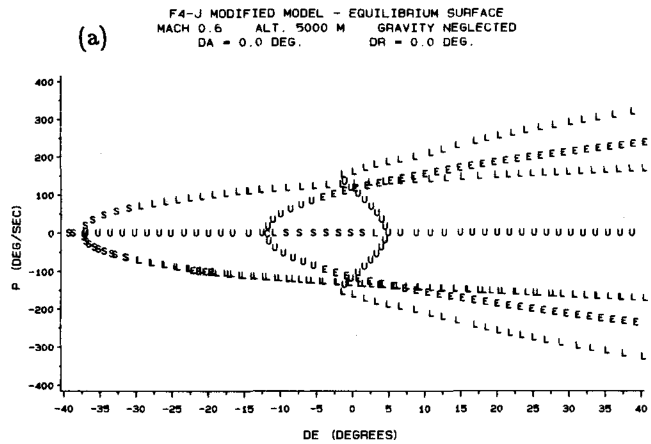

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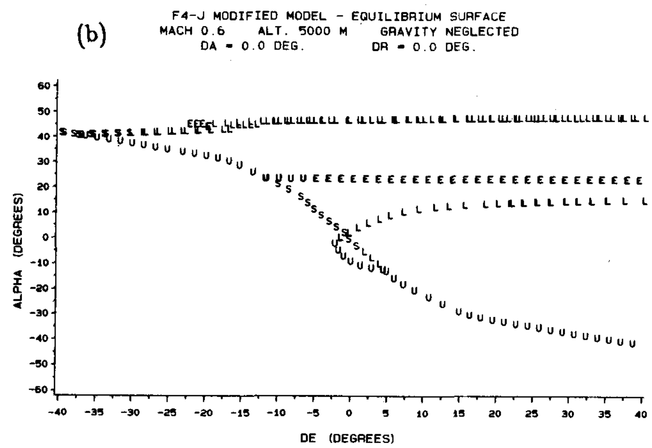

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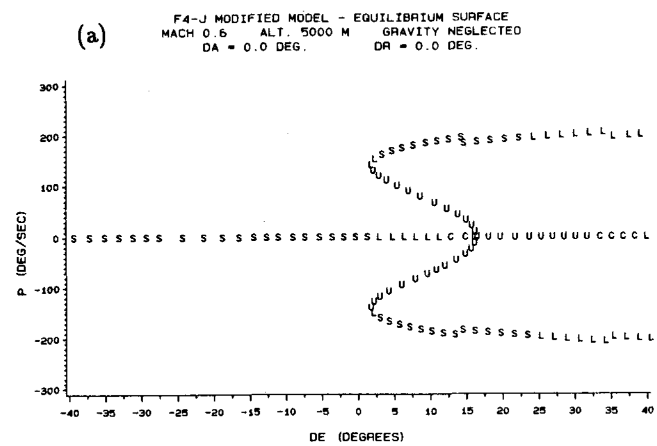

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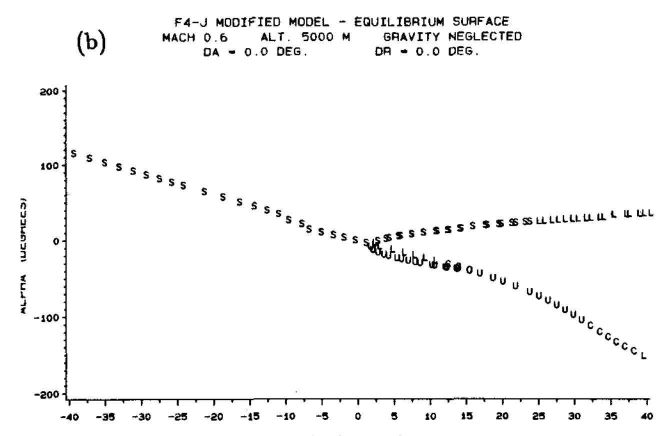

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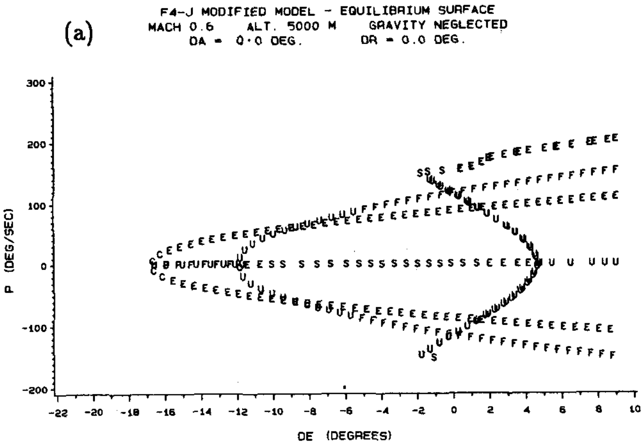

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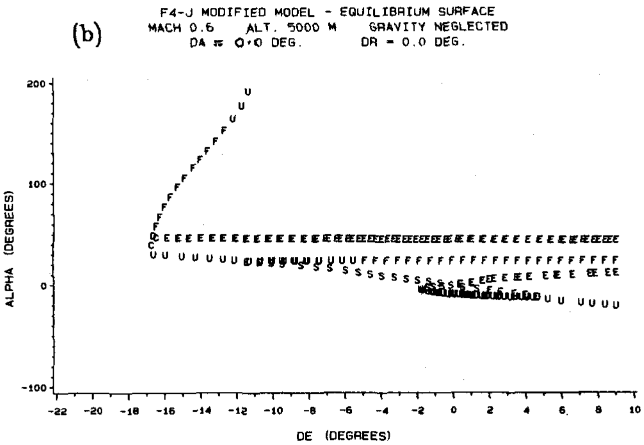

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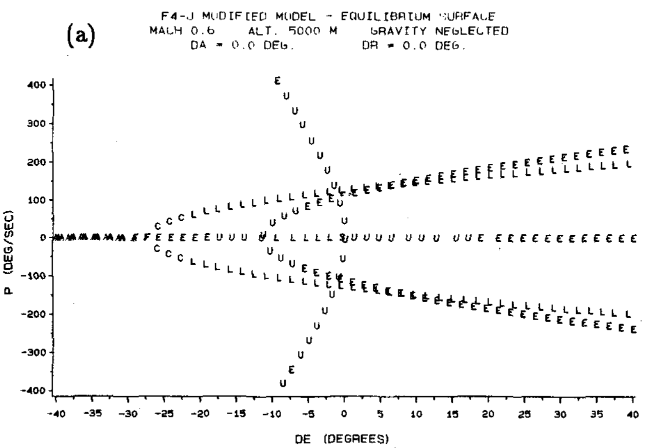

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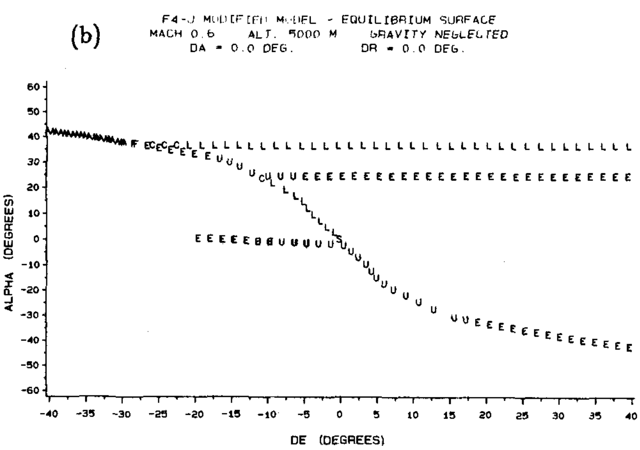

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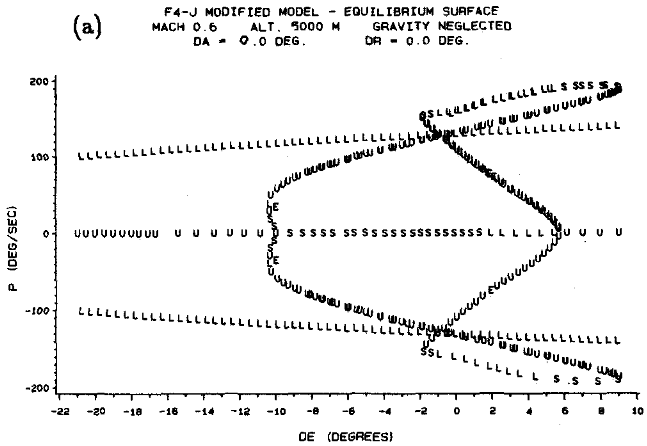

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