# SUPERSONIC FLUTTER OF COMPOSITE SKEW PANELS

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## Summary

Supersonic flutter of a composite skew panel is determined by the finite element method. An isoparametric 16-parameter conforming plate bending displacement model is adopted. It is demonstrated that the natural frequencies and the flutter parameter $\lambda_{cr}$ are functions of two matrices $[\Gamma]$ and $[\Theta]$.The first one $[\Gamma]$ depends exclusively on the skew angle between the sides of the skew plate; the second one $[\Theta]$ is dependent soley on the fibre orientation angle. The study is completed by a series of exemples .

## I. Introduction

Several authors have dealt with supersonic flutter study of rectangular panels made from isotropic material. Of these it is important to remember Earl H. Dowell [1] [2] [3] [4],who studied the subject both from the theoretical and experimental point of view.

According to T.Y. Yang [5], the aerodynamic forces due to supersonic flow are obtained from the exact theory for linearized two-dimensional unsteady flow where no limitations on the frequency order are imposed. The aerodynamic matrix becomes location dependent and must be formulated for each element. Mervyn D. Olson [6] [7] carried out an in-depth investigation with conforming and non conforming elements.J.G. Eisley and G. Luessen [8] also studied under combined normal and shear edge forces.Furthermore S. Durvasula [9] made an approximate analysis using Lagrange equations employing double Fourier sine series in oblique coordinates to represent the deflection surface. T.Y. Yang and A.D. Han [10] and Chun Mei [11] studied panel flutter considering the effect of large deformations.

As regards oblique panels made from isotropic material very little literature is available. It is interesting to note the papers by Kariappa,Somashekar and Shar [12] [13],and above all by G.A. Oyibo [14] [15] wich dealt with orthotropic panel. A closed-form rather than a modal approach has been used in the solution of the partial differential equation in the analysis to avoid convergence problems.

## II. Theoretical Formulation

In the method adopted the panel is subdivided into equal parallelogrammic plate bending elements (Fig.1).

At each node point there are four generalized displacements:

$w$ deflection in z-direction

$$\left\{ \begin{array}{l} \frac{a}{2}\frac{\partial w}{\partial x} \\ \frac{b}{2}\frac{\partial w}{\partial \bar{y}} \end{array} \right. \quad \text{the two oblique slopes}$$

$$\frac{a\,b}{4}\frac{\partial^2 w}{\partial \bar{y}\partial x} \quad \text{slope continuity}$$

Calculation procedures are simplified by means of the isoparametric formulation. It permits the passage from the real x y coordinate system to the so-called "natural" $\xi$ $\eta$ coordinate system through the topologic transformation:

$$\xi = -1 + \frac{2}{a}(x + t\,y)$$
$$\eta = -1 + \frac{2}{b}\frac{y}{c} \qquad 1)$$

On the $\xi$ $\eta$ plane, the parallelogrammic plate element becomes a square plate element with coordinates whose modulus does not exceed unity (Fig.2).

Using this formulation the nodal displacement vector results as:

$$\left\{ u_\omega \right\} = \left\{ w_1, w_{\xi 1}, w_{\eta 1}, w_{\xi\eta 1}, w_2, \ldots\ldots\ldots \ldots, w_4, w_{\xi 4}, w_{\eta 4}, w_{\xi\eta 4} \right\} \quad 2)$$

The deflection w and slopes are expressed in terms of isoparametric coordinates $\xi$ and $\eta$ :

$$w = \lfloor f \rfloor \left\{ a \right\} \quad 3)$$

$$\lfloor f \rfloor = \lfloor 1, \xi, \eta, \xi^2, \xi\eta, \eta^2, \xi^3, \xi^2\eta, \xi\eta^2, \eta^3, \xi^3\eta, \xi^2\eta^2, \eta^3\xi, \xi^3\eta^2, \xi^2\eta^3, \xi^3\eta^3 \rfloor \quad 4)$$

$$\left\{ a \right\} = \left\{ a_0, a_1, a_2, \ldots, a_{15} \right\} \quad 5)$$

where (5) is the vector of the costants $a_i$.

Replacing the nodal coordinates in (3) and in the expressions of its derivatives $\partial/\partial\xi$, $\partial/\partial\eta$, $\partial^2/\partial\xi\partial\eta$, $\{u_\omega\}$ may be expressed in the form:

$$\left\{ u_\omega \right\} = \left[ B \right] \left\{ a \right\} \quad 6)$$

hence

$$w = \lfloor f \rfloor \left[ B \right]^{-1} \left\{ u_\omega \right\} \quad 7)$$

Matrix $[B^{-1}]$ is listed in table 1.

For simplicity it is assumed that the air flowing above the panel is parallel to x-axis and that the effect of any air entrapped below may be neglected.

Assuming two-dimensional supersonic aerodynamic theory is sufficently accurate, the aerodynamic forces acting on the infinitesimal element dxdy of the panel, valid for $M > 1.7$, are:

$$\Delta F_a = -\frac{2q}{\beta} \frac{\partial w}{\partial x} \, dx \, dy \quad 8)$$

which is the real aerodynamic force, proportional to the $\partial w/\partial x$ slope and

$$\Delta F_d = -\left( \frac{2q}{V} \frac{M^2-2}{\beta^3} \frac{\partial w}{\partial t} \right) dx \, dy \quad 9)$$

is the aerodynamic damping due to the deflection w. $\Delta F_d$ can be considered as an addition to the inertial force

$$\Delta F_m = -\rho \frac{\partial^2 w}{\partial t^2} \, dxdy \quad 10)$$

In this case, only $\Delta F_a$ constitutes the aerodynamic force.

Furthermore, if $N_x$ is the initial in-plane loading per unit of length (tension positive) parallel to x-axis, it will create a force in z-direction

$$\Delta F_z = -N_x \frac{\partial^2 w}{\partial x^2} \, dx \, dy \quad 11)$$

The deflection w is assumed to be an exponential temporal function

$$w(t) = w \, e^{\omega t} \quad 12)$$

with $\omega = a+ib$ and w given by (3).

Therefore resorting to the isoparametric formulation:

$$dxdy = \det \mathcal{J} \, d\xi \, d\eta = \frac{ab}{4} \, d\xi \, d\eta \quad 13)$$

where

$$\mathcal{J} = \begin{bmatrix} \frac{dx}{d\xi} & \frac{dy}{d\xi} \\ \frac{dx}{d\eta} & \frac{dy}{d\eta} \end{bmatrix} = \begin{bmatrix} \frac{a}{2} & 0 \\ -\frac{bs}{2} & \frac{bc}{2} \end{bmatrix} \quad 14)$$

is the Jacobian operator.

Considering (12) and (13) the expression of the infinitesimal forces are:

$$\Delta F_m = -\det \mathcal{J} \, \rho \, \omega^2 w \, e^{\omega t} \, d\xi \, d\eta \quad 15)$$

$$\Delta F_a = -\det \mathcal{J} \, \frac{2q}{\beta} \frac{\partial \xi}{\partial x} \frac{\partial w}{\partial \xi} e^{\omega t} \, d\xi \, d\eta \quad 16)$$

$$\Delta F_d = -\det \mathcal{J} \, \frac{2q}{V} \frac{M^2-2}{\beta^3} \omega \, w \, e^{\omega t} \, d\xi \, d\eta \quad 17)$$

$$\Delta F_{\bar{n}} = -\det \mathcal{J} \, N_x \left( \frac{\partial \xi}{\partial x} \right)^2 \frac{\partial^2 w}{\partial \xi^2} e^{\omega t} d\xi \, d\eta \quad 18)$$

## II. Flutter Equation

In order to express the forces at the nodes, the principle of virtual work is used:

$$W_e + W_a + W_m + W_n = W_i \qquad 19)$$

where

- $W_e$ external work
- $W_a$ work due to aerodynamic forces
- $W_m$ work due to inertial forces and aerodynamic damping
- $W_n$ work due to $\Delta F_n$ created by $N_x$
- $W_i$ internal work

In [19] it is demonstrated that formula (19) governing motion can be written in the following adimensional form:

$$\left\{ \bar{F}_e \right\} = \frac{b}{4a} c \left( \lambda \left[ \bar{A}_e \right] - \mu \left[ \bar{M}_e \right] + R_x \left[ \bar{N}_e \right] + \left[ \bar{K}_e \right] \right) \left\{ u_e \right\} \qquad 20)$$

where

$$\lambda = \frac{2qL^3}{\beta D} \text{ and } \mu = -\frac{\rho L^4 \omega^2}{D} - \lambda \frac{L(M^2-2)\omega}{\beta^2 V}$$

are parameter and eigenvalues of adimensional aerodynamic pressure respectively, and

$$R_x = \left( L^2/n^2 D \right) N_x$$

$$\left[ \bar{A}_e \right] = 2/n^3 \left[ A_e \right]$$

$$\left[ \bar{M}_e \right] = 1/n^4 \left[ M_e \right]$$

$$\left[ \bar{N}_e \right] = \left[ N_e \right]$$

$$\left[ \bar{K}_e \right] = \frac{16}{D} \left[ K_e \right]$$

$$\left\{ \bar{F}_e \right\} = \left\{ F_e \right\} L^2/n^2 D$$

$\left\{ \bar{F}_e \right\}$ = generalized adimensional nodal forces corresponding to generalized displacements

Assembling finite elements and applying kinematic boundary conditions, the instability condition of the panel examined can be written in the following way:

$$\left[ \left[ \bar{K} \right] + \lambda \left[ \bar{A} \right] - \mu \left[ \bar{M} \right] + \left[ \bar{N} \right] \right] \left\{ u \right\} = 0 \qquad 21)$$

The eigenvalues can be put in a more convenient form:

$$\mu = -g_A \left( \frac{\omega}{\omega_o} \right) - \left( \frac{\omega}{\omega_o} \right)^2 \qquad 22)$$

with

$$g_A = \frac{(M^2-2)\rho_a V}{\beta^3 \rho \omega_o} \qquad 23)$$

damping aerodynamic adimensional parameter and

$$\omega_o = \sqrt{\frac{D}{\rho a^4}} \qquad 24)$$

where $\omega_o$ is a convenient frequency scale. $g_A$ varies approximately between 0 and 50 [17].

## Results and Concluding Remarks

Several application examples of the present method are presented in this section.In the case studied for a composite skew angle panel,bending stiffness is:

$$D = \sqrt{E_1 E_2} / 12(1 - \nu_{12}\nu_{21})$$

In particular for isotropic panel

$$E = E_1 = E_2 \qquad \nu = \nu_{12} = \nu_{21}$$

The numerical matrices $[\bar{A}_e]$, $[\bar{M}_e]$ and $[\bar{N}_e]$ are calculated and are listed in table 2.

The adimensional stiffness matrix $\bar{K}_e$ has been calculated for symmetric laminates with multiple orthotropic layers. All elements of $[K_e]$ are functions of six terms (Ref. Table 2)

$$
\begin{Bmatrix} \bar{k}_{4,4} \\ \frac{b}{a}\bar{k}_{4,5} \\ \frac{b^2}{a^2}\bar{k}_{4,6} \\ \frac{b^2}{a^2}\bar{k}_{5,5} \\ \frac{b^3}{a^3}\bar{k}_{5,6} \\ \frac{b^4}{a^4}\bar{k}_{6,6} \end{Bmatrix} = \frac{1024}{h^3\sqrt{E_1E_2}}[\Gamma]\sum_{k=1}^{N}(z_k^3 - z_{k-1}^3)[\Theta]_k \begin{Bmatrix} E_1 \\ E_2\nu_{12} \\ E_2 \\ G \end{Bmatrix}
$$

that depend on the angles $\gamma$ and $\theta$ by means of the matrices

$$
[\Gamma] = \begin{bmatrix} 1 & 2t^2 & t^4 & 4t & 4t^3 & 4t^2 \\ 0 & t/c & t^3/c & 1/c & 3t^2/c & 2t/c \\ 0 & 1/c^2 & t^2/c^2 & 0 & 2t/c^2 & 0 \\ 0 & 0 & t^2/c^2 & 0 & 2t/c^2 & 1/c^2 \\ 0 & 0 & t/c^3 & 0 & 1/c^3 & 0 \\ 0 & 0 & 1/c^4 & 0 & 0 & 0 \end{bmatrix}
$$

and for the k-th layer

$$
[\Theta]_k = (c_\theta^4)_k \begin{bmatrix} 1 & 2t_\theta^2 & t_\theta^4 & 4t_\theta^2 \\ t_\theta^2 & t_\theta^4+1 & t_\theta^2 & -4t_\theta^2 \\ t_\theta^4 & 2t_\theta^2 & 1 & 4t_\theta^2 \\ t_\theta & -t_\theta(1-t_\theta^2) & -t_\theta^3 & -2t_\theta(1-t_\theta^2) \\ t_\theta^3 & t_\theta(1-t_\theta^2) & -t_\theta & 2t_\theta(1-t_\theta^2) \\ t_\theta^2 & -2t_\theta^2 & t_\theta^2 & (1-t_\theta^2)^2 \end{bmatrix}_k
$$

The elements of $[\bar{K}_e]$ are shown in table 3.

The calculation method used is first of all to set the master aerodynamic-stiffness matrix of equation (21) for a particular assemblage of finite elements.

The eigenvalues are then calculated for increasing values of the dynamic parameter $\lambda$ until the first coalescence occurs.

The graphics in Figg. 3, 4 and 5 show the variation of $\lambda_{cr}$, $\sqrt{\mu}_{cr}$ and the natural frequencies $\sqrt{\mu}$ (for $\lambda = 0$) of various composite panels simply supported and clamped on all sides as $\gamma$ and $\theta$ vary.

An eight-layered symmetric lay-up ($\theta, -\theta, \theta, -\theta, -\theta, \theta, -\theta, \theta$) is used and the material properties of typical unidirectional composites are taken from [16]. The first natural frequency values may be compared with [16], and all parameter $\lambda_{cr}, \mu_{cr}$ and $\mu$ with [19], [20].

## IV. Symbols

| | |
|---|---|
| $\gamma$ | skew angle |
| c s t | cos $\gamma$ sin $\gamma$ tg $\gamma$ |
| $\vartheta$ | fibre orientation |
| $c_\vartheta$ $t_\vartheta$ | cos $\vartheta$ tg $\vartheta$ |
| x, y | rectangular coordinate axes |
| $\bar{y}$ | axis along the oblique side of parallelogrammic plate element |
| L | length of plate along x |
| $n^2$ | finite element grid |
| N | number of layers |
| $\xi$, $\eta$ | natural coordinate axes corresponding to x, $\bar{y}$ |
| a b | lengths of sides of parallelogrammic plate element |
| h | plate thickness |
| w | transverse displacement in the z-direction |
| $E_1$ $E_2$ | Young's moduli in the two principal directions |
| $\nu_{12}$ | major Poisson's ratio |
| $\nu_{21}$ | minor Poisson's ratio |
| $G_{12}$ | shear modulus in the panel plane |
| G = | $G_{12}(1-\nu_{12}\nu_{21})$ |
| V | speed of air flow |
| $\rho$ | density per surface unit of the pannel |
| $\rho_a$ | air density |
| q = | $\rho_a V^2/2$ |
| M | Mach number |
| $\beta$ = | $\sqrt{M^2-1}$ |
| $\lambda$, $\lambda_{cr}$ | nondimensional dynamic pressure and coalescence value, respectivly |

The other symbols are defined in the paper.

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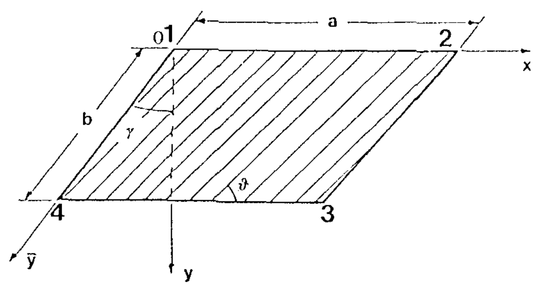



FIGURE 1

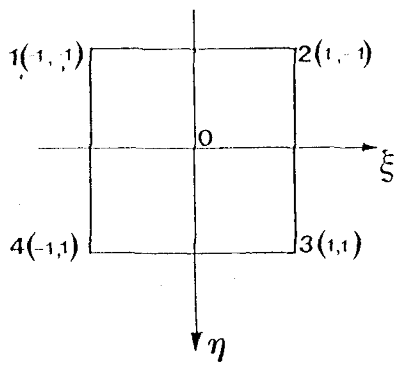



FIGURE 2

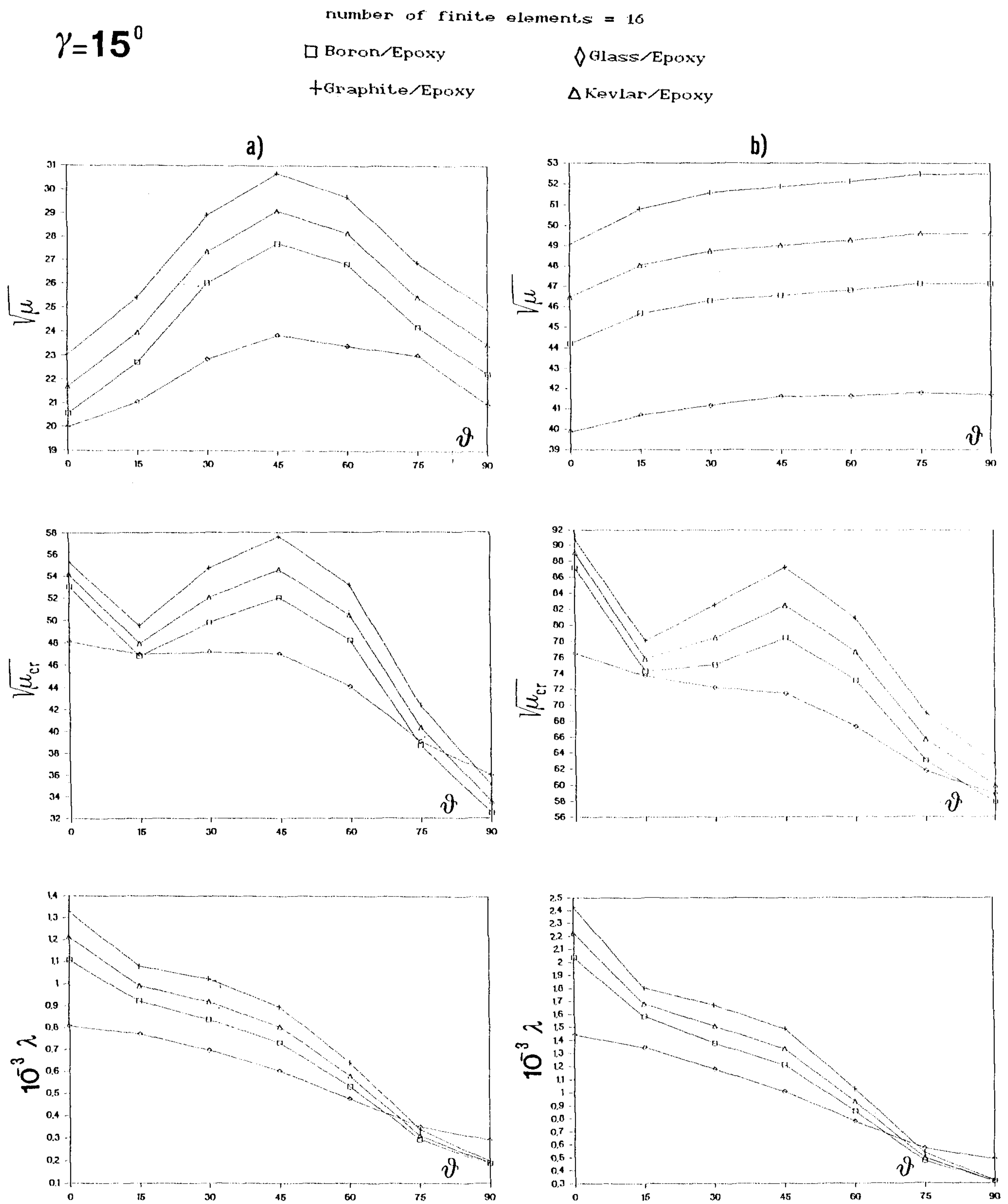



FIGURE 3 Simply supported rhombic panel (a)

Clamped panel (b)

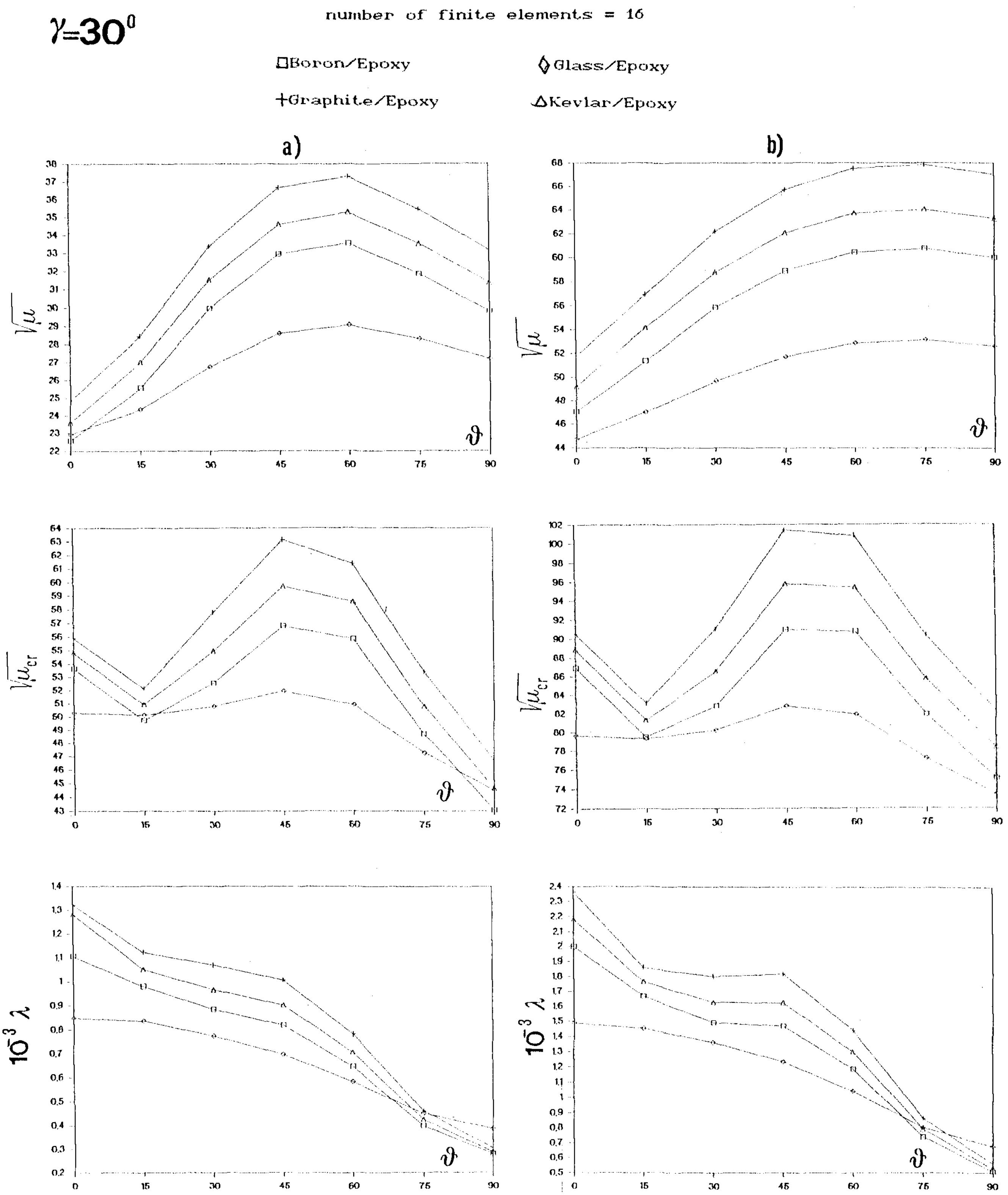



FIGURE 4 Simply supported rhombic panel (a)

Clamped panel (b)

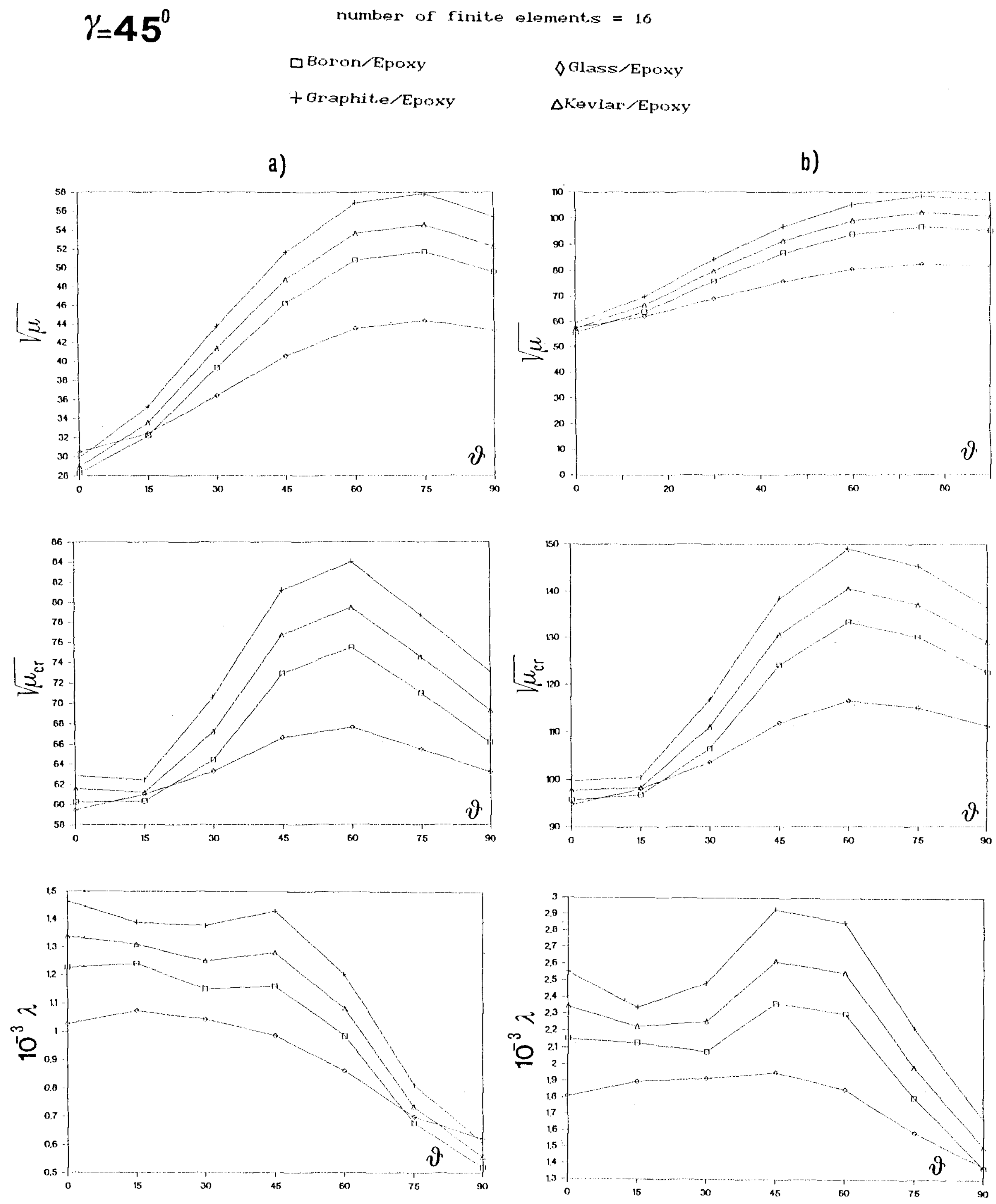


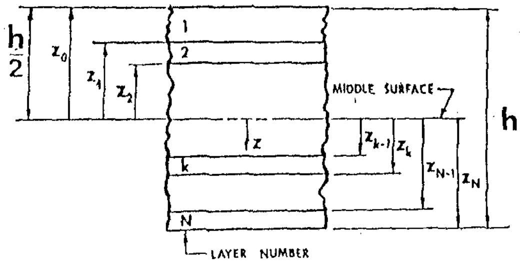

FIGURE 5 Simply supported rhombic panel (a)

Clamped panel (b)

TABLE 1

$$
[B]^{-1}=\frac{1}{16}\begin{bmatrix}
4 & 2 & 2 & 1 & 4 & -2 & 2 & -1 & 4 & -2 & -2 & 1 & 4 & 2 & -2 & -1\\
-6 & -2 & -3 & -1 & 6 & -2 & 3 & -1 & 6 & -2 & -3 & -1 & -6 & -2 & 3 & 1\\
-6 & -3 & -2 & -1 & -6 & 3 & -2 & 1 & 6 & -3 & -2 & 1 & 6 & 3 & -2 & -1\\
0 & -2 & 0 & -1 & 0 & 2 & 0 & 1 & 0 & 2 & 0 & -1 & 0 & -2 & 0 & 1\\
9 & 3 & 3 & 1 & -9 & 3 & -3 & 1 & 9 & -3 & -3 & 1 & -9 & -3 & 3 & 1\\
0 & 0 & -2 & -1 & 0 & 0 & -2 & 1 & 0 & 0 & 2 & -1 & 0 & 0 & 2 & 1\\
2 & 2 & 1 & 1 & -2 & 2 & -1 & 1 & -2 & 2 & 1 & -1 & 2 & 2 & -1 & -1\\
0 & 3 & 0 & 1 & 0 & -3 & 0 & -1 & 0 & 3 & 0 & -1 & 0 & -3 & 0 & 1\\
0 & 0 & 3 & 1 & 0 & 0 & -3 & 1 & 0 & 0 & 3 & -1 & 0 & 0 & -3 & -1\\
2 & 1 & 2 & 1 & 2 & -1 & 2 & -1 & -2 & 1 & 2 & -1 & -2 & -1 & 2 & 1\\
-3 & -3 & -1 & -1 & 3 & -3 & 1 & -1 & -3 & 3 & 1 & -1 & 3 & 3 & -1 & -1\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1\\
-3 & -1 & -3 & -1 & 3 & -1 & 3 & -1 & -3 & 1 & 3 & -1 & 3 & 1 & -3 & -1\\
0 & 0 & -1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 1\\
0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1\\
1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1
\end{bmatrix}
$$

TABLE 2




Geometry of an N layered laminate.

$\bar{k}_{4,11}=\bar{k}_{4,5}$

$\bar{k}_{4,13}=\bar{k}_{4,5}$

$\bar{k}_{5,11}=\bar{k}_{5,5}$

$\bar{k}_{5,13}=\bar{k}_{5,5}$

$\bar{k}_{6,11}=\bar{k}_{5,6}$

$\bar{k}_{6,13}=\bar{k}_{5,6}$

$\bar{k}_{7,7}=3\bar{k}_{4,4}$

$\bar{k}_{7,8}=2\bar{k}_{4,5}$

$\bar{k}_{7,9}=\bar{k}_{4,6}$

$\bar{k}_{8,8}=(\bar{k}_{4,4}+4\bar{k}_{4,5})/3$

$\bar{k}_{8,9}=2(\bar{k}_{4,5}+\bar{k}_{5,6})/3$

$\bar{k}_{8,10}=\bar{k}_{4,6}$

$\bar{k}_{9,9}=(\bar{k}_{6,6}+4\bar{k}_{5,5})/3$

$\bar{k}_{9,10}=2\bar{k}_{5,6}$

$\bar{k}_{10,10}=3\bar{k}_{6,6}$

$\bar{k}_{11,11}=(5\bar{k}_{4,4}+9\bar{k}_{5,5})/5$

$\bar{k}_{11,13}=\bar{k}_{4,6}+\bar{k}_{5,5}$

$\bar{k}_{13,13}=(5\bar{k}_{6,6}+9\bar{k}_{5,5})/5$

$\bar{k}_{4,12}=1/3(\bar{k}_{4,6}+\bar{k}_{4,4})$

$\bar{k}_{8,15}=\bar{k}_{4,4}/5+(4\bar{k}_{5,5}+\bar{k}_{4,6})/3$

$\bar{k}_{5,12}=1/3(\bar{k}_{5,6}+\bar{k}_{4,5})$

$\bar{k}_{9,15}=2\bar{k}_{4,5}/5+4\bar{k}_{5,6}/3$

$\bar{k}_{6,12}=1/3(\bar{k}_{6,6}+\bar{k}_{4,6})$

$\bar{k}_{10,15}=\bar{k}_{6,6}+3\bar{k}_{4,6}/5$

$\bar{k}_{11,12}=3/5\bar{k}_{5,6}+5/3\bar{k}_{4,5}$

$\bar{k}_{14,15}=8(\bar{k}_{5,6}+\bar{k}_{4,5})/5$

$\bar{k}_{12,13}=3/5\bar{k}_{4,5}+5/3\bar{k}_{5,6}$

$\bar{k}_{4,16}=\bar{k}_{4,5}$

$\bar{k}_{12,12}=(\bar{k}_{4,4}+\bar{k}_{6,6})/5+(16\bar{k}_{5,5}+2\bar{k}_{4,6})/9$

$\bar{k}_{5,16}=\bar{k}_{5,5}$

$\bar{k}_{7,14}=3/5\bar{k}_{4,6}+\bar{k}_{4,4}$

$\bar{k}_{11,16}=3(3\bar{k}_{5,5}+\bar{k}_{4,4}+\bar{k}_{4,6})/5$

$\bar{k}_{8,14}=2/5\bar{k}_{5,6}+4/3\bar{k}_{4,5}$

$\bar{k}_{12,16}=7(\bar{k}_{5,6}+\bar{k}_{4,5})/5$

$\bar{k}_{9,14}=\bar{k}_{6,6}/5+(4\bar{k}_{5,5}+\bar{k}_{4,6})/3$

$\bar{k}_{13,16}=3(3\bar{k}_{5,5}+\bar{k}_{6,6}+\bar{k}_{4,6})/5$

$\bar{k}_{10,14}=2\bar{k}_{5,6}$

$\bar{k}_{14,14}=\bar{k}_{6,6}/7+(3\bar{k}_{4,4}+12\bar{k}_{5,5}+2\bar{k}_{4,6})/5$

$\bar{k}_{15,15}=\bar{k}_{4,4}/7+(3\bar{k}_{6,6}+12\bar{k}_{5,5}+2\bar{k}_{4,6})/5$

$\bar{k}_{7,15}=2\bar{k}_{4,5}$

$\bar{k}_{16,16}=3(\bar{k}_{4,4}+\bar{k}_{6,6})/7+9(9\bar{k}_{5,5}+2\bar{k}_{4,6})/25$

$$
[\bar{K}_e]=\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
 & & & \bar{k}_{4,4} & \bar{k}_{4,5} & \bar{k}_{4,6} & 0 & 0 & 0 & 0 & \bar{k}_{4,11} & \bar{k}_{4,12} & \bar{k}_{4,13} & 0 & 0 & \bar{k}_{4,16}\\
 & & & & \bar{k}_{5,5} & \bar{k}_{5,6} & 0 & 0 & 0 & 0 & \bar{k}_{5,11} & \bar{k}_{5,12} & \bar{k}_{5,13} & 0 & 0 & \bar{k}_{5,16}\\
 & & & & & \bar{k}_{6,6} & 0 & 0 & 0 & 0 & \bar{k}_{6,11} & \bar{k}_{6,12} & \bar{k}_{6,13} & 0 & 0 & \bar{k}_{6,16}\\
 & & & & & & \bar{k}_{7,7} & \bar{k}_{7,8} & \bar{k}_{7,9} & 0 & 0 & 0 & 0 & \bar{k}_{7,14} & \bar{k}_{7,15} & 0\\
 & & & & & & & \bar{k}_{8,8} & \bar{k}_{8,9} & \bar{k}_{8,10} & 0 & 0 & 0 & \bar{k}_{8,14} & \bar{k}_{8,15} & 0\\
 & & & & & & & & \bar{k}_{9,9} & \bar{k}_{9,10} & 0 & 0 & 0 & \bar{k}_{9,14} & \bar{k}_{9,15} & 0\\
 & & & & & & & & & \bar{k}_{10,10} & 0 & 0 & 0 & \bar{k}_{10,14} & \bar{k}_{10,15} & 0\\
 & & & & & & & & & & \bar{k}_{11,11} & \bar{k}_{11,12} & \bar{k}_{11,13} & 0 & 0 & \bar{k}_{11,16}\\
 & & & & & & & & & & & \bar{k}_{12,12} & \bar{k}_{12,13} & 0 & 0 & \bar{k}_{12,16}\\
 & & & & & & & & & & & & \bar{k}_{13,13} & 0 & 0 & \bar{k}_{13,16}\\
 & & & & & & & & & & & & & \bar{k}_{14,14} & \bar{k}_{14,15} & 0\\
 & & & & & & & & & & & & & & \bar{k}_{15,15} & 0\\
 & & & & & & & & & & & & & & & \bar{k}_{16,16}
\end{bmatrix}
$$

TABLE 3

$$
\left[\bar{M}_e\right] = \frac{1}{11025}\begin{bmatrix}
6084 & 1716 & 1716 & 484 & 2106 & -1014 & 594 & -286 & 729 & -351 & -351 & 169 & 2106 & 594 & -1014 & -286 \\
 & 624 & 484 & 176 & 1014 & -468 & 286 & -132 & 351 & -162 & -169 & 78 & 594 & 216 & -286 & -104 \\
 & & 624 & 176 & 594 & -286 & 216 & -104 & 351 & -169 & -162 & 78 & 1014 & 286 & -468 & -132 \\
 & & & 64 & 286 & -132 & 104 & -48 & 169 & -78 & -78 & 36 & 286 & 104 & -132 & -48 \\
 & & & & 6084 & -1716 & 1716 & -484 & 2106 & -594 & -1014 & 286 & 729 & 351 & -351 & -169 \\
 & & & & & 624 & -484 & 176 & -594 & 216 & 286 & -104 & -351 & -162 & 169 & 78 \\
 & & & & & & 624 & -176 & 1014 & -286 & -468 & 132 & 351 & 169 & -162 & -78 \\
 & & & & & & & 64 & -286 & 104 & 132 & -48 & -169 & -78 & 78 & 36 \\
 & & & & & & & & 6084 & -1716 & -1716 & 484 & 2106 & 1014 & -594 & -286 \\
 & & & & & & & & & 624 & 484 & -176 & -1014 & -468 & 286 & 132 \\
 & & & & & & & & & & 624 & -176 & -594 & -286 & 216 & 104 \\
 & & & & & & & & & & & 64 & 286 & 132 & -104 & -48 \\
 & & & & & & & & & & & & 6084 & 1716 & -1716 & -484 \\
 & & & & & & & & & & & & & 624 & -484 & -176 \\
 & & & & & & & & & & & & & & 624 & 176 \\
 & & & & & & & & & & & & & & & 64
\end{bmatrix}
$$

$$
\left[\bar{A}_e\right] = \frac{1}{1575}\begin{bmatrix}
-1170 & 468 & -330 & 132 & 1170 & -468 & 330 & -132 & 405 & -162 & -195 & 78 & -405 & 162 & 195 & -78 \\
-468 & 0 & -132 & 0 & 468 & -156 & 132 & -44 & 162 & -54 & -78 & 26 & -162 & 0 & 78 & 0 \\
-330 & 132 & -120 & 48 & 330 & -132 & 120 & -48 & 195 & -78 & -90 & 36 & -195 & 78 & 90 & -36 \\
-132 & 0 & -48 & 0 & 132 & -44 & 48 & -16 & 78 & -26 & -36 & 12 & -78 & 0 & 36 & 0 \\
-1170 & -468 & -330 & -132 & 1170 & 468 & 330 & 132 & 405 & 162 & -195 & -78 & -405 & -162 & 195 & 78 \\
468 & 156 & 132 & 44 & -468 & 0 & -132 & 0 & -162 & 0 & 78 & 0 & 162 & 54 & -78 & -26 \\
-330 & -132 & -120 & -48 & 330 & 132 & 120 & 48 & 195 & 78 & -90 & -36 & -195 & -78 & 90 & 36 \\
132 & 44 & 48 & 16 & -132 & 0 & -48 & 0 & -78 & 0 & 36 & 0 & 78 & 26 & -36 & -12 \\
-405 & -162 & -195 & -78 & 405 & 162 & 195 & 78 & 1170 & 468 & -330 & -132 & -1170 & -468 & 330 & 132 \\
162 & 54 & 78 & 26 & -162 & 0 & -78 & 0 & -468 & 0 & 132 & 0 & 468 & 156 & -132 & -44 \\
195 & 78 & 90 & 36 & -195 & -78 & -90 & -36 & -330 & -132 & 120 & 48 & 330 & 132 & -120 & -48 \\
-78 & -26 & -36 & -12 & 78 & 0 & 36 & 0 & 132 & 0 & -48 & 0 & -132 & -44 & 48 & 16 \\
-405 & 162 & -195 & 78 & 405 & -162 & 195 & -78 & 1170 & -468 & -330 & 132 & -1170 & 468 & 330 & -132 \\
-162 & 0 & -78 & 0 & 162 & -54 & 78 & -26 & 468 & -156 & -132 & 44 & -468 & 0 & 132 & 0 \\
195 & -78 & 90 & -36 & -195 & 78 & -90 & 36 & -330 & 132 & 120 & -48 & 330 & -132 & -120 & 48 \\
78 & 0 & 36 & 0 & -78 & 26 & -36 & 12 & -132 & 44 & 48 & -16 & 132 & 0 & -48 & 0
\end{bmatrix}
$$

$$
\left[\bar{N}_e\right] = \frac{1}{25200}\begin{bmatrix}
-11232 & -20592 & -3168 & -5808 & 11232 & -1872 & 3168 & -528 & 3888 & -648 & -1872 & 312 & -3888 & -7128 & 1872 & 3432 \\
-1872 & -4992 & -528 & -1408 & 1872 & 1248 & 528 & 352 & 648 & 432 & -312 & -208 & -648 & -1728 & 312 & 832 \\
-2958 & -5598 & -1047 & -2007 & 2958 & -318 & 1047 & -87 & 1662 & -102 & -759 & 39 & -1662 & -3222 & 759 & 1479 \\
-318 & -1198 & -87 & -407 & 318 & 562 & 87 & 233 & 102 & 418 & -39 & -201 & -102 & -622 & 39 & 279 \\
11232 & 1872 & 3168 & 528 & -11232 & 20592 & -3168 & 5808 & -3888 & 7128 & 1872 & -3432 & 3888 & 648 & -1872 & -312 \\
-1872 & 1248 & -528 & 352 & 1872 & -4992 & 528 & -1408 & 648 & -1728 & -312 & 832 & -648 & 432 & 312 & -208 \\
2958 & 318 & 1047 & 87 & -2958 & 5598 & -1047 & 2007 & -1662 & 3222 & 759 & -1479 & 1662 & 102 & -759 & -39 \\
-318 & 562 & -87 & 233 & 318 & -1198 & 87 & -407 & 102 & -622 & -39 & 279 & -102 & 418 & 39 & -201 \\
3888 & 648 & 1872 & 312 & -3888 & 7128 & -1872 & 3432 & -11232 & 20592 & 3168 & -5808 & 11232 & 1872 & -3168 & -528 \\
-648 & 432 & -312 & 208 & 648 & -1728 & 312 & -832 & 1872 & -4992 & -528 & 1408 & -1872 & 1248 & 528 & -352 \\
-1662 & -102 & -759 & -39 & 1662 & -3222 & 759 & -1479 & 2958 & -5598 & -1047 & 2007 & -2958 & -318 & 1047 & 87 \\
102 & -418 & 39 & -201 & -102 & 622 & -39 & 279 & -318 & 1198 & 87 & -407 & 318 & -562 & -87 & 233 \\
-3888 & -7128 & -1872 & -3432 & 3888 & -648 & 1872 & 312 & 11232 & -1872 & -3168 & 528 & -11232 & -20592 & 3168 & 5808 \\
-648 & -1728 & -312 & -832 & 648 & 432 & 312 & 208 & 1872 & 1248 & -528 & -352 & -1872 & -4992 & 528 & 1408 \\
1662 & 3222 & 759 & 1479 & -1662 & 102 & -759 & 39 & -2958 & 318 & 1047 & -87 & 2958 & 5598 & 1047 & -2007 \\
102 & 622 & 39 & 279 & -102 & -418 & -39 & -201 & -318 & -562 & 87 & 233 & 318 & 1198 & -87 & -407
\end{bmatrix}
$$