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A GEOMETRICALLY NONLINEAR THEORY OF SHEAR DEFORMABLE LAMINATED COMPOSITE PLATES AND ITS USE IN THE POSTBUCKLING ANALYSIS

by

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Abstract

This paper is devoted to the formulation of a higher-order, geometrically nonlinear theory of anisotropic symmetrically-laminated composite plates and to the analysis, in this context, of their postbuckling behavior. Special attention is given to the postbuckling analysis of plates made of transversely isotropic layers for which case, the influence played by the degree of transversal-isotropy of the layers as well as by the geometrical parameters of the panel is investigated. Finally, the results obtained within the present higher-order theory are compared with their first order transverse shear deformation as well as with their classical (Kirchhoff) counterparts and a number of conclusions concerning their range of applicability and the influence of various parameters are presented.

1. Introduction

The increasing use of composite materials in the construction of aeronautical and aerospace structures has generated a special interest in the analysis of elastic stability of laminated composite structures subjected to compressive in-plane loadings. In this connection one may distinguish, on the one hand, the proccupation to examine their elastic stability in linear formulation which permits the finding of compressive buckling loads. On the other hand one may notice the interest to incorporate in the examination of this problem the geometrical (and/or physical) non-linearities. The non-linear approach of the elastic stability enables one to determine the behavior of the panel after buckling. In this sense it is well known [1] that the metallic panels are still capable to resist increased compressive loads well beyond the instant at which buckling occurs. This postbuckling strength experienced by the metallic panels played a great role in the design of aircraft structures in the sense that conventional aircraft structural elements are often designed to operate in the postbuckling range. However, the laminated composite plates, in contrast to their metallic counterparts exhibit a weak rigidity in transverse shear and in addition their material is characterized by high degrees of anisotropy. That is why, it is of a high practical importance to examine the postbuckling behavior of shear deformable composite laminated plates in order to assess the influence played on transverse shear deformation and anisotropy. This paper deals with an analytical investigation of the postbuckling behavior of shear deformable symmetrically-laminated composite plates. In this connection a simple geometrically nonlinear theory of symmetrically laminated flat panels is formulated. This theory developed in Lagrangian formulation retains the nonlinearities associated with the transversal displacement only. In addition, the theory incorporates transverse shear deformation, transverse normal stress as well as the higher order effects and fulfill the static conditions on the bounding planes of the panel. For more general geometrically nonlinear theories of shear deformable single-layered and composite laminated plates/shells the reader is referred to the monographs [2,3] and the articles [4-6].

It should also be mentioned that research on postbuckling behavior of shear deformable composite panels appears to be somewhat scarce, a fact which could clearly be inferred from the monograph [8] and the comprehensive survey-work [9].

2 Geometrically Non-Linear Theory of Shear Deformable Composite Plates

2a. General Considerations

Let us consider the case of a symmetrically laminated plate of uniform thickness h composed of 2m + 1 elastic orthotropic layers. It is assumed that the axes of orthotropy are parallel to the orthogonal in-plane geometrical axes x_ω to which the points of the undeformed mid-plane of the laminated are referred. The axis x_3 is perpendicular to the plane $x_3 = 0$. Throughout the paper the tensor quantities are referred to the orthogonal system of coordinates x_i associated with the undeformed body. In this system of coordinates there is no distinction between contravariant, covariant and mixed components of a tensor. We use the convention, that the Greek indices of a tensor range over the values, 1, 2 while the Latin ones range over the values 1, 2, 3.

2b. Displacement Representation and Strain Measures

In order to model the theory of laminated plates there are, roughly speaking, two main approaches: i) to start with some statical assumptions concerning e.g., the variation of transverse shear stresses across the laminate thickness or ii) to start with a certain representation of the displacement field through the entire laminate (or through each layer separately, by preserving however its continuity between the contiguous layers). Although the first approach was used in several papers (see [10,11]), the second one appears more promising. While the stress field exhibits jumps, the displacement field is to be assumed continuous through the

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laminate thickness. That is why, in the forthcoming developments, the latter option will be adopted. In this sense, the displacement components are represented as:

$$V_\alpha(x_\omega,x_3) = \overset{(0)}{V}_\alpha + x_3 \overset{(1)}{V}_\alpha + (x_3)^2 \overset{(2)}{V}_\alpha + (x_3)^3 \overset{(3)}{V}_\alpha . \tag{1}$$

$$V_3(x_\omega,x_3) = \overset{(0)}{V}_3$$

where

$$\overset{(m)}{V}_i \equiv \overset{(m)}{V}_i(x_\omega) .$$

The representation (1) has as a goal the exact fulfillment of tangential static coditions on the bounding planes $x_3 = \pm h/2$, expressed as:

$$[s_{3\alpha}]_{-h/2}^{h/2} = p_\alpha \to 0 \ , \ [s_{3\alpha}x_3]_{-h/2}^{h/2} = \tilde{p}_\alpha \to 0. \tag{2}$$

Here $s_{3\alpha}$ denote the transverse shearing components of the symmetric second Piola-Kirchhoff stress tensor s_{ij}; p_α and $\tilde{p}_\alpha$ denote the tangential loads and load couple components, respectively (measured per unit area of the reference surface). It is worth mentioning that henceforth the only nonlinearities which are retained there are the ones associated with the transverse displacement (and their gradients) only. This means that the product of any variable quantity with the <u>inplane</u> displacements (or their gradients) will <u>be neglected</u>.

Consistent with this assumption (which constitutes an extension of the one introduced by von Kármán in the classical plate theory), the Lagrangian strain tensor e_{ij} expressed in terms of the displacement components reads [3]:

$$2e_{ij} = V_{i,j} + V_{j,i} + V_{3,i}V_{3,j} \tag{3}$$

where $(\ )_i \equiv \partial(\ )/\partial x_i$. Herein the Einsteinian summation convention implied by the repetition of one index will be used. Employment of (1) and (3) into the constitutive equation

$$s_{\alpha 3} = 2E^{\omega 3}_{\alpha 3}\, e_{\omega 3} \tag{4}$$

considered in conjunction with (2) yields

$$\overset{(2)}{V}_\alpha = 0 \quad \text{and} \quad \overset{(3)}{V}_\alpha = -\frac{4}{3h^2}\left(\overset{(0)}{V}_{3,\alpha} + \overset{(1)}{V}_\alpha\right) \tag{5}$$

Here $s_{\alpha 3}$ and $e_{\omega 3}$ denote the transverse shear stress and transverse shear strain components, respectively, while $E^{\omega 3}_{\alpha 3}$ are the components of the tensor of elastic moduli E^{ij}_{mn} of an anisotropic body.

Although a cubic variation of the in-plane displacement components through the thickness was postulated, in light of (5), the displacement field contains the same number of dependent variables as within the first order transverse shear deformation theory, (FSDT), i.e. $\overset{(0)}{V}_3$, $\overset{(0)}{V}_\alpha$ and $\overset{(1)}{V}_\alpha$.

In terms of these basic unknowns the nontrivial strain tensor components write:

$$e_{\alpha\beta} = \overset{(0)}{e}_{\alpha\beta} + x_3 \overset{(1)}{e}_{\alpha\beta} + (x_3)^3 \overset{(3)}{e}_{\alpha\beta}$$
$$e_{\alpha 3} = \overset{(0)}{e}_{\alpha 3} + (x^3)^2 \overset{(2)}{e}_{\alpha 3} \tag{6}$$

where the <u>strain measures</u> $\overset{(i)}{e}_{\alpha\beta}$, $\overset{(i)}{e}_{\alpha 3}$ are

$$2\overset{(0)}{e}_{\alpha\beta} = \overset{(0)}{V}_{\alpha,\beta} + \overset{(0)}{V}_{\beta,\alpha} + \overset{(0)}{V}_{3,\alpha}\overset{(0)}{V}_{3,\beta}$$
$$2\overset{(1)}{e}_{\alpha\beta} = \overset{(1)}{V}_{\alpha,\beta} + \overset{(1)}{V}_{\beta,\alpha}$$
$$2\overset{(3)}{e}_{\alpha\beta} = -\frac{4}{3h^2}\left(2\overset{(0)}{V}_{3,\alpha\beta} + \overset{(1)}{V}_{\alpha,\beta} + \overset{(1)}{V}_{\beta,\alpha}\right) \tag{7}$$
$$2\overset{(0)}{e}_{\alpha 3} = \overset{(1)}{V}_\alpha + \overset{(0)}{V}_{3,\alpha}$$
$$2\overset{(2)}{e}_{\alpha 3} = -\frac{4}{h^2}\left(\overset{(1)}{V}_\alpha + \overset{(0)}{V}_{3,\alpha}\right)$$

Within FSDT, the non-zero components of the strain measures (being identical to their higher-order shear deformation theory (HSDT) counterparts), are $(7)_{1,2,4}$ while within the classical theory (based on Kirchhoff constraints) for which

$$\overset{(1)}{V}_\alpha = -\overset{(0)}{V}_{3,\alpha} \tag{8}$$

the only non-zero strain measures are

$$2\overset{(0)}{e}_{\alpha\beta} = \overset{(0)}{V}_{\alpha,\beta} + \overset{(0)}{V}_{\beta,\alpha} + \overset{(0)}{V}_{3,\alpha}\overset{(0)}{V}_{3,\beta}$$
$$2\overset{(1)}{e}_{\alpha\beta} = -2\overset{(0)}{V}_{3,\alpha\beta}. \tag{9}$$

2c. Determination of Transverse Normal Stress, and of Stress Resultants and Stress-Couples Governing Equations

From the third equation of equilibrium of the 3D elasticity theory

$$[s_{ir}(\delta_{jr} + V_{j,r})]_{,i} = 0 \tag{10}$$

i.e. from the equation corresponding in (10) to $j=3$, by using therein $s_{\alpha 3}$ expressed in terms of the basic unknowns, we may obtain through the integration over the segment $[0,x_3)$, the following expression of s_{33}

$$s_{33} = x_3 \overset{(1)}{s}_{33} + (x_3)^2 \overset{(2)}{s}_{33} + (x_3)^3 \overset{(3)}{s}_{33} \tag{11}$$

where

$$\overset{(1)}{s}_{33} = E^{\alpha 3}_{\lambda 3}(\overset{(1)}{V}_{\lambda,\alpha} + \overset{(0)}{V}_{3,\lambda\alpha})$$

$$\overset{(2)}{s}_{33} = \frac{4}{h^2} E^{\alpha 3}_{\lambda 3} \overset{(0)}{V}_{3,\alpha} \overset{(0)}{V}_{3,\lambda} \tag{12}$$

$$\overset{(3)}{s}_{33} = -\frac{4}{3h^2} E^{\alpha 3}_{\lambda 3} (\overset{(1)}{V}_{\lambda,\alpha} + \overset{(0)}{V}_{3,\lambda\alpha})$$

Now employment in conjunction with (7), (11) and (12) of constitutive equations

$$s_{\alpha\beta} = \tilde{E}^{\omega\rho}_{\alpha\beta} e_{\omega\rho} + \delta_A \frac{E^{33}_{\alpha\beta}}{E^{33}_{33}} s_{33}$$

$$s_{\alpha 3} = 2E^{\omega 3}_{\alpha 3} e_{\omega 3} \; ; \; (\tilde{E}^{\omega\rho}_{\alpha\beta} = E^{\omega\rho}_{\alpha\beta} - \frac{E^{\omega\rho}_{33} E^{33}_{\alpha\beta}}{E^{33}_{33}}) \tag{13}$$

followed by their integration throughout the laminate thickness yield the equations expressing the stress-resultants and stress-couples in terms of displacement quantities. These expressions will not be displayed here.

In order to obtain the governing equations expressed in terms of the displacement quantities we need the 2D equations of equilibrium. These are obtained by taking appropriately the moments of order zero and one of the equations of equilibrium of the 3D nonlinear elasticity theory.

Upon retaining the nonlinearities associated with the transverse deflection only, the pertinent 2D equations of equilibrium write as follows [3]:

$$\overset{(0)}{L}_{\alpha\beta,\alpha} = 0 \tag{14}$$

$$(\overset{(0)}{L}_{\alpha\beta} \overset{(0)}{V}_{3,\beta})_{,\alpha} + \overset{(0)}{L}_{\alpha 3,\alpha} + \overset{(0)}{p}_3 = 0 \tag{15}$$

$$\overset{(1)}{L}_{\alpha\beta,\beta} - \overset{(0)}{L}_{\alpha 3} = 0. \tag{16}$$

where $\overset{(0)}{p}_3$ denotes the transversal load. By virtue of (14), Eq. (15) becomes

$$\overset{(0)}{L}_{\alpha\beta} \overset{(0)}{V}_{3,\alpha\beta} + \overset{(0)}{L}_{\alpha 3,\alpha} + \overset{(0)}{p}_3 = 0. \tag{17}$$

As a result, the system of 2D equations of equilibrium is obtained by adjoining to Eq. (17) the equations (15) and (16). Substitution of macroscopic constitutive equations into Eqs. (14) and (15) yields a system of five equations in terms of the basic unknowns $\overset{(0)}{V}_\alpha$, $\overset{(1)}{V}_\alpha$ and $\overset{(0)}{V}_3$.

These equations which will be not displayed here, constitute the governing equations of shear-deformable cross-ply symmetrically laminated composite plate theory which incorporate the effect of large deflections.

3. Transversely-Isotropic Laminated Plates

3a. Basic Equations

Let us consider the case of a transversely-isotropic laminate. Let us assume that the plane of isotropy of each layer is parallel to the plate mid-plane and that the involved symmetry of the structure is both of a physical and geometrical nature. In this case, the tensors of elastic moduli are appropriately given by [12,3]

$$\hat{E}_{\alpha\beta\omega\rho} = \frac{E}{1+\mu} [\frac{1}{2} (\delta_{\alpha\omega}\delta_{\beta\rho} + \delta_{\alpha\beta}\delta_{\omega\rho}) + \frac{\mu}{1-\mu} \delta_{\omega\rho}\delta_{\alpha\beta}]$$

$$\frac{E^{33}_{\omega\rho}}{E^{33}_{33}} = \frac{\mu' E}{E'(1-\mu)} \delta_{\omega\rho}; \; E^{\omega 3}_{\alpha 3} = G'\delta^{\omega}_{\alpha} \tag{18}$$

where E, μ are the Young's modulus and Poisson's ratio associated with the isotropy plane; E', μ' and G' denote the Young's modulus, Poisson's ratio and transverse shear modulus in the planes normal to the isotropy plane while $\delta_{\alpha\beta}$ denotes the Kronecker's symbol.

Our first goal (whose sense will become evident later) is to express $\overset{(0)}{e}_{\alpha\beta}$ in terms of $\overset{(0)}{L}_{\alpha\beta}$ and $\overset{(0)}{V}_3$. Employment of $(7)_1$, (18) as well as the definition of $\overset{(0)}{L}_{\alpha\beta}$ allows one to obtain

$$\overset{(0)}{L}_{\alpha\beta} = \overset{(0)}{e}_{\alpha\beta} + \overset{(0)}{e}_{\omega\omega}\delta_{\alpha\beta} + \frac{4}{h^2} \delta_A \mathcal{T} \delta_{\alpha\beta} \overset{(0)}{V}_{3,\rho} \overset{(0)}{V}_{3,\rho} \tag{19}$$

where

$$\mathcal{A} = 2 [\frac{E_{<m+1>} h_{<m+1>}}{1 + \mu_{<m+1>}} + \sum_{r=1}^{m} \frac{E_{<r>}(h_{<r>} - h_{<r+1>})}{1 + \mu_{<r>}}]$$

$$\mathcal{B} = 2 [\frac{E_{<m+1>}\mu_{<m+1>}h_{<m+1>}}{1 - \mu^2_{<m+1>}} + \sum_{r=1}^{m} \frac{E_{<r>}\mu_{<r>}(h_{<r7} - h_{[r+1>})}{1 - \mu^2_{<r>}}] \tag{20}$$

$$\mathcal{T} = \frac{2}{3} [\frac{E_{<m+1>}\mu'_{<m+1>}G'_{<m+1>}}{E'_{<m+1>}(1 - \mu_{<m+1})} h^3_{<m+1>} + \sum_{r=1}^{m} \frac{E_{<r>}\mu'_{<r>}G'_{<r>}}{E'_{<r>}(1 - \mu_{<r>})} (h^3_{<r>} - h^3_{<r+1>})]$$

Here, a letter into the brackets "<>" affecting a quantity denotes the layer to which the respective quantity is associated. In order to invert (19), we postulate the following representation for $\overset{(0)}{e}_{\alpha\beta}$

$$\overset{(0)}{e}_{\alpha\beta} = \tilde{\mathcal{A}}\, \overset{(0)}{L}_{\alpha\beta} + \tilde{\mathcal{B}}\, \overset{(0)}{L}_{\omega\omega}\delta_{\alpha\beta} + \delta_A \tilde{\mathcal{T}}\delta_{\alpha\beta}\, \overset{(0)}{V}_{3,\rho}\, \overset{(0)}{V}_{3,\rho} \tag{21}$$

where $\hat{\mathcal{A}}$, $\hat{\mathcal{B}}$ and $\hat{\mathcal{T}}$ are undetermined coefficients.

Inserting (19) into (21) and identifying the coefficients of like terms we get

$$\hat{\mathcal{A}} = 1/\mathcal{A} \; ; \; \hat{\mathcal{B}} = -\frac{\mathcal{B}}{(\mathcal{A} + 2\mathcal{B})}$$
$$\hat{\mathcal{T}} = -\frac{4}{h^2}\frac{\mathcal{T}}{\mathcal{A} + 2\mathcal{B}} \, . \tag{22}$$

The equations (14) may be identically satisfied by expressing $\overset{(0)}{L}_{\alpha\beta}$ in the form

$$\overset{(0)}{L}_{\alpha\beta} = \epsilon_{\alpha\lambda}\epsilon_{\beta\mu}c_{,\lambda\mu} \tag{23}$$

where $c \equiv c(x_1, x_2)$ denotes a potential function while $\epsilon_{\alpha\lambda}$ stands for the permutation symbol ($\epsilon_{11} = \epsilon_{22} = 0$; $\epsilon_{12} = -\epsilon_{21} = 1$). In such a way, instead of Eq. (14) we are to use the compatibility equation (obtained by eliminating $\overset{(0)}{V}_\alpha$ from Eqs. $(7)_1$). This equation given by

$$\epsilon_{\alpha\pi}\epsilon_{\beta\lambda}(\overset{(0)}{e}_{\alpha\beta,\pi\lambda} + \frac{1}{2}\overset{(0)}{V}_{3,\alpha\beta}\overset{(0)}{V}_{3,\lambda\pi}) = 0 \tag{24}$$

is to be adjoined to Eqs. $(16)_3$ and (17).

3b. Governing Equations

Replacement of Eq. (23) and of the transversely-isotropic counterpart of constitutive equations for $\overset{(0)}{L}_{\alpha\beta}$ into Eq. (17) yields one of the governing equations

$$\epsilon_{\alpha\lambda}\epsilon_{\beta\mu}c_{,\lambda\mu}\overset{(0)}{V}_{3,\alpha\beta} + \mathcal{S}(\overset{(1)}{V}_{\omega,\omega} + \overset{(0)}{V}_{3,\omega\omega}) + \overset{(0)}{p}_3 = 0 \tag{25}$$

where

$$\mathcal{S} \equiv \mathcal{D} = 2\{G'_{<m+1>}h_{<m+1>} + \sum_{r=1}^{m} G'_{<r>}(h_{<r>} - h_{<r+1>}) - \frac{4}{3h^2}[G'_{<m+1>}h^3_{<m+1>} + \sum_{r=1}^{m} G'_{<r>}(h^3_{<r>} - h^3_{[r+1>})]\} \, . \tag{26}$$

The second governing equation may be obtained by replacing in Eq. (24) the expression of $\overset{(0)}{e}_{\alpha\beta}$ given by (21), considered in conjunction with Eq. (23). This equation reads

$$(\hat{\mathcal{A}} + \hat{\mathcal{B}})\, c_{,\lambda\lambda\rho\rho} + \frac{1}{2}(\overset{(0)}{V}_{3,\rho\rho}\overset{(0)}{V}_{3,\lambda\lambda} - \overset{(0)}{V}_{3,\lambda\rho}\overset{(0)}{V}_{3,\lambda\rho}) + 2\delta_A \tilde{\mathcal{T}}(\overset{(0)}{V}_{3,\rho\lambda\lambda}\overset{(0)}{V}_{3,\rho} + \overset{(0)}{V}_{3,\rho\lambda}\overset{(0)}{V}_{3,\rho\lambda}) = 0 \, . \tag{27}$$

Substitution of constitutive equations pertinent to the case of a transversely-isotropic body into (16) yields the two last governing equations

$$-A\overset{(0)}{V}_{3,\alpha\beta\beta} + B\overset{(1)}{V}_{\omega,\omega\alpha} + C\overset{(1)}{V}_{\alpha,\rho\rho} - D(\overset{(1)}{V}_\alpha + \overset{(0)}{V}_{3,\alpha}) - \delta_A M(\overset{(0)}{V}_{3,\pi\pi\alpha} + \overset{(1)}{V}_{\pi,\pi\alpha}) = 0 \tag{28}$$

The expressions of rigidity quantities A, ... M are not displayed here. By paralleling the procedure developed in [3] the equations (25) and (28) may be recast exactly into two uncoupled equations expressed in terms of $\overset{(0)}{V}_3$ and a potential function $\phi \equiv \phi(x_\omega)$. Towards this end $\overset{(1)}{V}_\alpha$ is expressed under the following form:

$$\overset{(1)}{V}_\alpha = -\frac{\mathcal{D}}{D}\Delta\overset{(0)}{V}_{3,\alpha} - (\frac{B+C}{D^2} - \delta_A\frac{M}{D^2})(\overset{(0)}{p}_{3,\alpha} + \overset{(0)}{L}_{\omega\beta}\overset{(0)}{V}_{3,\beta\omega\alpha}) - \overset{(0)}{V}_{3,\alpha} - \frac{1}{D}\epsilon_{\omega\alpha}\phi_{,\omega} \tag{29}$$

where Δ denotes the 2D Laplace operator. By virtue of (29), Eqs. (25) and (28) reduce to

$$(A+B+C)\overset{(0)}{V}_{3,\alpha\alpha\beta\beta} - \overset{(0)}{L}_{\omega\beta}[\overset{(0)}{V}_3 - (\frac{B+C}{D} - \delta_A\frac{M}{D})\overset{(0)}{V}_{3,\alpha\alpha}]_{,\omega\beta} - [\overset{(0)}{p}_3 - (\frac{B+C}{D} - \delta_A\frac{M}{D})\overset{(0)}{p}_{3,\alpha\alpha}] = 0 \tag{30}$$

and

$$\phi - \frac{C}{D}\phi_{,\lambda\lambda} = 0$$

In such a way the governing equations reduce to the equations (27) and (30). It should be remarked that Eqs. (27) and (30) define the interior solution while Eq. $(30)_2$ defines the boundary layer effect (its solution decaying rapidly when proceeding from the edges towards the interior of the plate). In connection with the boundary layer effect we are to underline the fact that its influence is a local one and may be neglected when dealing with global plate characteristics (as e.g., buckling, eigenfrequencies, etc.). In addition, as it will be shown later for simply-supported edge conditions the boundary-layer solution may exactly be eliminated.

In scalar form the governing equations (27) and (30) write as:

$$\begin{aligned} &\mathcal{c}_{,1111} + \mathcal{c}_{,2222} + 2\mathcal{c}_{,1212} \\ &+ \frac{\mathcal{A}(\mathcal{A} + 2\mathcal{B})}{2(\mathcal{A} + \mathcal{B})}(\overset{(0)}{V}_{3,11}\overset{(0)}{V}_{3,22} - (\overset{(0)}{V}_{3,12})^2) \\ &- \delta_A \frac{8}{h^2}\frac{\mathcal{A}\mathcal{T}}{\mathcal{A}+\mathcal{B}}(\overset{(0)}{V}_{3,111}\overset{(0)}{V}_{3,1} \\ &+ \overset{(0)}{V}_{3,221}\overset{(0)}{V}_{3,1} + \overset{(0)}{V}_{3,112}\overset{(0)}{V}_{3,2} \\ &+ \overset{(0)}{V}_{3,222}\overset{(0)}{V}_{3,2} + \overset{(0)}{V}_{3,11}\overset{(0)}{V}_{3,11} \\ &+ \overset{(0)}{V}_{3,22}\overset{(0)}{V}_{3,22} + 2\overset{(0)}{V}_{3,12}\overset{(0)}{V}_{3,12}) = 0 \end{aligned} \quad (31)$$

$$\begin{aligned} &\overset{(0)}{V}_{3,1111} + \overset{(0)}{V}_{3,2222} + 2\overset{(0)}{V}_{3,1212} - \frac{1}{\mathcal{D}}\{\mathcal{c}_{,22}\overset{(0)}{V}_{3,11} \\ &+ \mathcal{c}_{,11}\overset{(0)}{C}_{3,22} - 2\mathcal{c}_{,12}\overset{(0)}{V}_{3,12} - (\frac{B+C}{D} \\ &- \delta_A\frac{M}{D})(\mathcal{c}_{,11}\overset{(0)}{V}_{3,2222} + \mathcal{c}_{,11}\overset{(0)}{V}_{3,1212} \\ &+ \mathcal{c}_{,22}\overset{(0)}{V}_{3,1212} + \mathcal{c}_{,22}\overset{(0)}{V}_{3,1111} - 2\mathcal{c}_{,12}\overset{(0)}{V}_{3,1112} \\ &- 2\mathcal{c}_{,12}\overset{(0)}{V}_{3,2221}) - [\, p_3 - (\frac{B+C}{D} \\ &- \delta_A\frac{M}{C})(\overset{(0)}{p}_{3,11} + \overset{(0)}{p}_{3,22})]\} = 0 \end{aligned} \quad (32)$$

$$\phi - \frac{C}{D}(\phi_{,11} + \phi_{,22}) = 0. \quad (33)$$

Needless to say that for the actual problem the load terms connected with $\overset{(0)}{p}_3$ are to be dropped.

4. Postbuckling Analysis of Compressed Panels

4a. Several Considerations Related With the Boundary Conditions.

In the forthcoming development the static postbuckling behavior of simply-supported (SS) composite rectangular ($\ell_1 \times \ell_2$) panels will be investigated. The panel is assumed to be composed symmetrically of transversely-isotropic layers and subject to a system of uniform biaxial compressive edge loads $\overset{(0)}{\mathcal{L}}_{11}$ and $\overset{(0)}{\mathcal{L}}_{22}$. As a result of the finite-deflection model an inherent coupling between bending and extension appears requiring the fulfillment of both transverse and in-plane boundary conditions. Two different cases labelled as A and B will be considered next.

A) Biaxial compressive loads are acted on the plate whose edges are freely movable. As a result the following edge conditions are to be prescribed:

$$\underline{\overset{(0)}{V}_3} = \overset{(0)}{L}_{12} = \underline{\overset{(1)}{V}_2} = \underline{\overset{(1)}{L}_{11}} = 0$$

and

$$\overset{(0)}{L}_{11} = -\overset{(0)}{\mathcal{L}}_{11} \text{ at } x_1 = 0_1, \ell_1.$$

$$\underline{\overset{(0)}{V}_3} = \overset{(0)}{L}_{21} = \underline{\overset{(1)}{V}_1} = \underline{\overset{(1)}{L}_{22}} = 0 \quad (34)$$

and

$$\overset{(0)}{L}_{22} = -\overset{(0)}{\mathcal{L}}_{22} \text{ at } x_2 = 0, \ell_2.$$

B) Uniaxial compressive loads are acting on the free immovable edges $x_1 = 0, \ell_1$, the remaining ones being unloaded and immovable. In this case the edge conditions write:

$$\underline{\overset{(0)}{V}_3} = \overset{(0)}{L}_{12} = \underline{\overset{(1)}{V}_2} = \underline{\overset{(1)}{L}_{11}} = 0$$

and

$$\overset{(0)}{L}_{11} = -\overset{(0)}{\mathcal{L}}_{11} \text{ at } x_1 = 0, \ell_1 \quad (35)$$

$$\underline{\overset{(0)}{V}_3} = \overset{(0)}{V}_2 = \underline{\overset{(1)}{V}_1} = \underline{\overset{(1)}{L}_{22}} = \overset{(0)}{L}_{22} = 0 \text{ at } x_2 = 0, \ell_2$$

In Eqs. (34) and (35) the underlined terms are associated with the bending state of stress. It should be remarked that although the system of equations (30) is decoupled, the boundary conditions associated with the bending state of stress appear coupled (in the sense that by virtue of Eq. (29), $\overset{(1)}{V}_\alpha$ and $\overset{(1)}{L}_{\alpha\beta}$ are expressed in terms of both $\overset{(0)}{V}_3$ and ϕ). However, as it may readily be shown they may be reduced to the following decoupled form:

$$\overset{(0)}{V}_3 = 0;\quad D\overset{(0)}{V}_{3,11} + \left(\frac{B+C}{D} - \delta_A \frac{M}{D}\right)\left(\overset{(0)}{p}_3 + \overset{(0)}{L}_{11}\overset{(0)}{V}_{3,11}\right) = 0;\quad \phi_{,1} = 0$$

at $x_1 = 0, \ell_1$ and (36)

$$\overset{(0)}{V}_3 = 0;\quad D\overset{(0)}{V}_{3,22} + \left(\frac{B+C}{D} - \delta_A \frac{M}{D}\right)\left(\overset{(0)}{p}_3 + \overset{(0)}{L}_{22}\overset{(0)}{V}_{3,22}\right) = 0;\quad \phi_{,2} = 0$$

at $x_2 = 0, \ell_2$.

As it may be inferred, the governing equation (33) considered in conjunction with the B.C.$(36)_3$ and $(36)_6$ admits the trivial solution $\phi(\equiv\phi(x_1,x_2)) = 0$. This shows that for the case of SS edge conditions the discard of the boundary-layer solution does not constitute an approximation but an exact result which nevertheless yields a simplification of the problem, entailing the reduction of the order of the governing equations system from ten to eight.

The boundary conditions relative to the potential function are determined on an average. Towards this end, having in view that the representation

$$\overset{(0)}{V}_3 = \sum_{m,n} f_{mn}\sin\lambda_m x_1 \sin\mu_n x_2 \tag{37}$$

$$(\lambda_m = m\pi/\ell_1,\ \mu_n = n\pi/\ell_2)$$

fulfills exactly the B.C. associated with the bending state of stress, we may represent under the form

$$c(x_1,x_2) = c_1(x_1,x_2) - \frac{1}{2}\left(\overset{(0)}{L}_{11}(x_2)^2 + \overset{(0)}{L}_{22}(x_1)^2\right) \tag{38}$$

Here $c_1(x_1,x_2)$ is a particular solution of Eq. (31), (which is to be determined considered in conjunction with Eq. (37)) while $\overset{(0)}{L}_{11}$ and $\overset{(0)}{L}_{22}$ denote the in-plane normal edge loads (considered positive in compression). Upon imposing the conditions which concern the function

$$\begin{aligned}
&\int_0^{\ell_2} c_{1,22}\Big|_{x_1=0} dx_2 = \int_0^{\ell_2} c_{1,22}\Big|_{x_1=\ell_1} dx_2 = 0\\
&\int_0^{\ell_2} c_{1,11}\Big|_{x_2=0} dx_1 = \int_0^{\ell_2} c_{1,11}\Big|_{x_2=\ell_2} dx_1 = 0\\
&\int_0^{\ell_2} c_{1,12}\Big|_{x_1=0} dx_2 = \int_0^{\ell_2} c_{1,12}\Big|_{x_1=\ell_1} dx_2 = 0\\
&\int_0^{\ell_1} c_{1,12}\Big|_{x_2=0} dx_1 = \int_0^{\ell_1} c_{1,12}\Big|_{x_2=\ell_2} dx_1 = 0,
\end{aligned} \tag{39}$$

the parameters $\overset{(0)}{L}_{11}$ and $\overset{(0)}{L}_{22}$ acquire the meaning of average in-plane normal edge loads

$$\begin{aligned}
-\overset{(0)}{L}_{11} &= \frac{1}{\ell_2}\int_0^{\ell_2} c_{,22}\Big|_{x_1=0,\ell_1} dx_2\\
-\overset{(0)}{L}_{22} &= \frac{1}{\ell_1}\int_0^{\ell_1} c_{,11}\Big|_{x_2=0,\ell_2} dx_1
\end{aligned} \tag{40}$$

In the case of the plate loaded in the x_1-direction only, the condition of inmovability of the edges $x_2 = 0, \ell_2$ expressed in an average sense, involves the replacement of the conditions $(39)_{7,8}$ by the following one

$$\int_0^{\ell_1}\int_0^{\ell_2} \overset{(0)}{V}_{2,2}\, dx_1 dx_2 = 0 \tag{41}$$

By virtue of Eqs. $(7)_1$, (21) and (23) the condition (41) becomes:

$$\int_0^{\ell_1}\int_0^{\ell_2} \Big[(\tilde{A} + \tilde{B})\, c_{,11} + \tilde{B} c_{,22} + (\delta_A \tilde{T} - \tfrac{1}{2})\overset{(0)}{V}_{3,2}\overset{(0)}{V}_{3,2} + \delta_A \tilde{T}\, \overset{(0)}{V}_{3,1}\overset{(0)}{V}_{3,1}\Big] dx_1 dx_2 = 0. \tag{42}$$

4b <u>Postbuckling Behavior.</u>

For the sake of simplification we will restrain ourselves to the case of a square plate. Having in view [10] that in this case the minimum buckling load is obtained for $m = n = 1$ in which case

$$\overset{(0)}{V}_3 = f\sin\lambda_1 x_1 \sin\mu_1 x_2 \tag{43}$$

where $f_{11} \equiv f$, $\lambda_1 = \mu_1 = \pi/\ell$, and postulating, as usual, that the buckled form of the laminated plate remains unchanged in the postbuckling range, we obtain by following the procedured outlined before:

$$c(x_1,x_2) = f^2\Big[\frac{A(A + 2B)}{64(A + B)}\cos 2\lambda_1 x_1 + \frac{A(A + 2B)}{64(A + B)}\cos 2\lambda_1 x_2 + \delta_A \frac{1}{4h^2}\frac{AT}{A + B}\cos 2\lambda_1 x_1 \cos 2\lambda_1 x_2\Big] \tag{44}$$

$$-\frac{1}{2}\left(\overset{(0)}{L}_{11}(x_2)^2+\overset{(0)}{L}_{22}(x_1)^2\right).$$

The next and final step in the postbuckling analysis is the evaluation of the coefficient f in conjunction with the Eq. (32). This will be done by applying Galerkin's technique to the Eq. (32). To this end, substitution in Eq. (32) of $\overset{(0)}{V}_3$ and c given respectively by (43) and (44) followed by its successive multiplication by $\sin\lambda_1 x_1 \sin\lambda_1 x_2$ and integration of the obtained equation over the panel area, yields:

$$\overset{(0)}{L}_{11}+\overset{(0)}{L}_{22}=\frac{\pi^4 D}{\ell^2\Omega^2}+f^2\frac{\pi^4 A(A+2B)}{64\ell^2(A+B)\Omega^2}+f^2\frac{\pi^6 A}{32\ell^4}\frac{(A+2B)}{(A+B)\Omega^2}\left(\underline{\frac{B+C}{D}}-\delta_A\underline{\underline{\frac{M}{D}}}\right) \quad (45)$$

Equation (45) obtained for the case of a free movable contour gives the relationship between the average applied compressive loads and the maximum lateral deflection subsequent to the onset of buckling. However, in the case of innovable edges $x_2 = 0, \ell_2$, employment of Eqs.(43), (44) in Eq. (42) results in

$$\overset{(0)}{L}_{22}=-\frac{1}{\tilde{A}+\tilde{B}}\left(\frac{1}{8}\lambda_1^2 f^2-\frac{1}{2}\delta_A\hat{T}\lambda_1^2 f^2+\tilde{B}\overset{\sim(0)}{L}_{11}\right) \quad (46)$$

Insertion of (46) into (45) yields its counterpart for this case, i.e.

$$\overset{(0)}{L}_{11}=\frac{\hat{A}+\hat{B}}{\hat{A}}\left[\frac{\pi^4 D}{\ell^2\Omega^2}+f^2\frac{\pi^4 A(A+2B)}{64\ell^2(A+B)\Omega^2}+f^2\frac{\pi^6 A}{32\pi^4}\frac{(A+2B)}{(A+B)\Omega^2}\left(\underline{\frac{B+C}{D}}-\delta_A\underline{\underline{\frac{M}{D}}}\right)+\underline{f^2\frac{\pi^2}{8\ell^2}\frac{1}{\tilde{A}+\tilde{B}}-\delta_A f^2\frac{\pi^2}{2\ell^2}\frac{\tilde{T}}{\tilde{A}+\tilde{B}}}\right] \quad (47)$$

In Eqs. (45) and (47) we have denoted Ω^2 as

$$\Omega^2=\frac{\pi^2}{4}\left(1+\frac{2\pi^4}{\ell^2}\left(\underline{\frac{B+C}{D}}-\delta_A\underline{\underline{\frac{M}{D}}}\right)\right) \quad (48)$$

while, one and two solid lines identify the terms associated with transverse shearing effects and the ones arising from s_{33} effect, respectively.

The term $\frac{\pi^4 D}{\ell^2\Omega^2}$ identifies the Eulerian buckling load in uniaxial compression denoted henceforth as

$$(\overset{0}{L}_{11})_{cr}=\frac{\pi^4 D}{\ell^2\Omega^2} \quad (49)$$

The equations (45) and (47) determine in closed form the behavior of composite laminated plates subsequent to the onset of buckling. Needless to say that the panel begins to bend at the critical load predicted by the linear theory.

For the case of the first order transverse shear deformation theory (FSDT) the counterpart of Eqs. (45) and (47) may be obtained formally by considering therein $\delta_A = 0$; by suppressing in the stiffness quantities B, C, D and M the underlined terms and by replacing in the rigidity quantity D (simplified as mentioned before) G' by K^2G' where K^2 denotes a transverse shear correcton factor. The Kirchhoff (CL) counterpart of Eqs. (45) and (47) may be obtained formally by considering their FSDT variant and by specializing therein the rigidity D for $G' \to \infty$.

It should also be reminded that for the case of a single layered plate the previous equations remain unchanged. However the associated stiffness quantities are to be specialized by

considering therein

$h_{<m+1>} \to h/2$ and $\sum_{r=1}^{m}(\ldots) \to 0$.

5. Numerical Results

Numerical results for single-layered and three-layered square plates are presented. They allow one to infer about the influence played by transverse shear deformation, thickness ratio, heterogeneity, character of the in-plane boundary conditions, etc. Figures 2-5 display, (in terms of the dependence of the normalized central deflection vs. the nondimensional compressive load $\overset{(0)}{\underset{\sim}{L}}_{11}$, defined as $\overset{(0)}{\underset{\sim}{L}}_{11} \equiv \overset{(0)}{L}_{11}\ell^2/\pi^2\underset{\sim}{D}$ where $\underset{\sim}{D}$ denotes the bending rigidity of the plate and in the case of the composite, the bending rigidity of the mid-layer) the postbuckling behavior of single-layered transversely-isotropic square plates.

In Figs. 2 and 3 the case of biaxial compression with $\overset{(0)}{L}_{11} = \overset{(0)}{L}_{22}$) is considered all the edges being assumed freely movable while in Figs. 4 and 5 the case of the uniaxial ($\overset{(0)}{L}_{11} \neq 0$; $\overset{(0)}{L}_{22} = 0$) compression is considered, the edges parallel to the direction of applied load being considered both immovable and freely movable. Within these graphs, whenever the comparison with FSDT was made, a shear correction factor $K^2 = 5/6$ was taken.

Figures 6-8 display the postbuckling behavior for a three-layered plate whose mid-layer is two times thicker than the external ones (i.e. that $h_{<1>}/h$ ($\equiv h_{<3>}/h$) = 0.5 and $h_{<2>}/h$ = 0.25 - see Fig. 1). Three instances labelled as Case 1-Case 3 are considered:

Case 1

Within this case:

$$\frac{E_{\langle 2\rangle}}{G'_{\langle 2\rangle}} = 10;\ \frac{E_{\langle 1\rangle}}{G'_{\langle 1\rangle}} \left(\equiv \frac{E_{\langle 3\rangle}}{G'_{\langle 3\rangle}}\right) = 30;\ \frac{E_{\langle 1\rangle}}{E'_{\langle 1\rangle}} \left(\equiv \frac{E_{\langle 3\rangle}}{E'_{\langle 3\rangle}}\right) = 5$$

$$\frac{E_{\langle 2\rangle}}{E'_{\langle 2\rangle}} = 2;\ \frac{E_{\langle 1\rangle}}{E_{\langle 2\rangle}} = \frac{E_{\langle 3\rangle}}{E_{\langle 2\rangle}} = 10.$$

Case 2

Within this case:

$$\frac{E_{\langle 2\rangle}}{G'_{\langle 2\rangle}} = 30;\ \frac{E_{\langle 1\rangle}}{G'_{\langle 1\rangle}} \left(\equiv \frac{E_{\langle 3\rangle}}{G'_{\langle 3\rangle}}\right) = 10;\ \frac{E_{\langle 1\rangle}}{E'_{\langle 1\rangle}} \left(\equiv \frac{E_{\langle 3\rangle}}{E'_{\langle 3\rangle}}\right) = 5$$

$$\frac{E_{\langle 2\rangle}}{E'_{\langle 2\rangle}} = 2;\ \frac{E_{\langle 1\rangle}}{E_{\langle 2\rangle}} \left(\equiv \frac{E_{\langle 3\rangle}}{E_{\langle 2\rangle}}\right) = 10.$$

Case 3

Within this case:

$$\frac{E_{\langle 2\rangle}}{G'_{\langle 2\rangle}} = \frac{E_{\langle 1\rangle}}{G'_{\langle 1\rangle}} = \frac{E_{\langle 3\rangle}}{G'_{\langle 3\rangle}} = 30;\ \frac{E_{\langle 1\rangle}}{E'_{\langle 1\rangle}} \equiv \frac{E_{\langle 2\rangle}}{E'_{\langle 2\rangle}} = \frac{E_{\langle 3\rangle}}{E'_{\langle 3\rangle}} = 5$$

$$\frac{E_{\langle 1\rangle}}{E_{\langle 2\rangle}} = \frac{E_{\langle 3\rangle}}{E_{\langle 2\rangle}} = \lambda.$$

The last data imply that in this instance the three-layered plate may be viewed as a single layered plate. For all the previous cases it was considered invariably

$$\mu'_{\langle r\rangle} = \mu_{\langle r\rangle} = \mu'_{\langle m+1\rangle} = \mu_{\langle m+1\rangle} = 0.25$$

Within the last three instances the uniaxial compression case ($\overset{(0)}{L}_{11} \neq 0$; $\overset{(0)}{L}_{22} = 0$) is considered. Figure 6 displayes for the Case 2 the postbuckling response for three cases of thickness ratio as well as for freely movable and immovable edges (parallel to the direction of the compressive load $\overset{(0)}{L}_{11}$). Figure 7 exhibits for the same Case 2, the postbuckling behavior obtained within three different theories (namely HOT, FSDT and CL). The results associated with FSDT were determined for two values of the shear correction factor, i.e. $K^2 = 5/6$ and $2/3$.

Figure 8 presents a comparison of the postbuckling response of the composite panel with $h/\ell = 0.1$ for the three cases labelled as Case 1-Case 3. The results were obtained within HOT and FSDT, (where the shear correction factor $K^2 = 5/6$). However in order to get a better picture of the sensitivity of the postbuckling response (including the bifurcation one) obtained within FSDT to the variation of the shear correction factor, the following results are displayed in Tables 1-3.

Table 1

Postbuckling Behavior Obtained Within FSDT and HOT for a Composite Plate of $h/\ell = 0.1$ for Case 1.

HOT

f/h	$\overset{(0)}{L}_{11} \times 10^{-2}$
0.00	0.2613
0.10	0.2619
0.20	0.2636
0.30	0.2664
0.40	0.2704
0.50	0.2755
0.60	0.2818
0.70	0.2892
0.80	0.2978
0.90	0.3074
1.00	0.3182

Table 1 Continued

FSDT (K^2 = 5/6		FSDT (K^2 = 2/3)	
f/h	$\overset{(0)}{L}_{11} \times 10^{-2}$	f/h	$\overset{(0)}{L}_{11} \times 10^{-2}$
0.00	0.2986	0.00	0.2440
0.10	0.2992	0.10	0.2445
0.20	0.3010	0.20	0.2462
0.30	0.3040	0.30	0.2490
0.40	0.3083	0.40	0.2528
0.50	0.3137	0.50	0.2578
0.60	0.3204	0.60	0.2639
0.70	0.3283	0.70	0.2710
0.80	0.3373	0.80	0.2793
0.90	0.3476	0.90	0.2887
1.00	0.3591	1.00	0.2992

Table 2

Postbuckling Behavior Obtained Within FSDT and HOT for a Composite Plate of $h/\ell = 0.1$ for Case 2.

HOT

f/h	$\overset{(0)}{L}_{11} \times 10^{-2}$
0.00	0.5125
0.10	0.5133
0.20	0.5158
0.30	0.5198
0.40	0.5255
0.50	0.5329
0.60	0.5418
0.70	0.5524
0.80	0.5647
0.90	0.5785
1.00	0.5940

Table 2 Continued

FSDT ($K^2 = 5/6$)		FSDT ($K^2 = 2/3$)	
f/h	$\underset{\sim}{L}^{(0)}_{11} \times 10^{-2}$	f/h	$\underset{\sim}{L}^{(0)}_{11} \times 10^{-2}$
0.00	0.6215	0.00	0.5200
0.10	0.6224	0.10	0.5208
0.20	0.6252	0.20	0.5232
0.30	0.6298	0.30	0.5274
0.40	0.6363	0.40	0.5331
0.50	0.6445	0.50	0.5405
0.60	0.6547	0.60	0.5496
0.70	0.6667	0.70	0.5603
0.80	0.6805	0.80	0.5726
0.90	0.6962	0.90	0.5866
1.00	0.7137	1.00	0.6022

Table 3

Postbuckling Behavior Obtained Within FSDT and HOT for a Composite Plate of $h/\ell = 0.1$ for Case 3.

HOT	
f/h	$\underset{\sim}{L}^{(0)}_{11} \times 10^{-1}$
0.00	0.4631
0.10	0.4644
0.20	0.4683
0.30	0.4748
0.40	0.4840
0.50	0.4957
0.60	0.5100
0.70	0.5269
0.80	0.5464
0.90	0.5686
1.00	0.5933

Table 3 Continued

FSDT ($K^2 = 5/6$)		FSDT ($K^2 = 2/3$)	
f/h	$\underset{\sim}{L}^{(0)}_{11} \times 10^{-1}$	f/h	$\underset{\sim}{L}^{(0)}_{11} \times 10^{-1}$
0.00	0.4423	0.00	0.3639
0.10	0.4436	0.10	0.3651
0.20	0.4474	0.20	0.3685
0.30	0.4538	0.30	0.3742
0.40	0.4626	0.40	0.3823
0.50	0.4741	0.50	0.3926
0.60	0.4880	0.60	0.4052
0.70	0.5045	0.70	0.4200
0.80	0.5235	0.80	0.4372
0.90	0.5451	0.90	0.4567
1.00	0.5692	1.00	0.4784

6. Conclusions

The obtained numerical results allow one to infer the followings: a) With the increase of E/G' (i.e. when the composite plate becomes weaker in transverse shear) and of the thickness ratio, the capacity of carrying still compressive loads beyond the bifurcation point (manifested by the plates rigid in transverse shear) tends to diminish drastically (see Figs. 2-8). This tendency is also strongly manifested in the case of geometrically thin plates exhibiting high E/G' ratios. b) The effect of in-plane boundary conditions is very significant both in the case of shear deformable and of transversely rigid plates. The constraint introduced by the immovable edge conditions has a beneficial influence on the postbuckling behavior (see Figs. 4-6), independently on E/G'. c) The first order transverse shear deformation theory (FSDT) gives in the case of a single layered plate more conservative results both as concerns the bifurcation and the postbuckling responses (see Fig. 3 and Table 3). In such a case, as is well known, $K^2 = 5/6$ is reliable shear correction factor. However, in the case of a composite laminate, a high sensitivity to the selection of the shear correction factor is experienced (see Figs. 7 and 8 as well as Tables 1-3). In this last case $K^2 = 2/3$ appears to be a more reliable shear correction factor than $K^2 = 5/6$. In this connection Tables 1-3 reveal once more the importance of approaching the postbuckling problem in the framework of a higher-order plate theory. d) Both the bifurcation response and the postbuckling behavior reveal a tremendous improvement when the constituent layers of the composite experience high ratios of in-plane Young's moduli. Comparison of Tables 1 and 2 with Table 3 is relevant in this sense. Some of these results presented herein agree with the ones obtained in [13].

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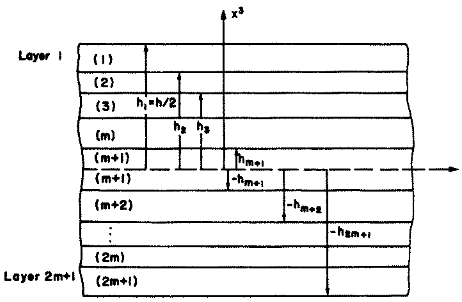

Figure 1. The geometry of the cross-section of the laminate.

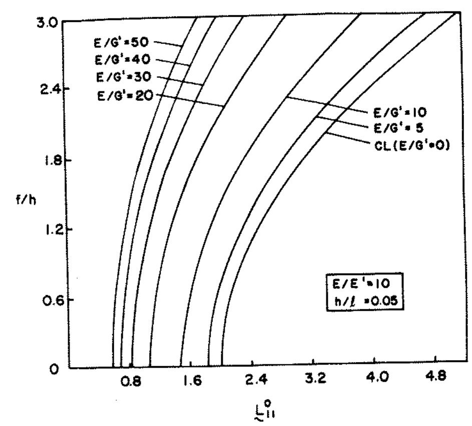

Figure 2. Postbuckling in biaxial compression of a transversely-isotropic square thin plate as predicted by HOT.

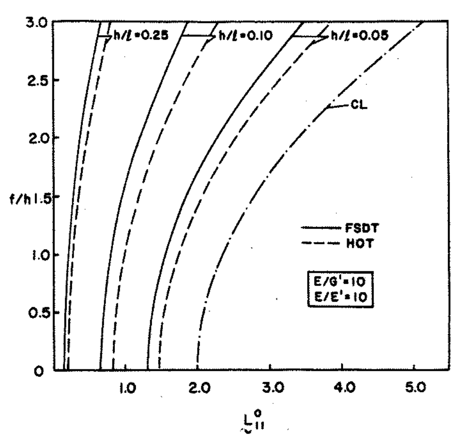

Figure 3. Postbuckling in biaxial compression of a transversely-isotropic square plate as predicted by FSDT, HOT and classical theories.

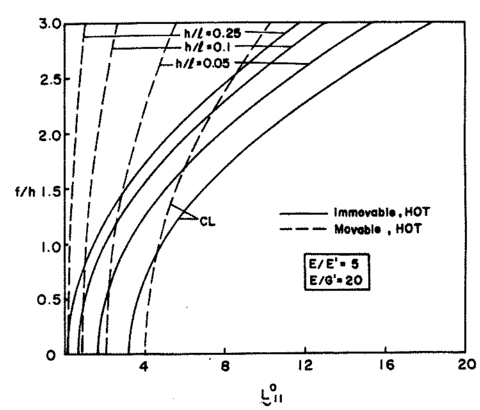

Figure 4. Postbuckling in uniaxial compression of a transversely-isotropic square plate as predicted by HOT and classical theories. The results concern both the case of movable and immovable edges $x_2 = 0, \ell$.

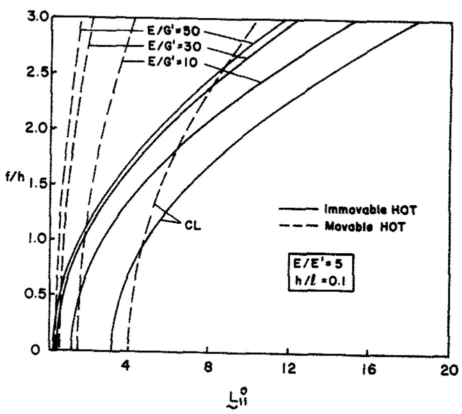

Figure 5. The influence of E/G' on the postbuckling behavior of a square plate whose edges $x_2 = 0,\ell$ are movable or immovable.

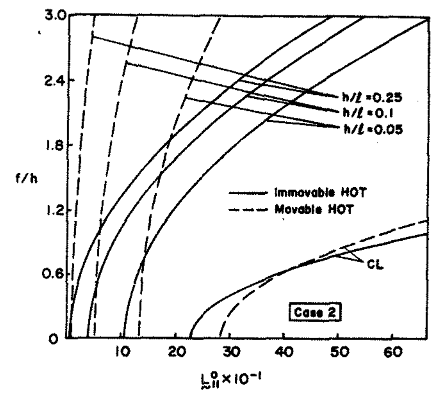

Figure 6. Postbuckling behavior of a three-layered composite square plate (Case 2) for three different values of the thickness ratio.

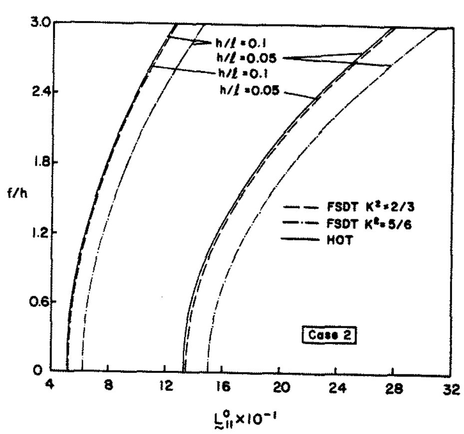

Figure 7. Postbuckling behavior of a three-layered composite square plate (Case 2). Comparison of the results obtained within HOT and FSDT.

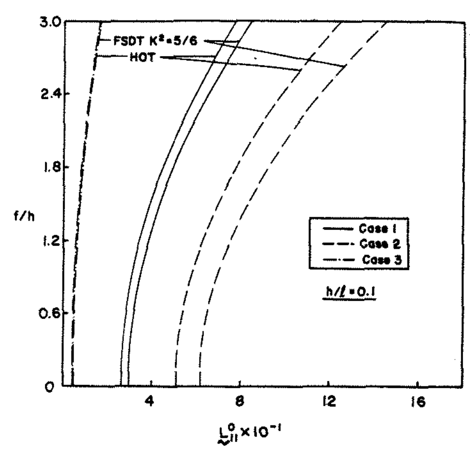

Figure 8. Postbuckling behavior of a moderate thick composite plate where HOT and FSDT are compared.