VARIOUS APPROACHES IN SOLVING STABILITY PROBLEMS FOR SYMMETRIC ANGLE-PLY LAMINATES UNDER COMBINED LOADING

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Abstract

In this paper many various elastic stability problems, for symmetric angle-ply laminated plates, are solved using the energy method and finite difference method. The energy method is applied in considering the buckling problem for rectangular plates that are simply supported along all edges and subjected to both uniform in-plane loads in the X- and Y-direction and constant shear load, simultaneously. This paper provides, also, the solution of buckling problem for simply supported plate under nonuniform compressive load in X-direction, by using the same method. The finite-difference method is applied in solving the buckling problem for rectangular plates under combined loading, with edges that are either simply supported or clamped. Many computer programs in PASCAL language have been developed, which enable calculation of the critical buckling loads for different symmetric angle-ply laminates and for different a/b ratios. At the end, the paper presents the comparison between results obtained by two methods for rectangular simply supported plates. The results are ingood agreement, but energy method provides values which converge a little more rapidly for the acceptable equivalent computing time.

1. Introduction

In this paper we consider stability problem of symmetrical laminated plates when the coupling terms are neglected. In particular the terms which couple twisting curvatures to normal moment resultants are included in the analyses. In this case, when the laminate possesses midplane symmetry, we get an important class of plates.

The stability problem of the symmetrical laminated plates has great interest in structural elements and very often is present in aeronautical structures where symmetrical plates have shown many advantages in comparison with unsymmetrical. The presentation of the used theory is based on the work of Jones(1), Ashton(2) and Whitney(2) and Agarwal(3). It is assumed that the general equations of the theory of laminated plates are known.

2. Stability problem of simply supported rectangular plate

For laminates such that the B_{ij} are all identically zero, for mid-plane symmetrical laminates, the expression for the strain energy can be written in the form

$$U = \frac{1}{2}\iint\Big\{ D_{11}\Big(\frac{\partial^2 w}{\partial x^2}\Big)^2 + 2D_{12}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + D_{22}\Big(\frac{\partial^2 w}{\partial y^2}\Big)^2 + 4D_{16}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial x\partial y} + 4D_{26}\frac{\partial^2 w}{\partial y^2}\frac{\partial^2 w}{\partial x\partial y} + 4D_{66}\Big(\frac{\partial^2 w}{\partial x\partial y}\Big)^2\Big\}\, dA + C, \quad (1)$$

where C represents constant strain energy due to displacement u^o and v^o.

From the theorem of stationary potential energy we hawe

$$\Pi = \iint_A \Big\{ D_{11}\Big(\frac{\partial^2 w}{\partial x^2}\Big)^2 + 2D_{12}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + D_{22}\Big(\frac{\partial^2 w}{\partial y^2}\Big)^2 + 4D_{16}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial x\partial y} + 4D_{26}\frac{\partial^2 w}{\partial y^2}\frac{\partial^2 w}{\partial x\partial y} + D_{66}\Big(\frac{\partial^2 w}{\partial x\partial y}\Big)^2\Big\}\, dA - \frac{1}{2}\iint\Big\{ N_x\Big(\frac{\partial w}{\partial x}\Big)^2 + N_y\Big(\frac{\partial w}{\partial y}\Big)^2 + N_{xy}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\Big\}\, dx\, dy + C. \quad (2)$$

The loads N_x, N_y and N_{xy}, in (2), can be arbitrary functions of coordinates along edges of the plate. In our case, Fig. 1, they are all uniform. Solving stability problem of such an anisotropic plate, under simultaneous action of three loads, it is of great importance to investigate the effects of D_{16} and D_{26} on the stability of the plate.

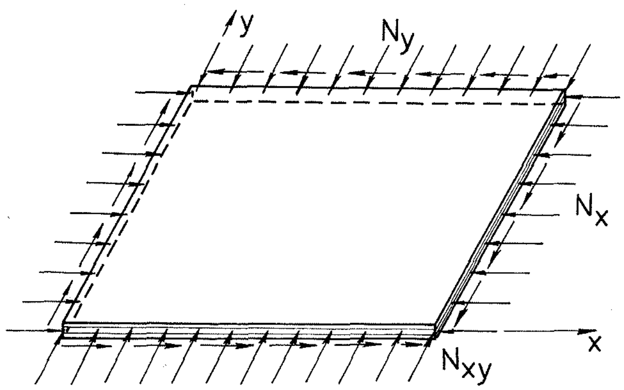


Figure 1. Combined Loading N_x, N_y, N_{xy}

The boundary conditions, for simply supported plate, are

for x=0 and x=a: $w = 0$, $M_x =$

$$= -(D_{11}\frac{\partial^2 w}{\partial x^2} + D_{12}\frac{\partial^2 w}{\partial y^2} + 2D_{16}\frac{\partial^2 w}{\partial x \partial y}) = 0$$

for y=0 and y=b: $w = 0$, $M_y =$

$$= -(D_{12}\frac{\partial^2 w}{\partial x^2} + D_{22}\frac{\partial^2 w}{\partial y^2} + 2D_{26}\frac{\partial^2 w}{\partial x \partial y}) = 0 \qquad (3)$$

When the deflection function is taken in the form of a double trigonometrical series

$$w = \sum_i \sum_j A_{ij} \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b} \qquad (4)$$

where A_{ij} are unknown coefficients, it is evident that (4) exactly satisfies the geometrical conditions $w = 0$ on the boundaries. To satisfy the boundary conditions of zero moments, one must add a term corresponding to the unbalanced edge moment. As it is $\partial^2 w/\partial x^2 = \partial^2 w/\partial y^2 = 0$, using the series (4), the additional term takes form

$$-\int_0^a 2D_{26}\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)_{y=b} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)_{y=o}\right]\frac{q\pi}{b}\sin\frac{p\pi x}{a}\,dx$$

$$-\int_0^b 2D_{16}\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)_{x=a} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)_{x=a}\right]\frac{p\pi}{a}\sin\frac{q\pi y}{b}\,dy. \qquad (5)$$

In this paper we have investigate the effect of the added term (5) in the case when the plate is submittedonly to the load N_x.

In our case, the potential energy is given in the form

$$\Pi = \frac{1}{2}(I_{11} + I_{12} + I_{12} + I_{22} + I_{16} + I_{26}) - \frac{1}{2}(I_x + I_y + I_{xy}) + C \qquad (6)$$

where, for instance, some of the integrals have values

$$I_{16} = -16D_{16}\frac{\pi^2}{a^2}\sum_i \sum_j \sum_p \sum_q A_{ij}A_{pq}\frac{i^3 j p q}{(i^2 - p^2)(j^2 - q^2)},$$

$$I_{26} = -16D_{26}\frac{\pi^2}{b^2}\sum_i \sum_j \sum_p \sum_q A_{ij}A_{pq}\frac{i j p q^3}{(i^2 - p^2)(j^2 - q^2)}, \qquad (7)$$

where $i \pm p$ and $j \pm q$ are odd numbers.

To find the solution of the problem it is necessary to form the expression $\partial\Pi/\partial A_{ij}$, which must be zero. General system of linear simultaneous equations in terms of A_{ij} is given by

$$\left\{\frac{\pi^2}{2}\left[D_{11}\frac{i^4}{c^3} + \frac{2(D_{12} + 2D_{66})\, i^2 j^2}{c} + D_{22}\, c j^4\right] - \frac{i^2}{2c}X - \frac{j^2 c}{2}Y\right\}A_{ij} - 16\sum_p \sum_q A_{pq}\frac{i j\, p q}{(i^2 - p^2)(j^2 - q^2)}\left[D_{16} \times \frac{i^2 + p^2}{c^2} + D_{26}(j^2 + q^2)\right] - \frac{N_{xy}}{\pi^2} b^2 \Big\} = 0 \qquad (8)$$

where $i = 1,2,3,\ldots m$ $\quad j = 1,2,3,\ldots n$.
$i \pm p$ and $j \pm q$ are all odd numbers at the same time.

The problem was how to solve the system (8) with three unknown loads N_x, N_y and N_{xy}, which act simultaneously. We applied the following process: two of three loads we expressed by means of the third

$$N_y = \alpha N_x \quad \text{and} \quad N_{xy} = \beta N_x \qquad (9)$$

The mathematical solution of the equations, represented by (8), gives mxn homogeneous simultaneous equations for unknown w_{ij}. In this work we have given the programme CXYT (Pascal) with four branches. The concept of the programme is that for $i = 1,2,\ldots 7$, i.e. 49 equations, we can obtain the smallest values for N_x, N_y or N_{xy}. The programme can easily be enlarged. First branch (X), corresponding to load N_x, the second (Y) to N_y, the third (XY) to N_{xy} and the fourth to the combined loads of N_x, N_y, and N_{xy} which act simultaneosly. First two branches calculate the critical values N_{xcr} or N_{ycr}, when two other loads are zero. The third branch gives two critical values $N_{xycr}(+)$ and $N_{xycr}(-)$, i.e. for different orientations of the shear load. In the fourth branch it is necessary to suppose the shear load and axial load in one direction, then as a solution we get another axial load. The obtained critical value with the two supposed gives the group of three critical values for the case of simultaneosly acting loads N_x, N_y and N_{xy}. Taking into consideration what is happening in the fourth branch, it can be seen that it is possible to work also with tension loads. For very high values of N_x and N_{xy} we can obtain small tension for N_y. Mathematically the results are correct, but it is the question how much the obtained results correspond physically. That is, the laminates of composite material show differencies in elastic modulus when occurs pressure or extension. This fact, of course, influences the variation in coefficients of laminates. Due to our experiance one should be very careful with the tension load, i.e. it is realistic to work with very low values of tension load.

In the programme CXYT the starting data were former defined D_{11}, D_{12}, D_{22}, D_{16}, D_{26}, D_{66}, dimensions of the plate and the thickness of lamenates.

The first case, in which we wanted to indicate the effect of the D_{16} and D_{26} terms on the buckling behaviour is the case of uniform axial compression N_x only. In this example we have considered simply supported laminated plate of 20 plies, boron-epoxy composite material, E_1=2,0684·10⁶bar, E_2/E_1 = 0,1, G_{12} = 0,03·E_1, ν_{12} = 0,3, t_1 = 0,01 cm, t = 0,2 cm, b = 10 cm, a = 11,3 cm, c = 1,13. The same example was given by Ashton and Whitney(2), who cosidered the problem using the term given by (5). In our work we have neglected the expresion (5) and the results are shown in the Fig. 2. Our results in the Fig. 2 are given with dotted line and also the numerical values in the TABLE 1. In this example two groups of plates are cosidered. The first case is when the principal material axes are oriented at (+ θ) to the plate edges. The second case is when orientation has alternating plies (+ θ) and (- θ), |10 plies have orientation (+ θ) and 10 (- θ)|. As it can be seen, the second group is not strictly symmetrical, in fact it is unsymmetrical, but the coefficients B_{ij} are so small that the case reduces to the special ortotropic plate.

The coefficient k, proporcional to the critical load is

$$k = \frac{N_x a^2}{E_1 \cdot t^3} \qquad (10)$$

for different θ, using the branch X of the programme CXYT, is given in the Table I.

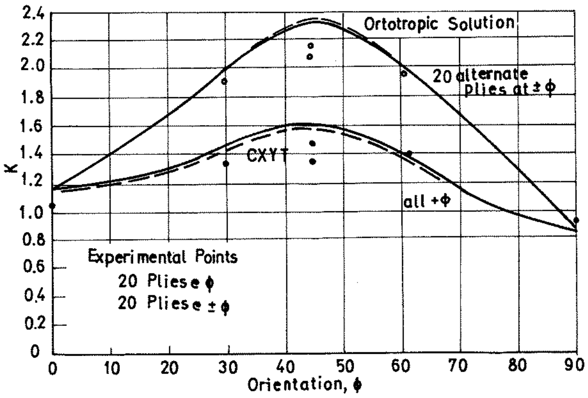


Figure 2. Compressive Buckling Coefficients, Simply-Supported Plates

Table I: Coefficients of flexural rigidities in |daNcm|, and the critical values N_{xcr}

θ	D_{11}	D_{22}	D_{12}	D_{66}	D_{16}	D_{26}	N_{xcr}	k
0	1391	139	42	41	0	0	150	1,15
15°	1227	143	122	121	295	17,8	159	1,23
30°	839	212	282	282	410	132	187	1,44
45°	445	445	362	362	313	313	203	1,57
60°	212	839	282	282	132	410	172	1,32
±15°	1227	143	122	121	-0,7629 x10⁻⁵	-0,7125 x10⁻⁶	202	1,56
±30°	839	212	282	282	-0,1315 x10⁻⁴	-0,6575 x10⁻⁵	258	1,99
±45°	445	445	362	362	-0,1643 x10⁻⁴	-0,1643 x10⁻⁴	305	2,35
±60°	212	838	282	282	-0,6575 x10⁻⁵	-0,263 x10⁻⁴	259	2,00

As it is possible to see from the Fig. 2 and the Table I, the results obtained using the programme CXYT, differ very little (2 %) from those obtained in (2). That leads to the conclusion that one does not produce essential mistake by neglecting the expression (5) in the case when the values for D_{16} and D_{26} are not small and when having for all plies the same orientation in θ. In the case of laminates with orientations +θ and -θ, when it is $D_{16} \simeq D_{26} \simeq 0$, the differences do not exist. The basic influence of the coefficients D_{16} and D_{26} is in decreasing the critical values of bucking loads, what of course must not be neglected. Points in the Fig. 2 show the experimental data for laminates (Mandell[15]), and they coincide very well with the theory.

Now we are going to present some application of our programme CXYT. We have considered the stability problem of simply supported anisotropic laminated rectangular plate under a) load N_x - the first branch, b) load N_{xy} - the third branch and c) the case of combined biaxial compression N_x and N_y and N_{xy} shear load - the fourth branch.

Properties of boron-epoxy composite material are:

$E_1 = 1{,}38 \cdot 10^6$ bar, $c = a/b = 2{,}0$
$E_2 = 0{,}145 \cdot 10^6$ bar, $t_{lam} = 0{,}0125$ cm
$G_{12} = 0{,}058 \cdot 10^6$ bar, $t = 0{,}3$ (total)
$\nu_{12} = 0{,}21$ 24 plies.
$b = 15$ cm
$a = 30$ cm.

The directions of the material θ, measured from x, are:
$/0^0_2, 30^0, 90^0, \pm 45^0, 30^0, 90^0, \pm 45^0, 0^0_2/s$,
where $\theta = 0^0_2$ means that the plies consist of two very thin elements, and index s means the symmetry of 12 plies. The plies are numerated from the external to the mid-plane. Due to given elastic and geometrical characteristics, for the coefficients of flexural rigidities of the plate we found

$$D = \begin{bmatrix} 1988{,}3 & 320{,}28 & 214{,}07 \\ 320{,}28 & 955{,}99 & 87{,}64 \\ 214{,}07 & 87{,}64 & 381{,}95 \end{bmatrix} \text{daN/cm}^2 \quad (10a)$$

The results for the cases a) and b) are given in Table 2.

Table 2: Values of the critical loads N_x or N_{xy}

c	N_x ($N_y=N_{xy}=0$) daN/cm	N_{xy} ($N_x=N_y=0$) daN/cm +	N_{xy} ($N_x=N_y=0$) daN/cm −
1,0	219	608	-433
1,1	213		
1,2	211	490	-352
1,3	213		
1,4	217	427	-311
1,5	224		
1,6	232	391	-289
1,7	240		
1,8	231	370	-278
1,9	224		
2,0	219	358	-270
2,1	216		
2,2	213	350	-268
2,3	212		
2,4	212	344	-257
2,5	212		
2,6	213	332	-249
2,7	215		
2,8	218	323	-244
2,9	221		
3,0	219	316	-240
3,1	217		
3,2	215	311	-238
3,3	213		
3,4	213	308	-237
3,5	212		
3,6	212	306	-236
3,7	212		
3,8	213	304	-234
3,9	214		
4,0	215	303	-231

At the end, in the Table 3 is given the case c), i.e. the critical values of N_x for supposed values for N_y and N_{xy}. Then, as we said before, three values represent the group of critical buckling values $(N_x, N_y, N_{xy})_{cr}$.

Table 3: Critical buckling values $(N_x, N_y, N_{xy})_{cr}$

$N_y = 0$ daN/cm		$N_y = 50$ daN/cm		$N_y = 25$ daN/cm		$N_y = 65$ daN/cm	
N_{xy} daN/cm	N_x daN/cm	N_{xy} daN/cm	N_x daN/cm	N_{xy} daN/cm	N_x daN/cm	N_{xy} daN/cm	N_x daN/cm
-285	-22	-150	- 8	-240	-32	-80	-10
-260	20	-130	11	-200	26	-60	4
-220	80	-110	28	-160	77	-40	10
-180	120	- 80	50	-120	118	-20	17
-140	154	- 50	66	- 80	149	0	21
-100	181	- 20	77	- 40	170	20	24
- 60	202	0	81	0	181	40	24
- 20	215	20	84	40	184	60	23
20	223	50	84	80	179	80	19
60	223	80	79	120	165	100	13
100	218	110	69	160	142	120	5
140	206	140	55	200	110	140	- 5
180	188	170	36	240	70		
220	165	200	12	280	23		
260	134	230	-16	320	-32		
300	86						
340	28						
380	-36						

The results from tha Table 3 are given as a diagramm, Fig. 3.

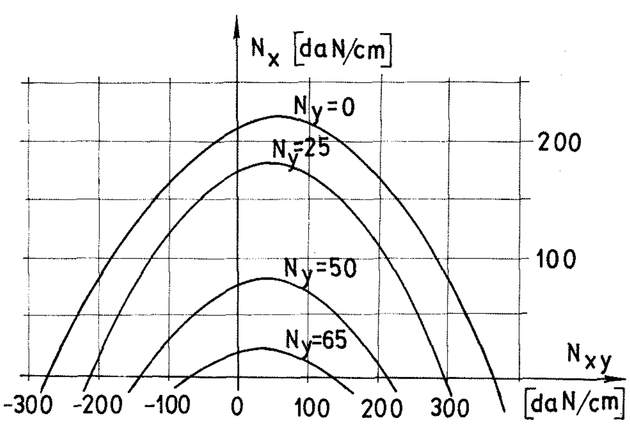


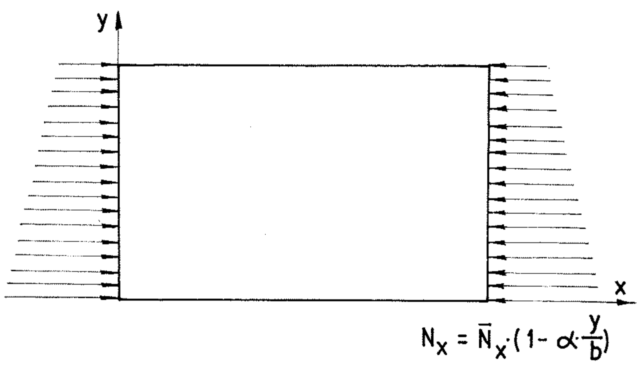
Figure 3. Critical buckling values $(N_x, N_y, N_{xy})_{cr}$

2.1. Buckling due to nonuniform compressive load

Let us now investigate the case when the plate is submitted to the forces their intensity being given by the equation

$$N_x = \overline{N}_x(1 - y/b) \tag{11}$$

The rectangular plate (Fig. 4) is also simply supported along all four sides. The load $\overline{N}_x$ is the intensity of compressive force at the edge y = 0. By changing α, various particular cases can be obtained. By taking $\alpha = 0$, we obtained the case when compressive load N_x is constant. As the plate is simply supported on all sides, the deflection function can be taken, as before, in the form (4).



Fugure 4. Buckling due to nonuniform compressive load

The work done by external forces, during buckling of the plate, is

$$U_e = -\frac{1}{2}\overline{N}_x \int_0^a \int_0^b (1 - \alpha\frac{y}{b})(\frac{\partial w}{\partial x})^2 \, dA \tag{12}$$

Now it is necessary to calculate the integral

$$I_\alpha = \int_0^a \int_0^b (1 - \alpha\frac{y}{b})(\frac{\partial w}{\partial x})^2 \, dA \tag{13}$$

i.e.

$$I_\alpha = \int_0^a \int_0^b (\sum_i \sum_j A_{ij} \frac{i\pi}{a} \cos\frac{i\pi x}{a} \sin\frac{j\pi y}{b})^2 \, dx\, dy -$$

$$- \frac{\alpha}{b} \int_0^a \int_0^b y\, (\sum_i \sum_j A_{ij} \frac{i\pi}{a} \cos\frac{i\pi x}{a} \sin\frac{j\pi y}{b})^2 \, dx\, dy =$$

$$= \frac{\pi^2}{a^2}\frac{ab}{4} \sum_i \sum_j A_{ij}^2\, i^2 - \frac{\alpha}{b} I' \,. \tag{14}$$

The value of the integral I' is

$$I' = I'' + I''' \tag{15}$$

where

$$I'' = \frac{\pi^2}{a^2}\frac{ab^2}{8} \sum_i \sum_j A_{ij}^2\, i^2 \,, \tag{16}$$

and

$$I''' = -\frac{2b^2}{a} \cdot \sum_i \sum_j \sum_q A_{ij} A_{iq} \frac{i^2 j q}{(j^2 - q^2)^2} \tag{17}$$

for $(j \pm q)$ odd number.

The final expression for I_α is

$$I_\alpha = \frac{\pi^2}{a^2}\frac{ab}{4}(1 - \frac{\alpha}{2}) \sum_i \sum_j A_{ij}^2\, i^2 +$$

$$+ \alpha \frac{2b}{a} \sum_i \sum_j \sum_q A_{ij} A_{iq} \frac{i^2 j q}{(j^2 - q^2)^2} \,, \tag{18}$$

for $(p \pm q)$ odd number.

The partial derivation of (18) is

$$\frac{\partial I_\alpha}{\partial A_{ij}} = \frac{\pi^2}{a^2}\frac{ab}{2}(1 - \frac{\alpha}{2})\, i^2 A_{ij} +$$

$$+ \frac{4\alpha b}{a} \sum_q A_{iq} \frac{i^2 j q}{(j^2 - q^2)^2} \,,$$

for $(j \pm q)$ odd number.

By taking derivatives of the expression (2) with respect to each coefficient A_{ij}, and equating these derivatives to zero, we finally obtain a system of linear equations in the following form:

$$\frac{\partial \Pi}{\partial A_{ij}} = D_{11}\frac{ab}{2}\frac{\pi^4}{a^4} i^4 A_{ij} + D_{22}\frac{ab}{2}\frac{\pi^4}{b^4} A_{ij}\, j^4 +$$

$$+ 2(D_{12} + 2D_{66})\frac{ab}{2}\frac{\pi^4}{a^2 b^2} i^2 j^2 -$$

$$- 16 D_{16}\frac{\pi^2}{a^2} \sum_p \sum_q A_{pq} \frac{(i^2 + j^2)\, i j p q}{(i^2 - p^2)(j^2 - q^2)} -$$

$$- 16 D_{26}\frac{\pi^2}{b^2} \sum_p \sum_q A_{pq} \frac{(j^2 + q^2)\, i j p q}{(i^2 - p^2)(j^2 - q^2)} -$$

$$- \overline{N}_x \Big\{ [\frac{\pi^2}{a^2}\frac{ab}{2}(1 - \frac{\alpha}{2})\, i^2 A_{ij}] +$$

$$+ 4\frac{\alpha b}{a} \sum_r A_{ir} \frac{i^2 j r}{(j^2 - r^2)^2} \Big\} \tag{19}$$

$i \pm p$, $j \pm q$, $j \pm r$ all odd numbers; $i = 1,2,\ldots m$;
$j = 1,2,\ldots n$.

or

$$\Big\{\frac{\pi^2}{2}\, [D_{11}\frac{i^4}{c^3} + \frac{2(D_{12} + 2D_{66})}{c} i^2 j^2 + D_{22}\, c\, j^4] -$$

$$- [\frac{i^2}{2c}(1 - \frac{\alpha}{2})\, X]\Big\} A_{ij} - \frac{4\alpha}{c\pi^2} X \sum_r A_{ir} \frac{i^2 j r}{(j^2 - r^2)} -$$

$$- 16 \sum_p \sum_q A_{pq} \frac{i p q r}{(i^2 - p^2)(j^2 - q^2)} [\frac{D_{16}(i^2 + p^2)}{c} +$$

$+ D_{26}(j^2 + q^2)] = 0,$ (20)

where $i \pm p$, $j \pm q$, $j \pm r$ are simultaneously odd number, and $i = 1,2,\dots m$; $j = 1,2,\dots n$.

For the solution of this problem we gave the programme CALFA (Pascal). In this case the programme has only one branche. It is evident that for $\alpha = 2$ the programme gives the case of pure bending, for $\alpha = 0$ we get the first branch of the former programme CXYT. The application of this programme is also given by one example. As in previous example, the laminated plate had 24 plies and was of the same material. In this case for $c = 2$ the matrix of the flexural rigidity coefficients is given by (10a). For α we have taken values $-0,5 \leqslant \alpha \leqslant 0,5$ with increment of 0,1. In the Table 4 are given the values for critical forces depending of the coefficient α. It can be seen that for $\alpha = 0$, the programme CALFA gives for $\overline{N}_{xcr}$ the same values as the programme CXYT.

Table 4: Critical buckling forces as a function of the coefficient α

α	-0,5	-0,4	-0,3	-0,2	-0,1	0	0,1	0,2	0,3	0,4	0,5
$\overline{N}_{xcr}$ $\frac{daN}{cm}$	176	183	191	200	209	220	231	244	258	274	292

3. The application of finite-difference equations

The problem of determining the critical buckling load can be solved by direct way, using the differential equation of the problem. In the general case of laminated anisotropic plates the differential equations are

$$A_{11}\frac{\partial^2 u^0}{\partial x^2} + 2A_{16}\frac{\partial^2 u^0}{\partial x \partial y} + A_{66}\frac{\partial^2 u^0}{\partial y^2} + A_{16}\frac{\partial^2 v^0}{\partial x^2} + (A_{12} + A_{66})\frac{\partial^2 v^0}{\partial x \partial y} + A_{66}\frac{\partial^2 v^0}{\partial y^2} = 0 ,$$

$$A_{16}\frac{\partial^2 u^0}{\partial x^2} + (A_{12} + A_{66})\frac{\partial^2 u^0}{\partial x \partial y} + A_{26}\frac{\partial^2 u^0}{\partial y^2} + A_{66}\frac{\partial^2 v^0}{\partial x^2} + 2A_{26}\frac{\partial^2 v^0}{\partial x \partial y} + A_{22}\frac{\partial^2 v^0}{\partial y^2} = 0 ,$$

$$D_{11}\frac{\partial^4 w}{\partial x^4} + 4D_{16}\frac{\partial^4 w}{\partial x^3 \partial y} + 2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} + 4D_{26}\frac{\partial^4 w}{\partial x \partial y^3} + D_{22}\frac{\partial^4 w}{\partial y^4} = q + N_x\frac{\partial^2 w}{\partial x^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} + N_y\frac{\partial^2 w}{\partial y^2} \quad (21)$$

For symmetric laminates it is necessary to solve only the last of equation (21). By direct solving of differential equations it is possible to solve only very few stability problems considering symmetric and unsymmetric laminates. In the case of unsymmetric laminates, for each side four boundary conditions must be satisfied. Along the edges we must satisfy additional following conditions:

for $x = 0$, and $x = a$: $u^0 = 0$, $N_{xy} = 0$ (22)

for $y = 0$, and $y = b$: $v^0 = 0$, $N_{xy} = 0$ (23)

All the other cases, which can not be solved in close mathematical form - finding the solution for differential equations, can be solved by using classical method of the finite-difference equations. In other words, the differential equations of the problem must be replaced by corresponding equations with finite increments. As the differential equations (21), which describe the equilibrium of symmetrical anisotropic plate, are linear and homogenous, analogous finite-difference equations will also be linear and homogeneous.

By application of the method of finite-differences we have solved the stability problem for symmetric generally ortotropic laminated plate under compressive load N_x and shear N_{xy}, which are uniformly distributed along the edges, Fig. 5. In this part we considered different boundary coditions of the plate. Applying this method for writing the finite-differences equations, we used the expression for the central differences. They give the best approximation for derivatives which appear in finite-difference equation of the problem. The shema for cosidered rectangular plates, with corresponding net, is shown in Fig. 6. As it can be seen the plate has m+1 fields in x and n+1 fields in y direction. In this way we got mxn nodal points.

If the plate is submitted only to uniform compressive load N_x and shear load N_{xy}, the differential equation is

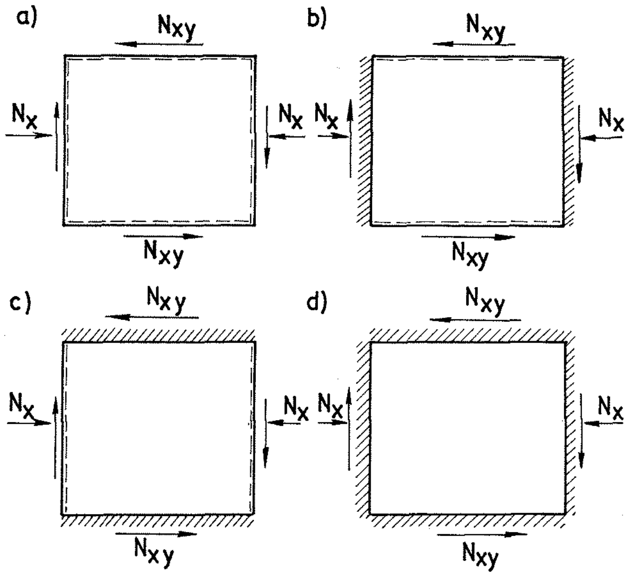


Figure 5. Four combinations of boundary conditions

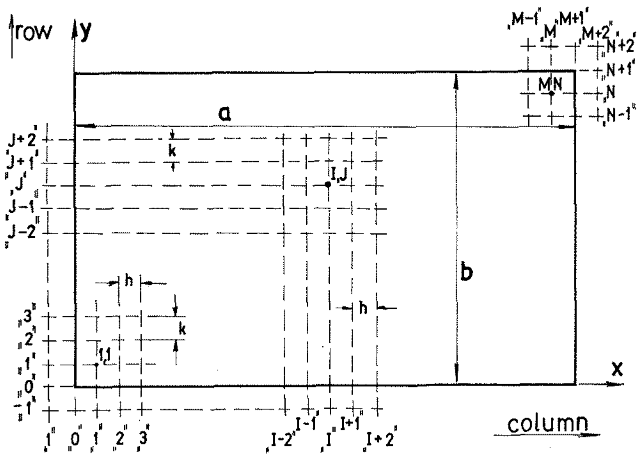


Figure 6. Shema of nodal points

$$D_{11}\frac{\partial^4 w}{\partial x^4} + 4D_{16}\frac{\partial^4 w}{\partial x^3 \partial y} + 2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} + 4D_{26}\frac{\partial^4 w}{\partial x \partial y^3} + D_{22}\frac{\partial^4 w}{\partial y^4} + N_x\frac{\partial^2 w}{\partial x^2} + N_{xy}\frac{\partial^2 w}{\partial x \partial y} = 0. \qquad (24)$$

Introducing the approximative values for partial derivations in (24), we come to the general form of the finite-difference equation

$$\frac{D_{16}}{h^3 k} w_{i+2,j+1} + \frac{D_{11}}{h^4} w_{i+2,j} + \left(-\frac{D_{16}}{h^3 k}\right) w_{i+2,j-1} + \frac{D_{26}}{hk^3} w_{i+1,j+1} + \left[-\frac{2D_{16}}{h^3 k} + \frac{2(D_{12} + 2D_{66})}{h^2 k^2} - \frac{D_{26}}{hk^3} + \frac{N_{xy}}{2hk}\right] w_{i+1,j+1} + \left[-\frac{4D_{11}}{h^4} - \frac{4(D_{12} + 2D_{66})}{h^2 k^2} + \frac{N_x}{h^2}\right] w_{i+1,j}$$

$$+ \left[\frac{2D_{16}}{h^3 k} + \frac{2(D_{12} + 2D_{66})}{h^2 k^2} + \frac{2D_{26}}{h k^3} - \frac{N_{xy}}{2hk}\right] w_{i+1,j-1} + \left(-\frac{D_{26}}{hk^3}\right) w_{i+1,j-2} + \frac{D_{22}}{k^4} w_{i,j+2} + \left[-\frac{4(D_{12} + 2D_{66})}{h^2 k^2} - \frac{4D_{22}}{k^4}\right] w_{i,j+1}$$

$$+ \left[\frac{6D_{11}}{h^4} + \frac{8(D_{12} + 2D_{66})}{h^2 k^2} + \frac{6D_{22}}{k^4} - \frac{2N_x}{h^2}\right] w_{i,j} + \left[-\frac{4(D_{12} + 2D_{66})}{h^2 k^2} - \frac{4D_{22}}{k^4}\right] w_{i,j-1} + \frac{D_{22}}{k^4} w_{i,j-2}$$

$$+ \left(-\frac{D_{26}}{hk^3}\right) w_{i-1,j+2} + \left[\frac{2D_{16}}{h^3 k} + \frac{2(D_{12} + 2D_{66})}{h^2 k^2} + \frac{2D_{26}}{hk^3} - \frac{N_{xy}}{2hk}\right] w_{i-1,j+1} + \left[-\frac{4D_{11}}{h^4} - \frac{4(D_{12} + 2D_{66})}{h^2 k^2} + \frac{N_x}{h^2}\right] w_{i-1,j}$$

$$+ \left[-\frac{2D_{16}}{h^3 k} + \frac{2(D_{12} + 2D_{66})}{h^2 k^2} - \frac{2D_{26}}{h k^3} + \frac{N_{xy}}{2hk}\right] w_{i-1,j-1} + \frac{D_{26}}{hk^3} w_{i-1,j-2} + \left(-\frac{D_{16}}{h^3 k}\right) w_{i-2,j+1} + \frac{D_{11}}{h^4} w_{i-2,j} + \frac{D_{16}}{h^3 k} w_{i-2,j-1} = 0 \qquad (25)$$

$$i = 1,2,\ldots m, \quad j = 1,2,\ldots n.$$

The boundary conditions for four combinations are given in Fig. 5:

a) All four sides are simply supported

$$w_{o,j} = w_{m+1,j} = w_{i,o} = w_{i,n+1} = 0$$
$$w_{-1,j} = -w_{1,j}$$
$$v_{m+2,j} = -w_{n,j}$$
$$w_{i,-1} = -w_{i,1}$$
$$w_{i,n+2} = -w_{i,k}$$
$$i = 1,2,\ldots m \; ; \; j = 1,2,\ldots n. \qquad (26)$$

b) Two sides are simply supported and two are clamped and submitted by compressive load:

$$w_{o,j} = w_{m+1,j} = w_{i,o} = w_{i,n+1} = 0$$
$$w_{-1,j} = w_{1,j}$$
$$w_{m+2,j} = w_{m,j}$$
$$w_{i,-1} = -w_{i,1}$$
$$w_{1,n+2} = -w_{i,n}$$
$$i = 1,2,\ldots m \; ; \; j = 1,2,\ldots n. \qquad (27)$$

c) Two sides are clamped and two are simply supported and submitted by compressive load:

$$w_{o,j} = w_{m+1,j} = w_{i,o} = w_{i,n+1} = 0$$
$$w_{-1,j} = -w_{1,j}$$
$$w_{m+2,j} = -w_{m,j}$$
$$w_{i,-1} = w_{i,1}$$
$$w_{i,n+2} = w_{i,n}$$
$$i = 1,2,\ldots m\ ;\ j = 1,2,\ldots n. \qquad (28)$$

d) All four sides are clamped:

$$w_{o,j} = w_{m+1,j} = w_{i,o} = w_{i,n+1} = 0$$
$$w_{-1,j} = w_{1,j}$$
$$w_{m+2,j} = w_{m,j}$$
$$w_{i,-1} = w_{i,1}$$
$$w_{i,n-2} = w_{i,n}$$
$$i = 1,2,\ldots m\ ;\ j = 1,2,\ldots n. \qquad (29)$$

Using anyone of four combinations for boundary conditions, the equation (25) becomes the system of mxn homogenous simultaneous linear algebraic equations with unknown deflections $w_{i,j}$.

As a result of our work we got four programmes in PASCAL language:

a) WSLOB - all four sides are simply supported;
b) WUKOP - compressive load is acting along the two clamped sides;
c) WSLOP - compressive load is acting along the two simply supported sides;
d) WUKLJ - all four edges are clamped.

Each of these programmes has three branches: compressive load, shear load and combined load of N_x and N_{xy}. In the third branch of each programme it is necessary to suppose N_{xy} in order to get the load N_x. In this way we always have the group of two critical buckling load $(N_x, N_{xy})_{cr}$.

The data for our programme are former values of D_{11}, D_{12}, D_{22}, D_{66}, D_{16}, D_{26}, with the same a, b, c and t. The number of net points is m in x and n in y direction. As it can be seen we could take any number of nodal points in order to form the difference equations. This is just necessary due to the anisotropy of the plate, because the obtained solution has not the character of uniform convergence by increasing the number of nodal points. The rectangular net corresponds nearly to the square net when it is $c \approx m+1/n+1$, but we are not sure that the solution will be the most possible correct. The applications of the programmes WSLOB, WUKOP, WSLOP, WUKLJ is presented ih the short form. We investigated the stability problem of symmetric 20-plied HT-S/4617 grafit-epoxy laminates submitted to the action of N_x and N_{xy} loads. The elastic constants are:

$E_1 = 1{,}3789\cdot10^6$ bar
$E_2 = 0{,}0896\cdot10^6$ bar
$G_{12} = 0{,}0448\cdot10^6$ bar
$\nu_{12} = 0{,}304$
$b = 12$ cm
$t_1 = 0{,}014$ cm
$t = 0{,}28$ cm.

The angles of orientation are:

$$\theta = /0^o_2,\ \pm45^o,\ 0^o_2,\ \pm45^o,\ 0^o_2/_s.$$

The coefficients of the geometrical characteristics are

$$[D] = \begin{bmatrix} 1919{,}9 & 250{,}29 & 35{,}59 \\ 250{,}29 & 382{,}35 & 35{,}59 \\ 35{,}59 & 35{,}59 & 282{,}11 \end{bmatrix} |\text{daNcm}|$$

Our results are given in four Tables:

Table 5: Critical buckling load $(N_x, N_{xy})_{cr}$
a) All four sides are simply supported.
Programme WSLOB

c = 1		c = 1,5		c = 2		c = 2,5	
N_{xy}	N_x	N_{xy}	N_x	N_{xy}	N_x	N_{xy}	N_x
$\frac{daN}{cm}$	$\frac{daN}{cm}$	$\frac{daN}{cm}$	$\frac{daN}{cm}$	$\frac{daN}{cm}$	$\frac{daN}{cm}$	$\frac{daN}{cm}$	$\frac{daN}{cm}$
-575	0	-378	0	-295	0	-266	0
-400	154	-300	113	-200	121	-200	109
-300	201	-200	154	-100	208	-100	198
-200	236	-100	204	0	242	0	231
-100	257	0	224	100	224	100	213
0	266	100	214	200	154	200	146
100	263	200	175	300	39	298	0
200	248	300	109	334	0		
300	221	400	19				
500	129	419	0				
400	181						
643	0						

Table 6: Critical buckling load $(N_x, N_{xy})_{cr}$,
b) Two clamped sides are compressed.
Programme WUKOP

c = 1		c = 1,5		c = 2		c = 2,5	
N_{xy}	N_x	N_{xy}	N_x	N_{xy}	N_x	N_{xy}	N_x
$\frac{daN}{cm}$	$\frac{daN}{cm}$	$\frac{daN}{cm}$	$\frac{daN}{cm}$	$\frac{daN}{cm}$	$\frac{daN}{cm}$	$\frac{daN}{cm}$	$\frac{daN}{cm}$
-945	0	-490	0	-343	0	-284	0
-800	192	-400	112	-300	69	-200	133
-600	382	-300	218	-200	186	-100	248
-400	518	-200	301	-100	273	0	297
-200	602	-100	357	0	311	100	270
0	635	0	380	100	292	200	175
200	618	100	370	200	221	300	29
400	550	200	322	300	111	317	0
600	434	300	257	381	0		
800	268	400	163				
1034	0	539	0				

Table 7: Critical buckling load $(N_x, N_{xy})_{cr}$,
c) Two simply supported sides are compressed.
Programme WSLOP

c = 1		c = 1,5		c = 2		c = 2,5	
N_{xy}	N_x	N_{xy}	N_x	N_{xy}	N_x	N_{xy}	N_x
$\frac{daN}{cm}$	$\frac{daN}{cm}$	$\frac{daN}{cm}$	$\frac{daN}{cm}$	$\frac{daN}{cm}$	$\frac{daN}{cm}$	$\frac{daN}{cm}$	$\frac{daN}{cm}$
-677	0	-471	0	-421	0	-385	0
-600	75	-400	103	-400	40	-300	131
-400	235	-300	233	-300	173	-200	257
-200	338	-200	339	-200	270	-100	340
0	379	-100	394	-100	332	0	374
200	356	0	413	0	357	100	358
400	272	100	406	100	346	200	293
600	132	200	374	200	298	300	184
746	0	300	284	300	217	426	0
		400	167	464	0		
		520	0				

Table 8: Critical buckling load $(N_x, N_{xy})_{cr}$.
d) All four edges are clamped.
Programme WUKLJ

c = 1		c = 1,5		c = 2		c = 2,5	
N_{xy}	N_x	N_{xy}	N_x	N_{xy}	N_x	N_{xy}	N_x
$\frac{daN}{cm}$	$\frac{daN}{cm}$	$\frac{daN}{cm}$	$\frac{daN}{cm}$	$\frac{daN}{cm}$	$\frac{daN}{cm}$	$\frac{daN}{cm}$	$\frac{daN}{cm}$
-986	0	-559	0	-448	0	-408	0
-800	224	-400	224	-400	83	-300	178
-600	425	-200	439	-300	242	-200	300
-400	580	0	534	-200	367	-100	383
-200	681	200	474	-100	434	0	418
0	723	400	286	0	463	100	402
200	702	600	20	100	451	200	338
400	621	613	0	200	400	300	232
600	487			300	300	450	0
800	307			400	154		
1070	0			492	0		

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