# EFFECTS OF MODEL REDUCTION ON STABILITY BOUNDARY AND LIMIT-CYCLE CHARACTERISTICS

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## Abstract

In this paper, the effects of model reduction on the stability boundaries of control systems with parameter variations, and the limit cycle characteristics of nonlinear control systems are investigated. In order to reduce these effects, a method is used which can approximate the original transfer function at S=0, S=∞, and also match some selected points on the frequency response curve of the original transfer function. Examples are given, and comparisons with the methods given in current literature are made.

## I. Introduction

In current literature, most of the methods for model reduction are based upon the assumption that the system has constant parameters and take care of the errors between the step-input (or frequency) responses of the reduced models and the original transfer function. Therefore, the methods for reducing the effects of model reduction on either the stability boundaries of systems with parameter variations or the limit cycle characteristics of nonlinear control systems (9) are still wanted. For example, the Padé approximations (1-5) are taking at S=0; thus the stability boundaries or the limit cycle characteristics of a closed-loop system with a reduced model may be deviated considerably from those of the original system. Although some other methods taking approximations at both S=0 and S=∞, or at S=a (a>0), were suggested (6-8), the aforementioned characteristics may also be deviated considerably from those of the original system when reduced models are used.

In this paper, first the effects of reduced models on the stability boundaries of systems with parameter variations and the limit cycle characteristics of nonlinear control systems are illustrated, and then a method for reducing these effects is proposed.

## II. Effects of Model Reduction on Stability Boundaries and Limit Cycle Characteristics

In this section, numerical examples are presented to show that, when reduced models are used both the stability boundaries and the limit cycle characteristics of control systems are deviated considerably from the original ones.

### (i) Stability Boundary

Example 1. Consider the system shown in Fig.1 where r=0.3, α and β are two

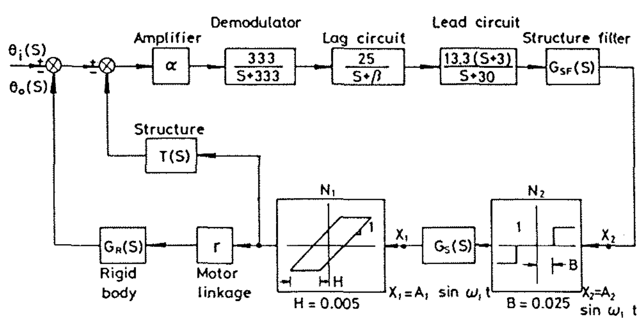


Fig.1 Bolck diagram of a missile control system.

parameters. The nonlinear elements $N_1$ and $N_2$ are neglected for this example. The transfer functions are defined as follows (9):

$$G_R(S) = \frac{7.21}{(S+1.6)(S-1.48)} \quad (2.1)$$

$$G_S(S) = \frac{2750}{S^2+42.2S+2750} \quad (2.2)$$

$$G_{SF}(S) = \frac{(S^2+70S+4000)(S^2+22S+12800)}{(S^2+30S+5810)(S^2+30S+12800)} \quad (2.3)$$

$$T(S)=[0.686(S+53)(S-53)(S^2-152.2S+14500)(S^2+153.8S+14500)]/[(S^2+S+605)(S^2+45.5S+2660)(S^2+2.51S+3900)(S^2+3.99S+22980)] \quad (2.4)$$

Applying the modified Padé approximation method, the reduced model of T(S) is found as

$$M_1(S)=R[3,5]_6^4(S) =[0.686(S-111.3723)(S^2+321.1828S+38950.873)]/[(S^2+1.2982S+401.8666)(S^2+152.1748S+28159.032)(S+107.7370)] \quad (2.5)$$

where the symbol $R[q,r]_j^i(S)$ represents a reduced model with q-zeros and r-poles, and it is obtained by continued-fraction expansion of T(S) about S=0 and S=∞ for i and j times, respectively. The Nyquist polts of T(S) and the reduced model $M_1(S)$ are shown in Fig.2. The stability

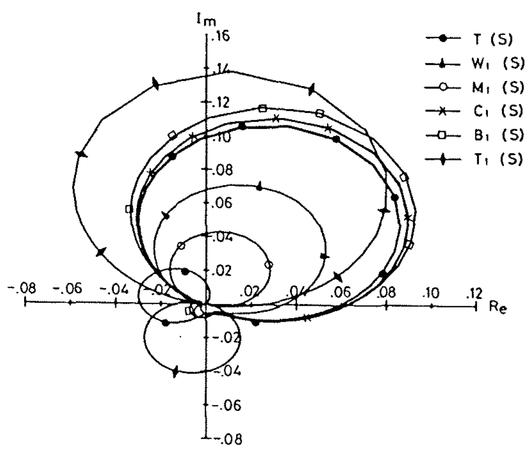


Fig.2 Nyquist plots of T(S) and its reduced models.

boundaries of the original closed-loop system and the system with reduced model $M_1(S)$ are shown in the α versus β plane as shown in Fig.3, where the stable region is to the right-side of each of the

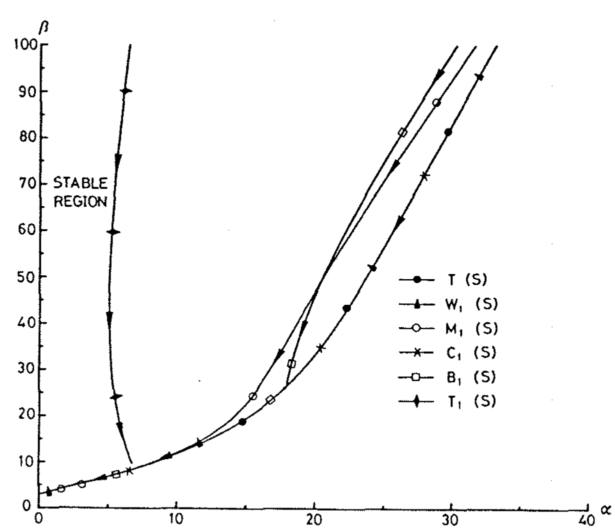


Fig.3 Stability boundaries of the system with T(S) and its reduced models.

boundaries along the arrow direction. It can be seen that the stable region is reduced when $M_1(S)$ is used. In other words, due to the effects of parameter variations a stable system may become unstable if an improper reduced model is used.

(ii) Limit Cycle

Example 2. Consider the system shown in Fig.1. Let the parameters α, β and r be 18, 90 and 0.3, respectively. By use of the describing function method and the parameter space method (9), a limit cycle can be found. The amplitude and frequency of the limit cycle are at

$$A_1 = 1.50256 \quad (2.6)$$

$$A_2 = 0.6841 \quad (2.7)$$

and $$\omega_1 = 54.844 \text{ rad/sec} \quad (2.8)$$

respectively, Replacing T(S) by $M_1(S)$ as defined in Eq.(2.5), the limit cycle characteristics are obtianed as shown in Table I. It can be seen that the limit cycle characteristics are changed when the reduced model is used. In other words, the result of limit cycle analysis my be incorrect if an improper reduced model is used.

Example 3. Consider the system shown

| limit cycle / Models | Amplitudes | | Frequency |
|---|---|---|---|
| | $A_2$ | $A_1$ | $\omega_1$ (rad/sec) |
| Original T(S) | 0.6841 | 1.50256 | 54.844 |
| $W_1(S)$ | 0.6841 | 1.50256 | 54.844 |
| $C_1(S)$ | 0.401 | 1.64019 | 50.191 |
| $D_1(S)$ | 0.813 | 1.60078 | 51.782 |
| $M_1(S)$ | 0.7775 | 1.38277 | 18.513 |
| $T_1(S)$ | 3.777 | 1.26043 | 61.32 |

Table I

in Fig.4, where $r=\delta=1$ and the transfer functions are defined as (10)

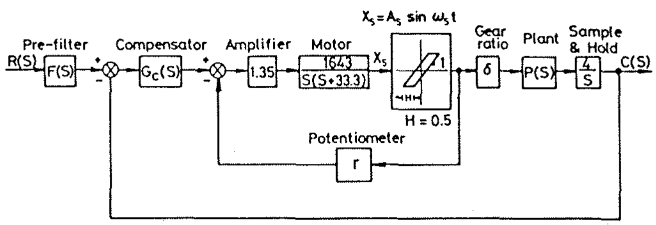


Fig.4 Block diagram of an altitude control system.

$$P(S)=\frac{80(S^2+0.6S+0.014)}{(S^2+0.018S+0.0053)(S^2+1.6S+80)} \quad (2.9)$$

$$F(S)=\frac{4504.5(S+4)}{(S+5.5)(S+6)^2(S+7)(S+13)} \quad (2.10)$$

$$G_c(S)=[(72.6(S+0.1)(S+2.5)^2(S+14)^2(S+15)(S^2+7.2S+144)]/[(S+1.5)(S+1.1)^2(S+30)(S+18)(S+35)(S+40)^2] \quad (2.11)$$

i.e.,

$$G_c(S)= 72.6[1-G_{cr}(S)] \quad (2.12)$$

where

$$G_{cr}(S)=[111.4(S+1.5700)(S+18.1854)(S+21.2885)(S^2+2.1137S+1.1687)(S^2+44.2324S+690.2429)]/[(S+1.5)(S+1.1)^2(S+30)(S+18)(S+35)(S+40)^2] \quad (2.13)$$

By use of the modified Padé approximation method, a reduced model of $G_{cr}(S)$ is obtained as

$$M_5(S)=R[4,5]_5^5(S)$$
$$=[111.4(S^2+2.0092S+1.0646)(S^2+45.6754S+684.6425)]/[(S+47.9733)(S^2+2.0330S+1.0555)(S^2+76.9885S+1611.3303)] \quad (2.14)$$

The Nyquist plots of $G_{cr}(S)$ and $M_5(S)$ are shown in Fig.5. The limit cycle characteristics of the original system and the system with reduced model $M_5(S)$ are shown in Table II, which indicates that the limit cycle characteristics are changed if the reduced model is used.

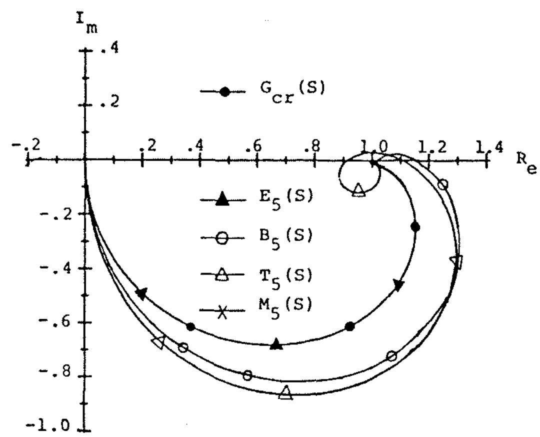


Fig.5 Nyquist plots of $G_{cr}(S)$ and its reduced models.

| Limit cycle / Modles | Amplitude | Frequency |
|---|---|---|
| | $A_5$ | $\omega_5$ (rad/sec) |
| Original $G_{cr}(S)$ | .335 | 17.8 |
| $E_5(S)$ | .335 | 17.8 |
| $B_5(S)$ | .2536 | 1.194 |
| $M_5(S)$ | .334 | 17.95 |
| $T_5(S)$ | * | * |

(*: system is unstable)

Table II

## III. The Proposed Method

Let the original transfer function and the reduced model be

$$G(S)=\frac{A_{21}+A_{22}S+A_{23}S^2+\ldots+A_{2n}S^{n-1}}{A_{11}+A_{12}S+A_{13}S^2+\ldots+A_{1,n+1}S^n} \quad (3.1)$$

and

$(\omega_1,\omega_2,\ldots,\omega_m)R[r-1,r]^i_j(S)$

$$= \frac{d_0+d_1S+d_2S^2+\ldots+d_{r-1}S^{r-1}}{c_0+c_1S+c_2S^2+\ldots+c_rS^r} \qquad (3.2)$$

respectively. In eqn.(3.2), r and r-1 represent the numbers of poles and zeros of R(S), respectively; i and j are the numbers of times of the continued-fraction expansion of G(S) about S=0 and S=∞, respectively; and $\omega_1,\omega_2,\ldots,\omega_m$ are the frequencies at which the frequency response of G(S) are matched by R(S). The procedure for finding the reduced models is as follows:

Step 1. Expand G(S) about S=0 for i (even number) times, i.e.

$$G(S) = \cfrac{1}{h_1 + \cfrac{1}{\frac{h_2}{S} + \cfrac{1}{\ddots + \cfrac{1}{\frac{h_i}{S} + \frac{H_N(S)}{H_D(S)}}}}} \qquad (3.3)$$

where

$$H_N(S) = A_{i+2,1}+A_{i+2,2}S+\ldots+S_{i+2,n-i/2}S^{n-1-i/2} \qquad (3.4)$$

and

$$H_D(S) = A_{i+1,1}+A_{i+1,2}S+\ldots+A_{i+1,n+1-i/2}S^{n-i/2} \qquad (3.5)$$

Step 2. Reverse the polynomial sequences in eqns.(3.4) and (3.5), and continue to expand eqn.(3.3) about S=∞ for j(even number) times yield

$$\frac{H_N(S)}{H_D(S)} = \cfrac{1}{E_1S + \cfrac{1}{E_2 + \cfrac{1}{\ddots + \cfrac{1}{E_j + \frac{F_N(S)}{F_D(S)}}}}} \qquad (3.6)$$

where

$$F_N(S) = A_{i+j+2,n-(i+j)/2}S^{n-1-(i+j)/2} + \ldots + A_{i+j+2,2}S+A_{i+j+2,1} \qquad (3.7)$$

and

$$F_D(S) = A_{i+j+1,n+1-(i+j)/2}S^{n-(i+j)/2} + \ldots + A_{i+j+1,2}S+A_{i+j+1,1} \qquad (3.8)$$

Step 3. Let

$$\frac{T_N(S)}{T_D(S)} = \frac{Y_{m-1}S^{m-1}+Y_{m-2}S^{m-2}+\ldots+Y_1S+Y_0}{X_mS^m+X_{m-1}S^{m-1}+\ldots+X_1S+X_0} \qquad (3.9)$$

and with

$$Y_{m-1} = 1 \qquad (3.10)$$

where m is the number of points on the frequency response curve of G(S) to be matched by R(S). Let

$$\left.\frac{T_N(S)}{T_D(S)}\right|_{S=j\omega_k} = \left.\frac{F_N(S)}{F_D(S)}\right|_{S=j\omega_k} = r_k + jm_k$$

$$k = 1,2,\ldots,m \qquad (3.11)$$

where $r_k$ and $m_k$ are the real part and the imaginary part of $F_N(S)/F_D(S)$ for $S=j\omega_k$, respectively. Separating the real part and the imaginary part in eqn.(3.11), one can obtain a pair of simultaneously independent equations. Therefore, the 2m unknowns $Y_{m-2},Y_{m-3},\ldots,Y_1,Y_0,X_m,\ldots,X_1,X_0$ can be obtained by solving the 2m simultaneously independent equations.

Step.4. Replace $\frac{F_N(S)}{F_D(S)}$ in eqn.(3.6) by eqn.(3.9) and invert the continued-fraction, the reduced model defined in eqn. (3.2) is obtained. The order of the denominator of the reduced model is

$$r = m+(i+j)/2 \qquad (3.12)$$

If both i and j are odd numbers, the above equations may have some minor differences.

## IV. Applications of the Proposed Method

The main purpose of this section is to apply the proposed method to those systems considered in Section II, and to show that the effects on the stability boundaries and the characteristics of limit cycles can be reduced. In addition, comparisons with the methods given in current literature are made.

(i) Stability Boundary

Example 4. Consider the system in Example 1. The proposed method is applied to reduce T(S) with frequency responses at $\omega_1$=54.844 rad/sec and $\omega_1'$=21.021 rad/sec (the phase-crossover frequency of the open-loop transfer function when the nonlinear elements $N_1$ and $N_2$ are neglected and the parameters $\alpha$, $\beta$ and r be 15,100 and 0.3, respectively) to be matched by the reduced model. The result is

$$W_1(S)=(\omega_1,\omega_1')R[3,5]_2^4(S) =[0.686(S+24.6442)(S-40.9717)(S+219.7768)]/[(S+28.5945)(S^2+1.2351S+593.7862)(S^2+10.0018S+3191.7797)] \quad (4.1)$$

By use of other methods, the reduced models of T(S) are found as:

(a) Continued-fraction method (1)

$$C_1(S)=R[4,5]_0^{10}(S) =[-2.44\times10^{-2}(S-61.3836)(S-67.5164)(S^2+81.5000S+1748.6381)]/[(S+41.3000)(S^2+0.9700S+605.9021)(S^2+25.4000S+2447.8033)] \quad (4.2)$$

(b) Stability-equation method and Padé approximation method (13)

$$B_1(S)=[-2.3873\times10^{-2}(S+38.1380)(S+130.4869)(S^2-101.2398S+2997.6391)]/[(S+72.5638)(S^2+0.91S+602.1438)(S^2+12.4262S+2902.0190)] \quad (4.3)$$

(c) Routh stability array method

$$T_1(S)=[-1.5899\times10^{-2}(S+53)(S+150.2440)(S-53)(S-157.0873)]/[(S+161.1435)(S^2+0.7348S+611.5744)(S^2+2.0104S+3807.4562)] \quad (4.4)$$

The Nyquist plots of the reduced models and the stability boundaries are shown in Figs.2 and 3, respectively. It can be seen that the stability boundaries are very close to that of the original system when the reduced model $W_1(S)$ is used.

(ii) Limit Cycle

Example 5. Consider the system in Example 2. Substituting T(S) by the reduced models defined from eqns.(4.1) to (4.4), the characteristics of limit cycles are obtained as shown in Table I. It can be seen that both the amplitudes and the frequency of the limit cycle can be preserved by the system with the proposed model $W_1(S)$.

Example 6. Consider the system in Example 3. The proposed method is applied to reduce $G_{cr}(S)$ with the frequency responses at $\omega_5$=17.8 rad/sec (the frequency of the limit cycle) to be matched by the reduced model. The reduced model is obtained as:

$$E_5(S)=(\omega_5)R[4,5]_2^6(S) =[111.4(S^2+2.0239S+1.0771)(S^2+44.7388S+609.5408)]/[(S+26.0942)(S^2+2.0490S+1.0685)(S^2+97.9296S+2635.9173)] \quad (4.5)$$

By use of other methods, the reduced models of $G_{cr}(S)$ are found as:

(a) Stability-equation method (11)

$$B_5(S)=[17.2106(S+3.7257)(S+2.5366)(S^2+1.9008S+0.9753)]/[(S+0.9689)(S^2+2.6431S+1.8027)(S^2+14.0581S+91.2592)] \quad (4.6)$$

(b) Routh stability array method

$$T_5(S)=[17.5216(S^2+1.6741S+0.7845)(S^2+7.2544S+15.5663)]/[(S+0.7660)(S^2+2.4813S+2.0201)(S^2+13.8462S+138.9465)] \quad (4.7)$$

(c) Continued-fraction method

$$C_5(S)=R[4,5]_0^{10}(S) =[-174.6522(S-73.4205)(S+1.5739)(S^2+2.1137S+1.1686)]/[(S+1.5043)(S^2+2.1994S+1.2094)(S^2-159.2948S+13027.0677)] \quad (4.8)$$

The Nyquist plots of the reduced models defined from eqns.(4.5) to (4.7) are shown in Fig.5. The characteristics of limit cycles for the system with the models are obtained as shown in Table II. The results indicate that the limit-cycle characteristics can be preserved when the reduced model $E_5(S)$ obtained by the proposed method is used. Note that $C_5(S)$ obtained by the continued-fraction method is unstable.

## V. Conclusions

The effects of model reduction on stability boundaries of control systems with parameter variations and the limit cycle characteristics of nonlinear control systems have been investigated. The applications of the proposed method to reduce these effects have been presented. In comparison with the results obtained by the methods given in current literature, the advantages of using the proposed method are quite evident.

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