BIFURCATION ANALYSIS OF CRITICAL AIRCRAFT FLIGHT REGIMES

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I. INTRODUCTION

The final phase in the flight dynamics study is the investigation of critical flight regimes; they are of ever increasing interest since determine the aircraft application anvelope and, to a great extent, their flight safety.

The critical regimes primarely include motions with a strong inertia coupling at a rapid aircraft roll (roll coupling), its dynamics at high angles of attack, when various forms of stability loss due to an aerodynamic characteristics degradation known as stall can occur, as well as spin at supernormal angles of attack.

The aircraft spatial motion dynamics with simultaneous control by ailerons, stabilizer and rudder is extremelly complex to investigate, and this problem can adequately be solved only using digital computers. The trends of such analysis are formulated in the book by Byushgens and Studnev[1], in which the theory of aircraft spatial motion at rapid roll is developed.

Recently a unified approach to investigate critical flight conditions based on general methods for analyzing nonlinear spatial motion equations has been shaped. The approach uses a numerical realization of qualitative analysis methods for nonlinear dynamic systems, recent advances in differential geometry, the bifurcation analysis, and the catastrophe theory. It is appropriate to mention here the contributions of Mehra, Carrol[2] and Guichetea[3].

Current efficient computational procedures permit to adequately calculate steady-state stationary and oscillatory motions, investigate their stability at small disturbances, predict bifurcation control parameters, at which changes of the solution structure of motion equations and the associated stability take place.

Critical regimes at rapid aircraft rolls, in stall and spin can be related, as a rule, to bifurcation singlarities of the motion equations and to changes of their solution structure. With deflected ailerons, e. g., the termination of a stable roll or the oscillation development may result in a disproportionate aircraft spin-up and entry into an uncontrollable critical inertial rotation[1]. Stalls are associated with a variation of disturbed lateral aircraft motions at high angles of attack. Oscillatory motions correspond to stable periodic solutions of motion equations (limit cycles or closed orbits); be it a rapid roll, stall, or spin there development is of universal nature governed by the so called Hopf bifurcation[4].

Possible stable motion regimes both stationary and periodic, the number and structure of which depend on the control deflections and flight variables have a decisive affect on the aircraft behaviour.

The bifurcation analysis efficiency, i. e. that of the determination of limit control deflections, at which the number of equilibrium motions and their stability varies, is maximum for quasi-stationary motions at a rather small control deflection rates. At rapid controls equilibrium motions may fail to realize themselves, but in this case, too, they would considerably affect the disturbed controllable motion and the type of transients. The quasi-stationary annalysis is supplemented by many results of numerical dynamics modeling.

An essential part of the general approach to the critical motion study and the bifurcation analysis is the continuation method for nonlinear parameter dependent equation systems[2,5,6]. Its interactive realization on a digital computer permits to turn the computer into a tool of a qualitative study for nonlinear dynamic systems. An important means of investigating the phase space structure of the motin equations having a high dimension is the Poincare point mapping technique[7,8]. Its numerical realization together with modified parameter continuation method used in this paper made it possible to calculate and study bifurcation features of the motion equation periodic solutions for a spinning aircraft, and to show the possibility of generating an invariant torus from a closed orbit.

II. EQUATIONS OF MOTION

For a rigid body aircraft with constant mass and inertia parameters, full force and moment equations with respect to principal and central body-fixed axes are given as (fig. 1):

$$\dot{\alpha} = \omega_z - [(F_x - \omega_y sin\beta)sin\alpha + (F_y + \omega_x sin\beta)cos\alpha]/cos\beta \quad (1)$$

$$\dot{\beta} = F_z cos\beta - (F_x sin\beta - \omega_y)cos\alpha + (F_y sin\beta + \omega_x)sin\alpha \quad (2)$$

$$\dot{V} = V(F_x cos\alpha\ cos\beta - F_y sin\alpha\ cos\beta + F_z sin\beta) \quad (3)$$

where

$$F_x = -(\rho VS/2M)\ c_x - (g/V)sin\theta$$

$$F_y = (\rho VS/2M)\ c_y - (g/V)\ cos\theta\ cos\gamma$$

$$F_z = (\rho VS/2M)\ c_z + (g/V)\ cos\theta\ sin\gamma$$

$$\dot{\omega}_x = ((I_y - I_z)/I_x)\ \omega_y\omega_z + (\rho V^2 SL/2I_x)\ m_x \quad (4)$$

$$\dot{\omega}_y = ((I_z - I_x)/I_y)\ \omega_z\omega_x + (\rho V^2 SL/2I_y)\ m_y \quad (5)$$

$$\dot{\omega}_z = ((I_x - I_y)/I_z)\ \omega_x\omega_y + (\rho V^2 Sb_a/2I_z)\ m_z \quad (6)$$

The aircraft attitude relative to inertial Earth reference system is found by kinematic equations with respect to Euler's angles

$$\dot{\theta} = \omega_y sin\gamma + \omega_z cos\gamma \quad (7)$$

$$\dot{\gamma} = \omega_x - tan\theta\ (\omega_y cos\gamma - \omega_z sin\gamma) \quad (8)$$

The flight altitude variation H with density ρ dependent on it is determined by the equation

$$\dot{H} = V(\sin\theta \cos\beta \cos\alpha - \cos\theta \cos\gamma \cos\beta \sin\alpha - \cos\theta \sin\gamma \sin\beta) \qquad (9)$$

Nondimensional coefficients of aerodynamic forces c_x, c_y, c_z and moments m_x, m_y, m_z as functions of kinematic motion parametrs and control deflections are presented as tabular functions with a linear interpolation or that by spline functions on the bacis of aerodynamic experimental data using the static method, the forced oscillation method, and the method of equilibrium rotation[9].

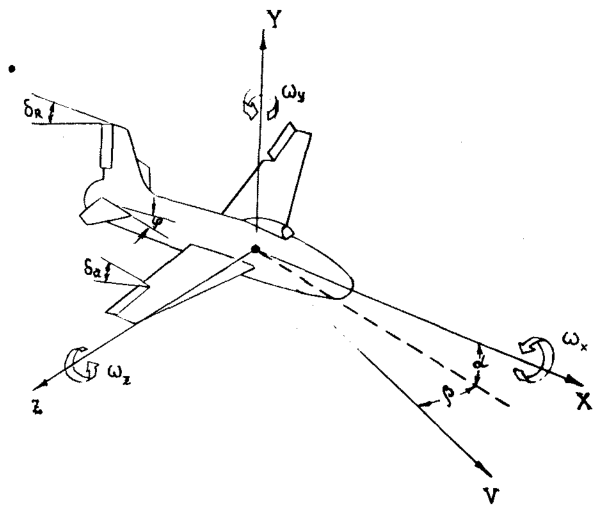


Figure 1.

The dynamics investigation based on quasisteady spatial motions depending on the problem under study necessitates the consideration of variour phisically justifiable motion models: autonomous equation groups.

E. g., aircraft motion at a rapid roll including inertial, kinematic and aerodynamic coupling of longitudinal and lateral motions is investigated assuming that: a) V = const, H = const; b) α, β are small, so that $\sin\alpha \sim \alpha$, $\sin\beta \sim \beta$; c) the effect of the path curvature due to gravity on the dynamics is neglegted i. e. it is assumed that $g/V = 0$. These assumptions enable to separately examine five autonomous equations (1), (2), (4), (5), (6). Digital computers and numerical methods (see III) allows to use arbit – rary rather smooth functions of aerodynamic coefficients. The same equation group taking account of weighting terms may be also utilized for aircraft stall.

The aircraft spin is of spatial nature both about the centre of mass and in the path motion. The equilibrium spin is characterized by an essentially vertical spiral descent trajectory with constant characteristics from one turn to another. The effect of gravitational terms on the dynamics may be considerable and when investigating the spin eqs. (1)–(8) are treated simultaneously. Equations (1)–(9) as a whole are not autonomous. The density constancy assumption allows to consider the first eight equations as an autonomous system. The altitude variation would lead to a quasistatic alteration of spin parameter.

Thus, motion equations are reduced to a general vector form

$$\dot{X} = F(X, U) \qquad (10)$$

where F is the vector-function, X is the vector of the dimension n, and U is the control parameter vector of the dimension m.

The equation number n may be different depending on the problem to be solved, e. g. for the spin problem: $X = (\alpha, \beta, \omega_x, \omega_y, \omega_z, V, \vartheta, \gamma)^T$; $U = (\varphi, \delta_a, \delta_R)^T$ where φ, δ_a, δ_R are the deflections of the stabilizer, ailerons and the rudder.

Stationary solutions, when $\dot{X} = 0$, are reduded to that of nonlinear control parameter-dependent equations: $F(X, U) = 0$, an efficient tool here being the continuation method.

III. DIFFERENTIAL CONTINUATION METHOD

With a continuously changing parameter the solution $X(U)$ implicitly specified by the vector equation

$$F(X, U) = 0 \qquad (11)$$

where $F \in R^n$, $X \in R^n$, will also continuously vary, and U with no generality limitation may be considered a scalar: $U \in R$. Let functions F_i, $i = 1, 2 \ldots n$ be continuous and have continuous partial derivatives in all varibles X_i and U.

Paper[5] suggests to reduce the calculation of $X(U)$ of system (11) to an integration of the differential equation set

$$\frac{dX}{dU} = -\left(\frac{\partial F}{\partial X}\right)^{-1} \frac{\partial F}{\partial U} \qquad (12)$$

at initial conditions $X(U_0) = X_0$, where $F(X_0, U_0) = 0$,

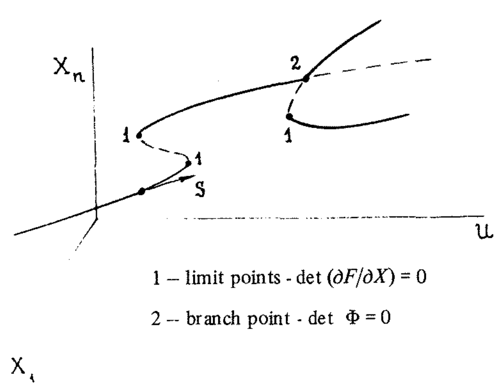
$$\left(\frac{\partial F}{\partial X}\right) = \left\{\frac{\partial F_i}{\partial X_j}\right\}$$

is the Jacobian matrix,

$$\frac{\partial F}{\partial U} = \left(\frac{\partial F_1}{\partial U}, \frac{\partial F_2}{\partial U}, \ldots, \frac{\partial F_n}{\partial U}\right)^T$$

To solve $X(U)$ in a range (U_0, U_1) it is necessary that the functional determinant $J = \det(\partial F/\partial X)$ differ from zero. The Jacobian degeneration in parameter limited points makes it impossible to continue the solution using equations (12).

Papers[2,6] contain algorithms which permit to correct the solution and pass parameter limited points of the curve which are bifurcation points of system (10).



1 – limit points - det $(\partial F/\partial X) = 0$

2 – branch point - det $\Phi = 0$

Figure 2.

The representation of $X(U)$ solution trajectory in space $Z=(X^T, U)$ (see fig. 2) by its length l enables to eliminate this singularity and construct a symmetric, with respect to variables, differential equation system of type (12) where the sought trajectory will be an asymptotically stable phase path.

When proceeding along the solution curve component increments of ΔX state vector and ΔU control parameter will be defined by the projections of a unit vector $S \in R^{n+1}$ tangential to the curve. This vector is orthogonal with respect to all gradients of function F_i, $i=1,\dots n$ which are the components of vector F. Each function F_i specifies a hypersurface of a unit codimension in Z space, and their intersection forms a solution trajectory.

If among the vectors $\partial F_i/\partial Z$ there are no linearly dependent, i. e. rank $(\partial F/\partial Z)=n$, then together with the tangent vector S they form a basis in Z space, and the matrix Φ of the dimenstion $(n+1, n+1)$ made up of them will be undegenerate

$$\Phi = \begin{pmatrix} \frac{\partial F}{\partial Z} \\ S^T \end{pmatrix} \tag{13}$$

The components of S vector are expressed through cofactors of the lower row of matrix Φ

$$S_i = \frac{A_{n+1,i}}{\det \Phi} \quad \text{where} \quad \det \Phi = \left(\sum_{j=1}^{n+1} A^2_{n+1,j}\right)^{1/2} \tag{14}$$

Now, like in (12) the solution trajectory of (11) can be found in a differential form as follows:

$$\frac{dZ}{dL} = S$$

or

$$\frac{dX_i}{dL} = S_i, \quad i=1,2,\dots,n$$

$$\frac{dU}{dL} = S_{n+1} = \det(\partial F/\partial X)/\det \Phi \tag{15}$$

at initial conditions of $X=X_0$, $U=U_0$.

The numerical integration of (15) will result in an error causing a deviation of the sought solution from an exact one. The solution correction may be orthogonal to S vector and eqs. (15) in combination with the correction will be described by

$$\Phi \frac{dZ}{dL} = \begin{pmatrix} -\lambda F \\ 1 \end{pmatrix} \quad \text{or} \quad \frac{dZ}{dl} = \Phi^{-1}\begin{pmatrix} -\lambda F \\ 1 \end{pmatrix} \tag{16}$$

where λ is the positive parameter.

Eqs. (16) provide for a sought solution with an automatic reduction of the error ($\|F\| \neq 0$) in the integration with a finite step Δl, and in an incorrect choice of the initial point Z_0, which does not satisfy (11). The sought solution of (11) is an asymptotically stable phase trajectory of differential equations (16), since the solution $F=F_0 e^{-\lambda l}$ is valid. System (16) is degenerate only in solution branch points where $\det \Phi = 0$. To pass branch points one may use the structural instability of branch points relative to small distortions of vector field (11) at which they form a number of nonsingular trajectories. Algorithms are also known to determine various branch orientations in a branch point[2].

Fig. 3 shows a calculation example using the continuation method of the surface of steady roll rates at different stabilizer and aileron deflections. The aircraft parameters are chosen so that ω_x values form a continuous surface. In the (φ, δ_a) plane there are regions with a different number of equilubrium roll motions, which varies from 1 to 5. The projections of limit surface points onto the control parameter plane form bifurcation boundaries which are termed ,,butterfly"[11, 12].

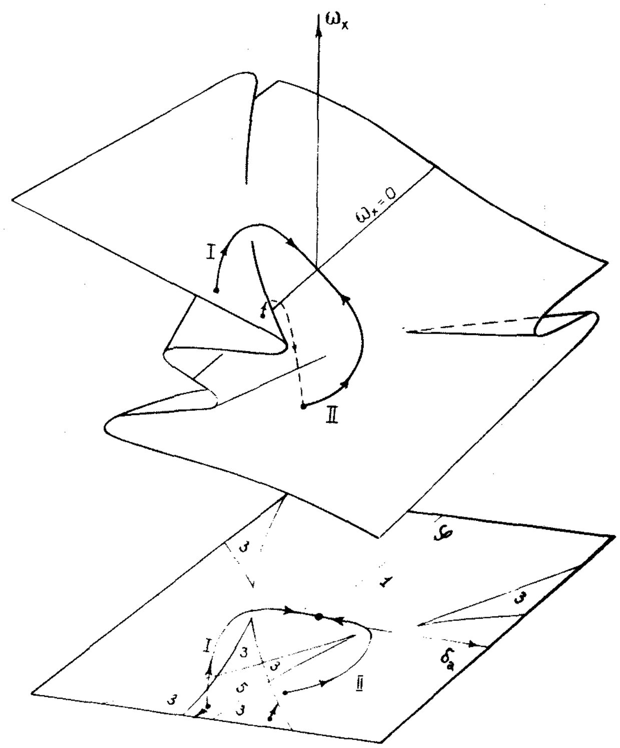


Figure 3.

The equilibrium surface ω_x contains envelopes of critical inertial roll motion which may ocur at neutral aileron deflections $\delta_a = 0$. These motions are characterized by a sharp rise of normal and lateral load factors and by an opposite response to φ and δ_R control deflections. The aileron deflection at positive stabilizer values φ in counter rotation does not result in its termination.

One of the critical motion investigation problems such as inertial roll or spin, is to assess the possibility of the aircraft recovery from these regimes and the termination of its roll by a sequence of control deflections.

In some cases this problem can be solved using quasistatic control with rather slow control deflections.

This procedure may be formalized by minimizing a quadratic functional of the kinetic energy type

$$W = \tfrac{1}{2}MV^2(\alpha^2 + \beta^2) + \tfrac{1}{2}(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2) \quad (17)$$

provided that phase state vector $X = (\alpha, \beta, \omega_x, \omega_y, \omega_z)^T$ and control vector U should satisfy motion stationarity conditions (11).

To reduce the functional W the direction of changing the control vector U by gradient descent method is defined as follows

$$dU = k\left[\frac{\partial W}{\partial X}\left(\frac{\partial F}{\partial X}\right)^{-1}\frac{\partial F}{\partial U} - \frac{\partial W}{\partial U}\right] \quad (18)$$

where k is the positive constant.

According to (18) with limitations of (11) the control may be changed by the continuation method, where k is chosen as a scalar control parameter.

For this example controls minimizing functional W are calculated for two initial values in the inertial roll (fig. 3).

Trajectory I avoids all bifurcation singularities and reduces smoothly the motion parameters to zero values. The variation of φ is not characteristic of ordinary piloting procedure, for the normal load factor in inertial roll is already very great. The aileron deflections coincide with aircraft roll direction which, either, is unnatural for the pilot.

Trajectory II (fig. 4) arrives at the bifurcation boundary of inertial roll region, followed by a dynamic jump achieved by integrating motion equations (10) at fixed control, at which opposite direction rotation establisches. A further variation of φ, δ_a by (18) results in zero motion parameters.

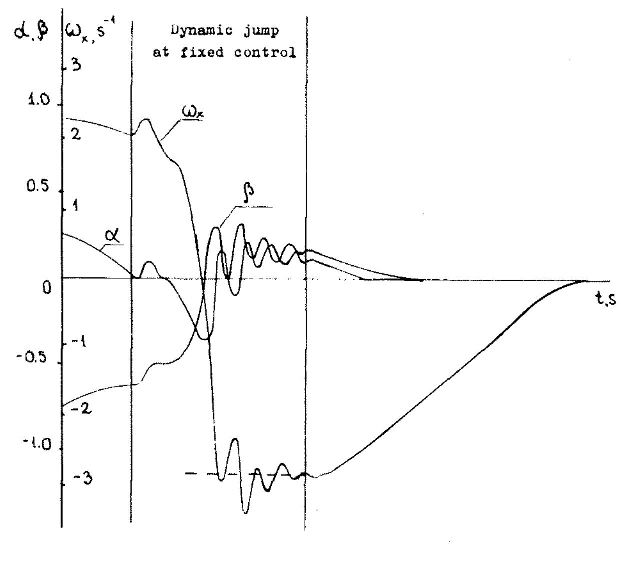


Figure 4.

A similar procedure may be used to calculate control deflections for an aircraft spin recovery.

IV. CALCULATION OF PERIODIC SOLUTIONS

The calculation of periodic solutions of motion equations (10) and their stability analysis may be performed by Poincare point mapping method[7, 8].

To use the point mapping method a secant manifold, i. e. an S – hyperplane in space $X \in R^n$ of a unit codimension ($\dim S = n - 1$), is specified. The hyperplane is at least twice crossed by a closed and proximate phase trajectories (fig. 5). The point mapping

$$\overline{X} = P(X), \text{ where } \overline{X}, X \in S$$

is numerically integrated using motion equations (10)

$$\overline{X} = \int_0^T F(X, U)dt + X$$

where T is defined from the second crossing of S – hyperplane.

The dynamic system structure is one-to-one governed by the structure of point mapping generated by it on the secant S – hyperplane[7].

If X_* is a fixed mapping point on the secant S – hyperplane, then phase trajectory emanating from it returns to the point X_* after a finite time peroid. Thus, closed phase trajectories (closed orbit) of the dynamic system (10) correspond to fixed points of mapping. As a rule, a preliminary analysis of the solution structure (10), in particular, of stationary ones, permits to choose a successful secant S – hyperplane.

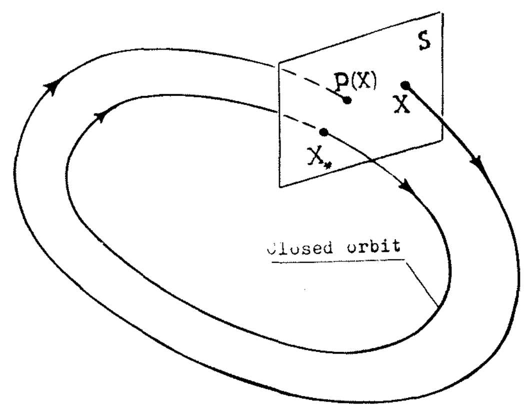


Figure 5.

The search of the fixed point X_* of mapping P on the S – hyperplane is reduced to a solution of nonlineare equations specified by mapping P

$$X_* - P(X_*, U) = 0$$
$$X_* \in S \quad (19)$$

The continuation method is used equally efficiently to solve eqs. (19).

The phase space dimension of the mapping is $n - 1$, but when the Jacobian is calculated, $\partial\overline{X}/\partial X$ is performed by varying X – point in all its n coordinates which enable it to leave S – hyperplane. This results in a $\partial\overline{X}/\partial X$ matrix degeneration. Note that the Jacobian of the nonlinear system (19) does not become degenerate here.

The orbital stability of the periodic solution (10) is directly related to that of respective fixed points[7]. The periodic solution will be stable, if all Jacobian eigenvalues of the point mapping

$$H = \frac{\partial \bar{X}(X, U)}{\partial X} \tag{20}$$

lie within a unit circle. These eigenvalues coincide with multiplicators from Floquet – Lyapunov therory[7].

Just as the Jacobian eigenvalues of (10) determine the structure of phase trajectoris in the vicinity of a stationary point, the Jacobian eigenvalues of point mapping (20) define the benaviour of phase trajectories in the proximity of the period ic solution of eqs. (10).

A real positive eigenvalue makes the stationary point unstable, the behaviour of the trajectories becoming saddle-type. The transition of a pair of complex conjugate numbers to the right semiplane results in an instability of oscillatory type (divergence). In this case the generation of periodic solutions called Hopf bifurcation is possible[4].

Similarly, a positive eigenvalue more than 1 in matrix H (20) causes a stability loss in the periodic solution and its structure becomes saddle-type. The arrival of the pair of complex conjugate eigenvalues behind the unit circle causes an oscillatory stability loss in the periodic solution; here, the bifurcation of a higher order, i. e. the generation of a stable attracting torus manifold is possible (fig. 6)[4, 8].

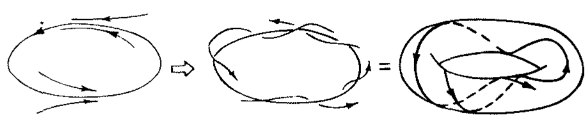


Figure 6.

V. BIFURCATIONS IN SPIN

Fig. 7 shows a relationship of the aircraft angle-of-attack in equilibrium spin with different rudder and aileron deflections at $\varphi = -10°$.

There are two solution groups for a flat ($\alpha \sim 65–75°$) and a steep ($\alpha \sim 40–50°$) spin with stationary parameters. Spin conditions are shown to exist not at all control deflections.

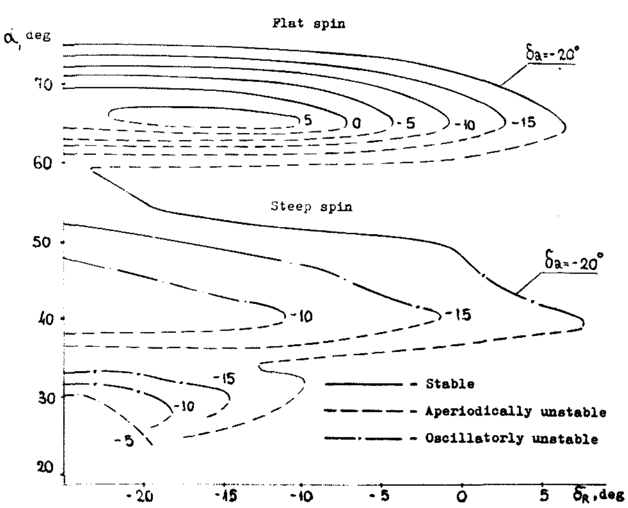


Figure 7.

The local analysis of the stationary solution stability in the branch $\varphi = -10°$, $\delta_a = -20°$, δ_R = var for a steep spin shows that the complex pair of eigenvalues with $\Omega \sim 1.8\ s^{-1}$, crosses the right semiplane at $\delta_R = 2°$, and at $\delta_R = 6.2°$ they become stable with $\Omega \sim 1.4\ s^{-1}$; and at $\delta_R = 6.5°$ occurs an oscillatory mode dedamping with lower $\Omega \sim 0.8\ s^{-1}$.

Using the continuation method with varying δ_R in the vicinity of this stationary solution branch the periodic solutions of motion equations (1)–(8) were obtained which are clearly seen in the plane (β, δ_R) of (fig. 8) where sideslip angle amplitudes (maximum and minimum values) are plotted in respective closed orbits. The plot shows different cycle stabilities governed by H-matrix eigenvalues (20).

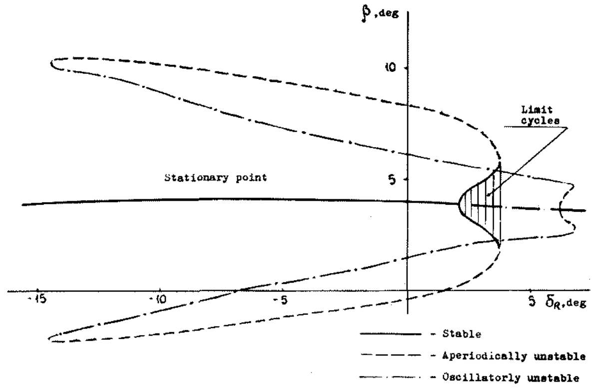


Figure 8.

Within a great deflection range of $\delta_R = (-15°, 2°)$ there are two closed orbits, one being a saddle-type, and the other oscillatorly unstable, as well as a stable stationary point.

The stationary point at $\delta_R = 2°$ becomes oscillatorly unstable; and there appear limit cycles, annihilated when combined with saddle-type closed orbits at $\delta_R = 3.7°$.

Oscillatorly unstable closed orbits annihilated when combined with saddle-type cycles at $\delta_R = -14.5°$ and $\delta_R = 6.7°$. The second branch of saddle-type cecles is generated at $\delta_R = 6.2°$, when stationary point eigenvalues return to the left semiplane.

Thus, the continuation method made it possible to plot a whole family of phase trajectories with different stabilities using δ_R – parameter.

Limit cycles correspond to oscillatory aircraft motions. Saddle-type trajrctories can not physically be realized but they play an important role in forming a stability region of a locally stable stationary point.

Fig. 9 and 10 present a family of limit cycles and saddle-type closed orbits projected onto the (β, ω_y) and (β, ω_x) parameter plane.

There are markers in fixed points of the mapping on a secant hyperplane to determine a continuous trajectory obtained by the continuation method.

As mentioned above, oscillatorly unstable closed orbits may result in an invariant torus[4, 8]. This can be realized with a certain phase space structure of (10).

Fig. 11 shows time history of transients obtained by integrating full motion equations (1)–(8) in the proximity of stationary points, one of which is oscillatorly unstable, and the other a saddle-type point.

Variations of α, β, ω_y parameters are of unsteady oscillatory nature. From a local analysis of the stationary point stability a small oscillation period $T \sim 4$ s corresponds to a natural unstable mode. The integration for a long time period of ~ 160 s showed that the parameter

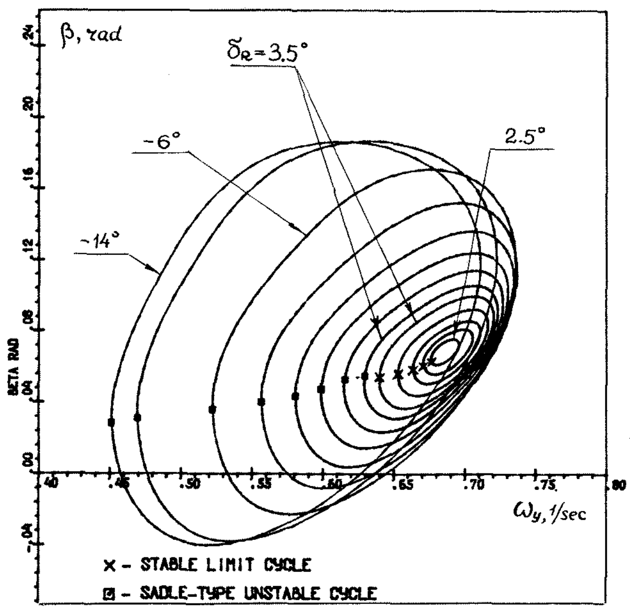


Figure 9.

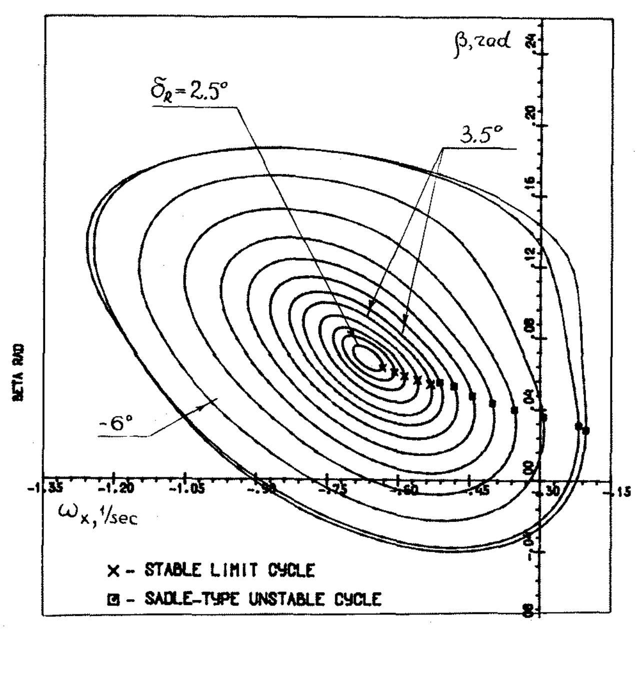


Figure 10.

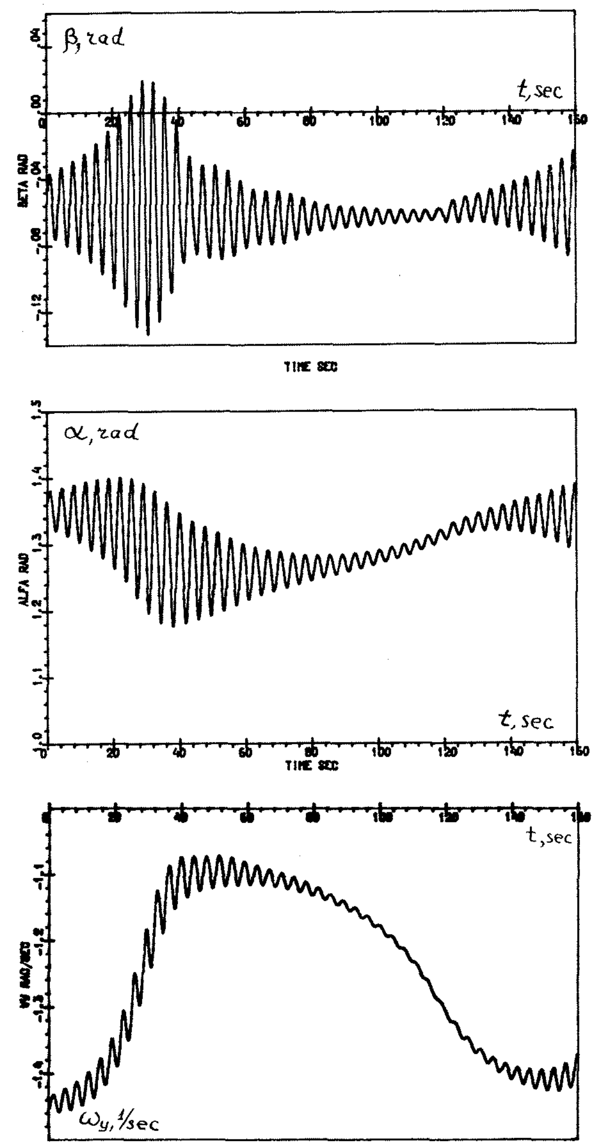


Figure 11.

change behaviour begins to repeat. If one considers short periodes of this transient ~ 20 s (several natural periods), the transient behaviour differs significantly, which indicates that the motion strongly depends on *initial conditions*.

Fig. 12 shows a phase trajectory projections onto (ω_x, β) and (ω_y, β) parameter planes, which points unambiguously to the fact that the phase trajectory encompasses the toroidal surface. The closed orbit inside the torus which is oscillatory unstable has been calculated. It is denoted by markers in fig. 12.

Such invariant manifolds may be responsible for a number of complex aperiodic beating modes observed in aircraft spin flight tests[13].

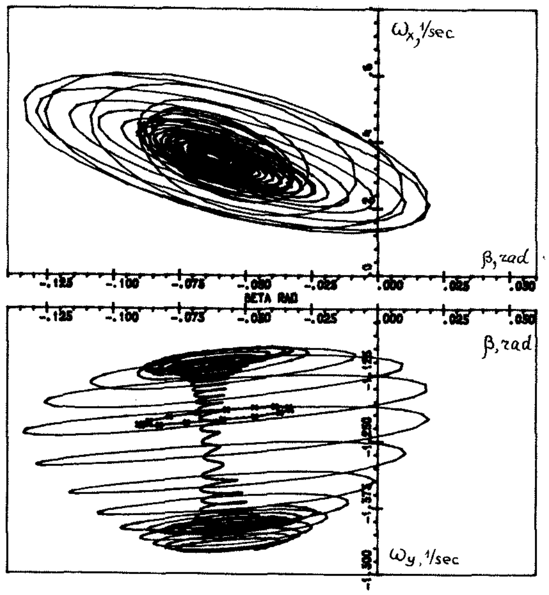


Figure 12.

CONCLUSIONS

A general investigation technique for critical aircraft flight regimes is presented using an analysis of quasiequilibrium motions and their stability. The procedure is based on a qualitative theory of nonlinear dynamic systems, differential geometry, bifurcation analysis and catastrophe theory as well as on efficient numerical calculations for equilibrium stationary and oscillatory motions, their stability and bifurcations in control.

The numerical procedure of the continuation method is refined to provide for the convergence and crossing of complex bifurcations.

A control search alforithm for aircraft recovery from critical envelopes is suggested.

The efficiency of the Poincare point mapping method in combination with the continuation method is shown by a specific example of spin conditions to calculate and analyze the periodic solution stability of aircraft motion equations.

The posibility of complex aperiodic aircraft oscillations in spin associated with an invariant toroidal manifold of phase trajectories is presented. This behaviour is often observed for a spinning aircraft.

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