EVALUATION OF SINGULARLY PERTURBED PURSUIT-EVASION GAMES

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Abstract

A horizontal interception problem, formulated as a pursuit-evasion game, is studied in order to evaluate the accuracy of suboptimal feedback strategies obtained by singular perturbation technique. The exact solution of the problem is generated by an algorithm based on differential dynamic programming. The aircraft models used in the study are of realistic aerodynamical and propulsion characteristics. Results indicate that for initial ranges larger then 8-10 turning radii the suboptimal strategies provide a reasonable accuracy.

1.Introduction

The aerial combat is known to be a very complex nonlinear process and in the course of its analysis many difficulties are encountered. One class of interesting air combat problems is the air-to-air interception, which can be formulated as a zero-sum pursuit-evasion game. This class of differential games was more successfully analysed in the past by using a multiple-time-scale singular perturbation technique (SPT) as described in Refs 1-3. The most important feature of the SPT approach is that the control strategies are obtained in an explicit feedback (closed-loop) form. These calculations can easily be performed on-line by an airborne computer onboard of aircraft and missiles. This solution is, however, only a suboptimal (zeroth-order) approximation of the exact solution and its accuracy has not been fully evaluated until now. The exact solution has to be obtained by numerical methods like those described in Refs 4 and 5. These algoritms provide open-loop controls by iterative off-line computations. Computer programs based on the method of Ref 4 seem to be the best practical ones today. The purpose of this paper is to evaluate the accuracy of the SPT approximations of Refs 1 and 2 by comparison to the correct optimal solution obtained by the method of Ref 4.

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2.The Differential Game Problem

The horizontal air-to-air interception problem of interest is formulated as a perfect information, fixed final time (t_f), zero-sum pursuit-evasion game with the cost function

$$V(x_o) = r_f \qquad (1)$$

where x_o is the initial state vector at time t_o and r_f is the distance of separation between the pursuer and the evader at "t_f". Such a formulation is selected because it suits the DDP algorithm of Ref 4 best.

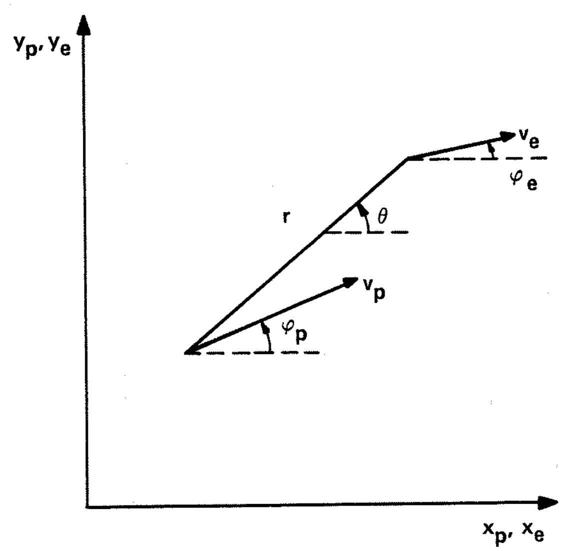

Figure 1. Interception geometry

The geometry of the interception is shown in Fig 1 and the equations of motion, are

$$\dot{r} = -v_p \cos(\varphi_p - \theta) + v_e \cos(\varphi_e - \theta) \quad , r(0)=r_o \qquad (2)$$

$$\dot{\theta} = (-v_p \sin(\varphi_p - \theta) + v_e \sin(\varphi_e - \theta))/r \quad , \theta(0)=\theta_o \qquad (3)$$

$$\dot{v}_p = \phi_p(v_p, \zeta_p) - G_p(v_p)\, u_p^2 \quad , v_p(0)=v_{po} \qquad (4)$$

$$\dot{v}_e = \phi_e(v_e, \zeta_e) - G_e(v_e)\, u_e^2 \quad , v_e(0)=v_{eo} \qquad (5)$$

$$\dot{\varphi}_p = u_p \quad , \varphi_p(0) = \varphi_{po} \qquad (6)$$

$$\dot{\varphi}_e = u_e \quad , \varphi_e(0) = \varphi_{eo} \qquad (7)$$

where ζ_p, ζ_e are the respective thrust controls bounded by

$$0 \leq \zeta_i \leq 1 \quad i = p,e \tag{8}$$

and the turning rates u_p and u_e are constrained by

$$|u_i| \leq \hat{u}_i(v_i) \quad i = p,e \tag{9}$$

Detailed derivation of Eqs (4)-(9) is given in the Appendix.

We introduce the adjoint variables (the gradient components of the optimal cost) V_r, V_θ, V_{v_p}, V_{v_e}, V_{φ_p}, V_{φ_e} and form the Hamiltonian

$$H = V_r \dot{r} + V_\theta \dot{\theta} + V_{v_p} \dot{v}_p + V_{v_e} \dot{v}_e + V_{\varphi_p} \dot{\varphi}_p + V_{\varphi_e} \dot{\varphi}_e + \mu_p (|u_p| - \hat{u}_p) + \mu_e (|u_e| - \hat{u}_e) \tag{10}$$

where μ_p and μ_e are multipliers[4]. The adjoint variables have to satisfy the following differential equations with the respective transversality conditions

$$\dot{V}_r = - H_r \quad , V_r (t_f) = 1 \tag{11}$$
$$\dot{V}_\theta = - H_\theta \quad , V_\theta (t_f) = 0 \tag{12}$$
$$\dot{V}_{v_p} = - H_{v_p} \quad , V_{v_p}(t_f) = 0 \tag{13}$$
$$\dot{V}_{v_e} = - H_{v_e} \quad , V_{v_e} (t_f) = 0 \tag{14}$$
$$\dot{V}_{\varphi_p} = - H_{\varphi_p} \quad , V_{\varphi_p} (t_f) = 0 \tag{15}$$
$$\dot{V}_{\varphi_e} = - H_{\varphi_e} \quad , V_{\varphi_e} (t_f) = 0 \tag{16}$$

The optimal saddle-point control strategies are obtained from the min-max (max-min) of the separable Hamiltonian Eq (10) subject to the constraints (8) and (9)

$$\zeta_p, u_p, \zeta_e, u_e = \arg \min_{\zeta_p, u_p} \max_{\zeta_e, u_e} H = \arg \max_{\zeta_e, u_e} \min_{\zeta_p, u_p} H \tag{17}$$

Eqs (2)-(17) express a nonlinear two point boundary value problem which has to be solved numerically by some iterative method as of Refs 4 and 5. The solution is obtained in an open-loop form. For a "real time" airborne implementation a suboptimal but closed-loop (feedback) solution may be of advantage. Such an approximation can be obtained by using a multiple-time-scale forced singular perturbation approach described in detail in Refs 1-3.

3. Singular Perturbation Technique

In Refs 1 and 2 it was shown that the nonlinear TPBVP of Eqs (2)-(17) can be solved in a closed form if a time scale separation can be assumed to exist between the state variables of the problem. Since such time scale separation is never perfect the analytical solution of Refs 1 and 2 expressed in a feedback form, is only a zeroth-order approximation of the exact solution.

If the interception starts at relatively long ranges compared to the best turning radius of the aircraft,"R_{min}",the rotation of the line of sight "θ" is very slow compared to the aircraft turning rates. Therefore in medium-range interceptions relative geometry varies indeed were slowly. Based on this observation in Refs 1 and 2 "r" and "θ" are considered as the slowest variables. The time-scale separation between relative geometry and aircraft turning dynamics can be expressed by the ratio

$$\varepsilon_g = (R_{min}/r_o) \tag{18}$$

which is called the geometrical singular perturbation parameter. In Ref 1 it is assumed that velocity dynamics are slower than turning dynamics. This SPT model, called Model A, is based on the observation that the longitudinal acceleration of an airplane are much smaller than the lateral accelerations used for turning maneuvers. Based on this assumption the optimal strategies of both aircraft (pursuer and evader) can be expressed by the following simple control laws.

$$\zeta^o = 1 \tag{19}$$

$$u^o = u_s(v)\sqrt{\frac{2\,v}{v_{max}-v}}\,\sin\left(\frac{\theta-\varphi}{2}\right) \tag{20}$$

where $u_s(v)$ is the sustained turning rate achieved whith T = D.

This control strategy has an apparent deficiency by always requiring full thrust. It is known, however, that in some cases zero thrust can be optimal. In such a case the turning maneuver is maximal and consequently the longitudinal deceleration can be important. This effect can be expressed by a different time scale separation SPT model assuming that speed dynamics is faster than turning. Such a model, called in Ref 2 "Model B" in contrast with the previous "Model A" of Ref 1, allows optimal zero thrust and predicts in closed-loop when the throttle has to be switched to full thrust. This model is useful for initial conditions of high speed and large turning requirement. When full throttle is switched on the assumptions of Ref 1 became valid and for further maneuvering the strategy of "Model A" has to be used.

Since the real time scale separation betwen speed dynamics and turning is certainly not a perfect one, the accuracy of the approximations obtained either by "Model A" or by "Model B" have to be carefully evaluated by a comparison to the exact game solution obtained numerically.

4. The Numerical Optimization Method

The background to the DDP method is described in Ref 4 , but a short description of the procedure will be given here. The program used for this particular problem is the same as in Ref 5.

The iterative computational procedure is initiated by assuming nominal control histories $\bar{\zeta}_p(t)$, $\bar{u}_p(t)$, $\bar{\zeta}_e(t)$, $\bar{u}_e(t)$, subject to the constraints Eqs (8), (9) generating corresponding nominal trajectories by integrating Eqs (2)-(7) forward in time from t_o and up to t_f. Store the nominal controls and trajectories as well as the calculated cost value Eq (1).

The adjoint variables are evaluated along the nominal trajectory. This means that the end point boundary are now known and the adjoint Eqs (11)-(16) can be integrated backward in time, while calculating the optimal controls from Eq (17). These new control histories must be stored. In parallel with the integration of the adjoint equations a predicted cost change will be calculated. Assume that the Hamiltonian, Eq (10), is separable in the components of each control variable,

$$H = \sum_{i=1}^{4} H^i(\bar{x}, u_i, V_x; t) \qquad (21)$$

where $u_1 = \zeta_p$, $u_2 = u_p$, $u_3 = \zeta_e$, $u_4 = u_e$ furthermore $\bar{x}$ and V_x represent the state and adjoint vectors respectively. The differential equation for the predicted cost change will then be

$$\dot{a}_i(t) = H^i(\bar{x}, \bar{u}_i, V_x; t) - H^i(\bar{x}, u_i^*, V_x; t),$$

$$a_i(t_f) = 0, \quad i = 1, 2, 3, 4 \qquad (22)$$

The predicted cost change is obtained at t_o as

$$a(t_o) = \sum_{i=1}^{4} a_i(t_o) \qquad (23)$$

Apply the new control histories and integrate Eqs (2)-(7), subject to the constraints, forward in time generating a trajectory which should be better than the nominal one. Also calculate the new cost value and form the cost change, ΔV, by subtracting the nominal cost value from the new one obtained.

By comparing $a(t_o)$ with ΔV we will obtain comprehension of the convergence status of the iteration just fulfilled. If the status is not satisfactory, which is very common, a convergence control technique must be applied.

Convergence control

The reason for bad convergence is too large changes in the optimization of the controls. These changes can be restricted by introducing penalty terms in the Hamiltonian, Eq (10), which then are used in Eq(17). The modified Hamiltonian will then be

$$H = \sum_{i=1}^{4} (H^i (\bar{x}, u_i^*, V_x; t) \pm C_i (\bar{u}_i - u_i^*)^2) \qquad (24)$$

where the + is for the minimizer and - for the maximizer and C_i is a convergence control parameter (CCP).This technique means that the contribution from a particular u_i^* is strongly related to the cost change by $a_i(t_o)$. Accordingly the effects of an individual control variable can be observed and the corresponding C_i can be used to prevent bad influence or to speed up the convergence for this particular control variable. It is very important to have this device, especially when the number of control variables is large. Otherwise it would be very difficult to obtain practical acceptable convergence of such an iterative procedure. How to use the CCP technique see Ref 4. The CCP-technique covers singular control problems, which is the case for the throttle setting, see Eq (A-8) which gives H^i linear in ζ. Additionally, a proper choice of C_i will always make it possible to attain H_{uu} positive definite with respect to the pursuer and negative definite with respect to the evader.

5. Numerical Examples

The interception scenario, depicted in Fig 2, is the following. A fighter-bomber (the evader) on an air-to-ground mission, flying at low altitude and at a high subsonic speed, is detected by a high performance interceptor (the pursuer). The pursuit-evasion game starts when the fighter-bomber becomes aware of the situation. Then it drops its bomb load an initiates an evasive maneuver.

The characteristic parameters of the two participating aircraft are summarized in Table 1.

Aircraft	Pursuer	Evader
Weight (kg)	9000	20000
Wing area (m^2)	30	50
Thrust to weight ratio	0.91	0.68
Maximum speed at h=0 (m/s)	419.8	339.2
Maximum load factor (g's)	6	6
Maximum lift coefficient	1.32	.9

Table 1. Aircraft parameters

Since the maximum speed of the interceptor is higher, capture (defined by reaching the range required for effective missile firing) always takes place. In this case the objective of the evader should be either to maximize the time of capture or, as in the present formulation, to maximize the separation distance for a given (fixed) final time.

The aerodynamics and thrust models of both aircraft are derived from experimental data and expressed by curve fitting as polynomials and other analytical functions. The performance characteristics of the aircraft are best illustrated by their respective horizontal "domain of maneuverability" shown in Fig 3. The dashed lines in this figure indicate the sustained turning performance with full thrust.

A set of six examples with different initial ranges and respective fixed final times were solved numerically by using the DDP algorithm(4) as well as the two SPT models of Refs 1 and 2. The set of values $r_o \in \{8,10,12,14,16,18\}$ km was enough to cover the range of interest for the geometric singular perturbation parameter (Eq 18) related to the best turning radius of the pursuer at t=0 (R_{min}=1878 m). The fixed final times, necessary for a convenient application of DDP, were selected in order to keep the final range $r_f = r(t_f)$ in the order of 1.5 - 3.5 kms, compatible with the firing range of air-to-air missiles. The, results can easily be recalculated using the fact that both aircraft have reached their stationary (maximum) velocity. This makes it possible to estimate the final time given 2.0 km as the separation distance for the reference case (both aircraft using DDP). This final time is then used in the other strategy combinations for the particular r_o considered.

In all examples several similar features were found. All interceptions terminate by a maximum speed "tail-chase", regardless of the method of solution. The pursuer's optimal maneuver (obtained by DDP) is of full trust for the whole interception, therefore SPT "Model B" is not applicable. This is the consequence of the favourable initial conditions (see Fig 2). On the contrary the evader starts its optimal maneuver with zero thrust and swithes to full throttle only a few seconds "t_{sw}" later.

The optimal values of "t_{sw}" derived by the DDP solution are plotted in Fig 4 as a function of "r_o". The SPT approximation for "t_{sw}", obtained from "Model B", is also shown on the same figure.

The accuracy of the SPT approximate feedback strategies for the evader as well as for the pursuer is measured by the relative gain or loss in the final time in order to reach the "optimal" final range "r_f^*", obtained from the exact DDP saddle-point game solution. The error parameter "δ" can be thus defined as

$$\delta = \frac{\Delta t}{t_f} 100 = \frac{r(t_f) - r_f^*}{(v_{p_{max}} - v_{e_{max}})\, t_f} 100 \quad \% \qquad (25)$$

The outcomes of six different games (played with different control strategies) are presented for each set of initial range (r_o) and final time (t_f) in Tables 2a - 2f. The respective error parameters are also plotted as a function of "r_o" for the various SPT strategies in Figs 5-7.

6. Discussion of Results

Let us concentrate first on the accuracy of the SPT strategy (Model A) for the pursuer. Due to the favourable initial conditions the turning requirements is very moderate. For short initial ranges the rotation of the line of sight is important and certainly not negligible compared to the pursuers actual turning rate. Therefore the SPT approximation creates a relatively large error in the performance index. For longer initial ranges, i.e. smaller values of the singular perturbation parameter ε_g defined in Eq (18), the approximation becomes better and better and the value of the accuracy parameter is smaller than 1 % for r_o > 14 km, (see Fig 5). This result confirms a previous accuracy assessment (Ref 6) carried out by a different method for the same SPT "Model A".

The evaluation of the SPT strategies of the evader is more complex. The unfavourable initial conditions require an important "hard" turning maneuver starting with zero thrust. This result obtained from the exact DDP solution justifies use of the SPT approximation "Model B" (Ref 2) As is shown in Fig 4 the SPT analysis overpredicts the value of "t_{sw}" (the time using zero thrust). The turning strategy is, however, very similar to the optimal one. The error in the prediction of "t_{sw}" has different effect for different initial ranges. It is clearly demonstrated by using for a "Model A" (t_{sw} = 0) strategy comparison. For the shortest initial range r_o = 8 km the optimal value is t_{sw}^*=2.4 sec. while "Model B" predicts t_{sw} = 6.6 sec. Consequently the error caused by "Model A" is smaller than the error of "Model B". For all other initial conditions "Model B" performs better and for r_o = 18 km (ε_g= 0.103) the error parameter "δ" is reduced to 1.5 %.

Using SPT strategies for both aircraft the errors partially cancel each other (see Fig 7) and consequently for relativly short initial ranges (ε_g < 0.2 "Model B") the zeroth-order SPT game solution presents a reasonably accurate approximation.

7. Conclusions

The present paper extends the evaluation of suboptimal SPT game strategies for cases where the initial optimal maneuver requires zero thrust. The SPT model allowing such a strategy (Model B) overpredicts the time flown with zero thrust. However, for initial ranges of the order of 8-10 turning radii, the error in the performance index due to a suboptimal SPT feedback strategies is small enough to justify its use for an airborne implementation.

The SPT approximation has an additional merit. Serving as the first guess in the DDP algorithm it reduces the number of iterations required for a converged game solution.

References

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Appendix: Derivation of Eqs (4)-(7)

The longitudinal acceleration of an aircraft in horizontal flight, is given, neglecting the thrust deflection due to the angle of attack by

$$\dot{v} = [T(v,h,\zeta) - D(v,h,n)]/m \qquad (A\text{-}1)$$

where "T" is the maximum available thrust in the given flight conditions "v" is the velocity and "h" is the altitude, "ζ" is the throttle setting parameter controlled by the pilot, "D" is the aerodynamic drag depending on the flight conditions "h,v" and the load factor "n" and "m" is aircraft mass.

The drag force is generally expressed by the nondimensional drag coefficient C_D

$$D(v,h,n) = q\ S\ C_D \qquad (A\text{-}2)$$

where "S" is the aircraft reference wing area and "q" is the dynamic pressure depending on speed and air density $\rho(h)$

$$q = \frac{1}{2}\rho(h)\ v^2 \qquad (A\text{-}3)$$

The drag coefficient consists of two parts, the zero-lift drag coefficient, which is only Mach number dependent, $C_{Do}(M)$ and the induced drag coefficient C_{Di} which also depends on the lift coefficient C_L, given by

$$C_L = \frac{n\ mg}{q\ S} \qquad (A\text{-}4)$$

In the most simple aerodynamic models the induced drag coeifficient is assumed to be proportional to $C_L{}^2$ yielding for horizontal flight at a constant altitude

$$C_D = C_{D_o}(v) + K(v)\ C_L{}^2 = C_D\ (v,n) \qquad (A\text{-}5)$$

In horizontal flight the turning rate of the aircraft, used as a control variable in the present paper, is expressed by

$$\dot{\varphi} \triangleq u = g\sqrt{(n^2 - 1)}\ /v \qquad (A\text{-}6)$$

This control variable is limited either by the maximum structural load factor "n_{max}" or by the maximum value of the lift coefficient "$C_{L_{max}}$" given the maximum turnrate "$\hat{u}(v)$".

Substituting Eqs(A-2)-(A-6) into (A-1) yields the form used in Eqs. (4)-(5), namely

$$\dot{v} = \Phi(\zeta, v) - G(v)\ u^2 \qquad (A\text{-}7)$$

with

$$\Phi(\zeta,v)=[\zeta T(v)-q\ S\ C_{D_o}(v)-K(v)m^2\ g^2/(q\ S)]/m \qquad (A\text{-}8)$$

and

$$G(v) = 2\ K(v)\ m/(S\rho) \qquad (A\text{-}9)$$

The maximum horizontal speed of the aircraft is limited either by the maximum allowable dynamic pressure "q_{max}" or by the equilibrium of thrust and drag. In the model used for the present paper the thrust model is structured in a way that the "q_{max}" limit is automatically respected.

CASE a
R_o = 8.0 km, t_f = 25.8 sec, ϵ = 0.235

E \ δ	DDP	SPT-A
DDP	r_f = 2000 δ = ±0	r_f = 2285 δ = +13.7
SPT-A	r_f = 1885 δ = −5.5	r_f = 2192 δ = +9.2
SPT-B	r_f = 1842 δ = −7.6	r_f = 2058 δ = +2.8

CASE b
R_o = 10.0 km, t_f = 48.1 sec, ϵ = 0.188

E \ δ	DDP	SPT-A
DDP	r_f = 2000 δ = ±0	2153 δ = +4.0
SPT-A	r_f = 1776 δ = −5.8	r_f = 1995 δ = −0.12
SPT-B	r_f = 1804 δ = −5.0	r_f = 1948 δ = −1.4

CASE c
R_o = 12.0 km, t_f = 71.2 sec, ϵ = 0.156

E \ δ	DDP	SPT-A
DDP	r_f = 2000 δ = ±0	r_f = 2099 δ = −1.7
SPT-A	r_f = 1762 δ = −4.1	1893 δ = −1.9
SPT-B	r_f = 1822 δ = −3.1	r_f = 1903 δ = −1.7

CASE d
R_o = 14.0 km, t_f = 94.9 sec, ϵ = 0.134

E \ δ	DDP	SPT-A
DDP	r_f = 2000 δ = ±0	r_f = 2060 δ = +0.8
SPT-A	r_f = 1731 δ = −3.5	r_f = 1819 δ = −2.2
SPT-B	r_f = 1827 δ = −2.3	r_f = 1870 δ = −1.7

CASE e R_o = 16.0 km, t_j = 118.8 sec, ϵ = 0.116		
E \ δ	DDP	SPT-A
DDP	r_f = 2000 δ = ±0	r_f = 2046 δ = +0.5
SPT-A	r_f = 1711 δ = −3.0	r_f = 1777 δ = −2.3
SPT-B	r_f = 1821 δ = −1.9	r_f = 1861 δ = −1.5

CASE f R_o = 180 km, t_f = 143 sec, ϵ = 0.103		
E \ δ	DDP	SPT-A
DDP	r_f = 2000 δ = ±0	r_f = 2031 δ = +0.3
SPT-A	r_f = 1693 δ = −2.7	r_f = +0.3 δ = −2.2
SPT-B	r_f = 1827 δ = −1.5	r_f = 1843 δ = −1.3

Table 2a-2f. Game results for 6 different strategy combinations

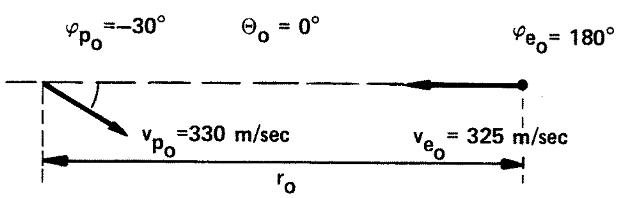

Figure 2. Initial interception conditions

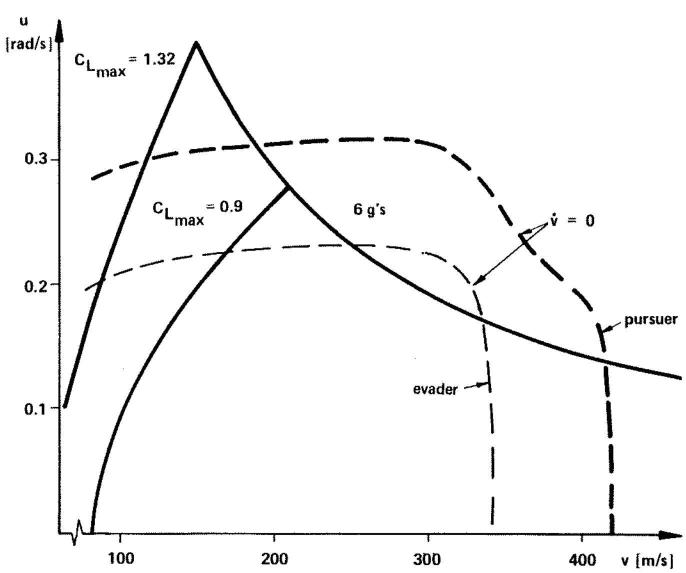

Figure 3. Sustained and limiting turnrate vs velocity

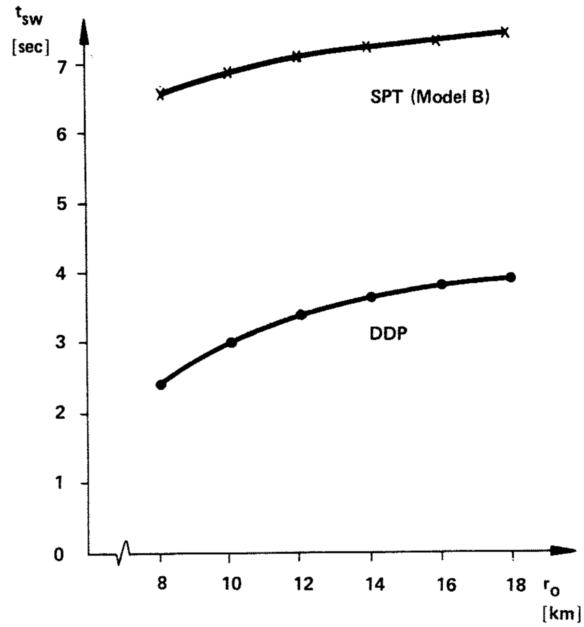

Figure 4. Optimal throttle off vs initial range

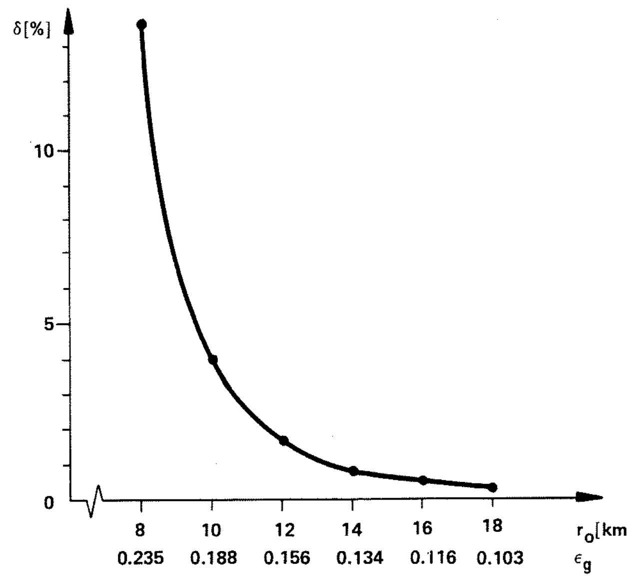

Figure 5. Error parameter due to suboptimal Pursuer strategy

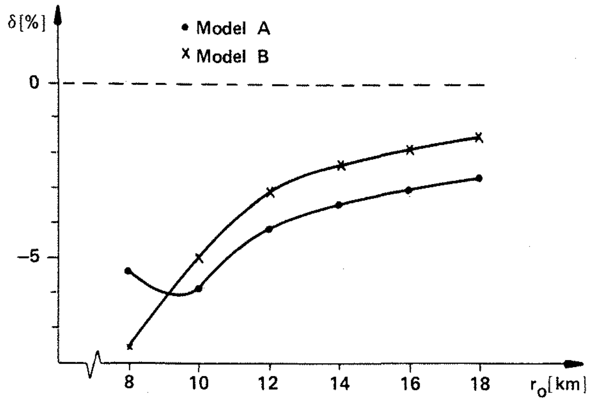

Figure 6. Error parameters due to suboptimal Evader strategies

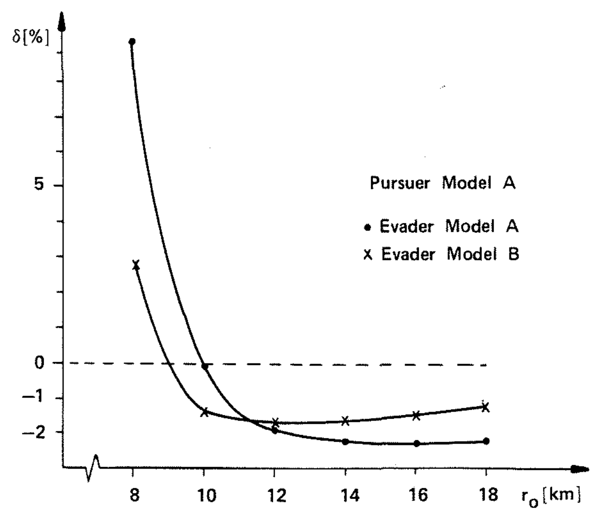

Figure 7. Error parameters in zerothorder SPT game