ICAS-84-3.3.4

SOLUTION OF THE NONLINEAR DYNAMIC PROBLEM OF STRUCTURAL BEHAVIOUR OF A COUPLED MODEL OF AN AIRPLANE WING WITH AN AILERON

M. J. Josifović, Z. V. Bojanić and J. M. Janković
Mechanical Faculty of the University of Belgrade
Belgrade, Yugoslavia

Abstract

The paper describes a procedure of modelling the dynamics of elastic systems in the presence of geometric nonlinearities caused by large displacements of the structure. The formulation of the problem of modelling the dynamics of elastic structure systems is presented in a general form, starting from the variation principle defined on a variable domain with a variable contour, which is a results of geometric nonlinearity.

In the further procedure of approximative modelling of the system, use has been made of the fact, already well-known in practice, that dynamic systems having an elastic structure behave like slightly nonlinear systems, where the small parameter method is used in a modified form. There is also a complete algorithm of synthesis of the reduced dynamic model of an elastic structure system for the case of large displacements.

I. Introduction

The procedure of synthesis of the dynamic model of an elastic structure system accompanied by geometric nonlinearities, as described in the paper, has been conducted in three stages. The first consists in forming a variation principle for the elastic structure under large displacements, that is, with a variable domain and a variable contour defined by the displacements of the structure. This approach to the consideration of the structural dynamics of elastic systems gives a nonlinear matrix formulation of the dynamic model of the system, which is, without introducing certain assumptions, practically useless for further analysis.

The basic assumption, already well-known and accepted in engineering practice, is that the geometric nonlinearity of a system does not change to a high degree the character of its oscillation, i.e. the oscillations of the system remain of approximately harmonic form. Here is introduced the assumption of slow-varying amplitudes and phases with time. To obtain a system of nonlinear ordinary differential equations which determine the amplitudes and phases of system modes the authors make use of a modified variant of the small parameter method.

Since the procedure of applying the small parameter method is extremely difficult in case of large-sized systems, that is, those having a large number of generalized coordinates, a procedure of approximative reduction of the system must be developed first, as described in reference [1]. This procedure makes it possible to synthesize the dynamic model of the system with larger steps of numerical integration.

II. Synthesis of the Nonlinear Dynamic Model of an Elastic Structure

In the procedure of synthesis of the dynamic model of an elastic structure for the case of occurrence of large displacements, i.e. geometric nonlinearities, a start is made either from Hamilton's principle or the principle of the minimum of total potential energy. In the first variant, the functional whose variation is sought consists of three terms, i.e. the work of external volume and surface forces, potential energy of elastic deformations and kinetic energy due to the total displacements of the structure. If the principle of the minimum of total potential energy is applied, the last term does not exist explicitly in the functional, but it is necessary to include the inertial load as well in the total volume forces acting on the structure.

For the case of occurrence of geometric nonlinearities only, the stress-strain relation is still subject to Hooke's law, and therefore the potential energy due to elastic deformations may be written in the known form

$$U = \frac{1}{2}\int_V (\sigma_x \cdot \varepsilon_x + \sigma_y \cdot \varepsilon_y + \dots + \tau_{zx} \cdot \gamma_{zx}) dV \qquad (1)$$

where $\sigma_x, \dots, \tau_{zx}$ are stress components, and $\varepsilon_x, \dots, \gamma_{zx}$ strain components. The work of total external volume and surface forces, including

Copyright © 1984 by ICAS and AIAA. All rights reserved.

inertial forces due to the displacements, may be also expressed in the known form

$$W = \frac{1}{2}\int_V \vec{R}\cdot\vec{u}\,dV + \frac{1}{2}\int_A \vec{F}\cdot\vec{u}\,dA \qquad (2)$$

where $\vec{R}$ is the vector of total volume forces, $\vec{F}$ the vector of external surface forces, and $\vec{u}$ the radius vector of the elastic displacements of an elementary mass of the elastic structure in a body axis coordinate system. Let us suppose that the elastic properties of the material extend over the whole volume of the body. If there are large displacements of the elastic structure, the integration domain and its contour are changed in position with time, and it is therefore necessary to introduce, in the expression for the total potential energy variation, some additional terms which correspond to the variations of the above domain and its contour.

If the following symbols are introduced

$$U = \int_V U'dV \qquad W = \int_V W'dV + \int_A W''dA \qquad (3)$$

the total potential energy variation may be written in the form

$$\delta\Pi = \delta(\int_{V+\delta V}(W' - U')dV) + \delta(\int_{A+\delta A}W''dA) \qquad (4)$$

or in the following expanded form

$$\delta\Pi = \int_V \delta(W' - U')dV + \int_A \delta W''dA + \int_A (W' - U')\delta n\,dA + \int_A \nabla(W''\cdot\vec{n})\delta n\,dA \;. \qquad (5)$$

The additional terms of expression (5) on the surface of contour A of the domain V refers to the variation of the domain and its contour.

The variation of the normal δn is a result of displacement, i.e. a variation of elastic displacements of the structure. The first two terms of relation (5) correspond to the conventional linear theory, which disregards the influence of large displacements. The third and fourth terms refer to the domain variation and contour variation due to an arbitrary variation of the normal δn.

If the displacement of the domain and its contour is equivalent to the displacement of the structure itself, then it is possible to write out the variation of the normal onto the elastic domain contour in the form

$$\delta n = \delta u_n = \delta\vec{u}\cdot\vec{n} \quad . \qquad (6)$$

Upon introducing the matrix relations which define the relations between the elastic displacement of an arbitrary elementary mass and node displacements, as well as relations between strains, that is, stersses and node displacements, given in the form

$$\{u\} = [N]\{q\} \quad \{\varepsilon\} = [B]\{q\} \quad \{\sigma\} = [C][B]\{q\}, \qquad (7)$$

where $\{q\}$ - vector of node displacement, $\{u\}$ - vector of elastic displacement of an elementary mass of the body, $\{\varepsilon\}$ and $\{\sigma\}$ vectors of strains and their corresponding stresses, relation (6) will then have the following form

$$\delta n = \{\delta q\}^T[N]^T\{n\}, \qquad (8)$$

where $\{n\}$ denotes the vector of the normal with respect to the axes of the appropriate coordinate system of the body. The fourth term of expression (5), considering the first term of relations (7), is given in its final matrix form

$$\int_A \nabla(W''\vec{n})\delta n dA = \{\delta q\}^T[Q(\phi)]\{q\} \qquad (9)$$

where the matrix $[Q(\phi)]$ assumes the following form

$$[Q(\phi)] = \frac{1}{2}\int_A [N]^T\{n\}\{\nabla\}^T(\{n\}\{\phi\}^T[N])dA \quad . \qquad (10)$$

Relation (10) defines the additional matrix of rigidity which is a result of geometric nonlinearities. The matrix coefficients are proportional to the external surface forces acting on the structure observed. In the case of the action of a time-nonvarying load $\{\phi\}$, or if it is an arbitrary function of time, the additional matrix of rigidity is a constant or time-dependent matrix. In the given cases this matrix does not make the dynamic model of the elastic structure nonlinear, which is very important for the further analysis of the model itself.

Before we begin to consider the third term of expression (5), let us determine the magnitude of volume forces arising from inertial loading. The assumption is that the observed elastic body moves freely through space, where the mentioned motion is described by the motion of the appropriate coordinate system of the observed body and by elastic displacements in relation to the body axis coordinate system.

Without going into details of obtaining the following relation, it is possible to express the volume load, due to the arbitrary motion of the appropriate coordinate system of the body defined by the vector of translational speed {V} and the vector of angular speed {ω}, in the form

$$
\begin{aligned}
-\frac{1}{\rho}\{R\} &= \{\dot{V}\} + [N]\{\ddot{q}\} + [F(\omega)]\{V\} + \\
&+ 2[F(\omega)][N]\{\dot{q}\} + [G(\omega)]\{r\} + [R]\{\dot{\omega}\} + \\
&+ [G(\omega)][N]\{q\}
\end{aligned} \tag{11}
$$

where the above-mentioned inertia matrices are given in the form

$$
[R] = \begin{bmatrix} o & z & -y \\ -z & o & x \\ y & -x & o \end{bmatrix} \quad [F(\omega)] = \begin{bmatrix} o & -\omega_z & \omega_y \\ \omega_z & o & -\omega_x \\ -\omega_y & \omega_x & o \end{bmatrix}
$$

$$
[G(\omega)] = \begin{bmatrix} -(\omega_y^2 + \omega_z^2) & \omega_x \ \omega_y & \omega_x \ \omega_z \\ \omega_x \ \omega_y & -(\omega_x^2 + \omega_z^2) & \omega_y \ \omega_z \\ \omega_x \ \omega_z & \omega_y \ \omega_z & -(\omega_x^2 + \omega_y^2) \end{bmatrix} \ . \tag{12}
$$

Considering relations (2), (3), (5) and (11), we can obtain the following form of the third term of relation (5) as follows

$$
\begin{aligned}
\int_A (W' - U')\delta n \ dA &= \\
&= \{\delta q\}^T(([D_1(R)] - [D_2(q)] - [D_3(V,\omega)])\{q\} - \\
&- [D_4(q)]\{\dot{q}\} - [D_5(q)]\{\ddot{q}\}),
\end{aligned} \tag{13}
$$

where the matrices are given by the relations

$$
\begin{aligned}
[D_1(R)] &= \frac{1}{2}\int_A [N]^T\{n\}\{R_o\}^T \ dA \\
[D_2(q)] &= \frac{1}{2}\int_A [N]^T\{n\}(\{q\}^T[B]^T[C][B] + \\
&\quad + \rho\{q\}^T[G(\omega)][N]) \ dA \\
[D_3(V,\omega)] &= \frac{1}{2}\int_A \rho[N]^T \{n\}(\{\dot{V}\}^T + \{V\}^T[F(\omega)]^T + \\
&\quad + \{r\}^T[G(\omega)]^T + \{\dot{\omega}\}^T[R]^T) \ dA \\
[D_4(q)] &= \int_A \rho[N]^T\{n\}\{q\}^T[F(\omega)][N] \ dA \\
[D_5(q)] &= \frac{1}{2}\int_A \rho[N]^T\{n\}\{q\}^T[N] \ dA
\end{aligned} \tag{14}
$$

The vector R_o represents the vector of external volume forces.

Relation (13) defines the additional nonlinear terms of the system which are a result of geometric nonlinearities. All the matrices given in expressions (14), which are dependent on the vector of node displacements of the structure, represent their linear function.

The first two terms of relation (5), after being expanded by means of expressions (7) and (11), receive the final form

$$
\begin{aligned}
\int_V \delta U' \ dV &= \{\delta q\}^T[K]\{q\} \\
\int_V \delta W' \ dV &= \{\delta q\}^T (\{Q_o\} + \{T(V,\dot{V},\omega,\dot{\omega})\} + \\
&+ [E(\omega,\dot{\omega})]\{q\} + [E(\omega)]\{\dot{q}\} + [E]\{\ddot{q}\}) \\
\int_A \delta W'' \ dA &= \{\delta q\}^T\{Q\}
\end{aligned} \tag{15}
$$

where the coefficient matrices are given in the form

$$
\begin{aligned}
[K] &= \int_V [B]^T \ [C][B] \ dV \\
\{Q_o\} &= \int_V [N]^T\{R_o\} \ dV \\
[E] &= -\int_V \rho[N]^T[N] \ dV \\
[E(\omega)] &= -2\int_V \rho[N]^T[F(\omega)][N] \ dV \\
[E(\omega,\dot{\omega})] &= -\int_V \rho[N]^T([G(\omega)] + \frac{d}{dt}[F(\omega)])[N] dV \\
\{T(V,\dot{V},\omega,\dot{\omega})\} &= -\int_V \rho[N]^T(\{\dot{V}\} + [F(\omega)]\{V\} + \\
&\quad + [F(\omega,\dot{\omega})]\{r\}) \ dV \\
\{Q\} &= \int_A [N]^T\{\phi\} \ dA \ .
\end{aligned} \tag{16}
$$

On the basis of relations derived here, the variational principle corresponding to relation (5) may be written in its final form, where the equilibrium conditions have the following matrix nonlinear form

$$
\begin{aligned}
&(-[E] + [D_5(q)])\{\ddot{q}\} + (-[E(\omega)] + [D_4(q)])\{\dot{q}\} + \\
&+ ([K] - [E(\omega,\dot{\omega})] - [Q(\phi)] - [D_1(R)] + \\
&+ [D_2(q)] + [D_3(V,\omega)])\{q\} = \\
&= \{Q_o\} + \{Q\} + \{T(V,\dot{V},\omega,\dot{\omega})\} \ .
\end{aligned} \tag{17}
$$

The dynamic model of the elastic structure system, given by relation (17), is nonlinear and therefore extremely unsuitable and complicated for further analysis, on which basis one can conclude that it is, in the given form, practically useless for the case of large-sized systems. However, the text below develops an approximative algorithm to analyze the nonlinear dynamic model obtained, which is based on the character of behaviour of real structures.

As it is well known, real structures behave, in the case of large deflections of a wing, approximately equivalently to the behaviour found in small displacements, i.e. a system of structure with large displacements may be considered as slightly nonlinear. It is on this assumption that the further presentation of this investigations is based.

If the given assumption is to satisfy the form of system (17) completely, it is also necessary to satisfy the following condition of coupling the external dynamics of the body axis coordinate system and the structural dynamics itself, which consists in the fact that the motion of an airplane does not exert an essential influence on the change in system parameters, but only on the particular integral which represents the structure deflection under the action of primary inertial load. This assumption is well known in practice.

The first step in the algorithm of analysis of the dynamic model of an elastic structure system, as given in relation (17), consists in determining the particular integral of the system which corresponds to the linearized model of system (17) for the case of small displacements. Such a linearized model, already quite usual in practice, is obtained when all matrices of the system are defined for the state $\{q\} = 0$, as well as $\{V\} = \text{const.}$ and $\{\omega\} = 0$, that is, when the transient inertial forces equal zero.

To the dynamic model, synthesized in this manner, it is necessary to apply the procedure of dynamic reduction of system [1]. This considerably simplifies the further procedure, since the reduction procedure substantially lowers the order of the system.

In addition to the given assumptions, system (17) must also satisfy the assumption on a slow influence of geometric nonlinearities and primary inertial load upon the change of phase and amplitude characteristics of the system. The slower the influence, the larger step may be used in the numerical integration of the system of conditional differential equations, which will be given in the further text of this paper. Practical considerations show that the amplitudes and phases may be regarded as constant during the time which corresponds to the period of the slowest mode. With structures characterized by greater rigidity the integration step may be taken as the time corresponding to the product of several slowest periods of the system.

The given procedure is used to simulate the parameters of the system, which actually means the simulation of changes in the phases and amplitudes of the reduced modes, and not a direct simulation of the motion of the system itself, which may be considered as practically unfeasible since the system parameters must be calculated in real time several hundred or even thousand times more than usual, because the largest integration step must be adopted as several times smaller than the smallest period of the system, which is practically impossible of accomplishing for multidimensional systems, that is, for high frequencies. The procedure of a previous reduction of the system may facilitate the application of the above procedure to a certain degree, but the outcome is relatively small when compared with the procedure described in this paper.

III. Reduction of the System

Let the linearized form of system (17) be given in the matrix form

$$-[E]\{\ddot{q}\} + [K]\{q\} = \{Q_0\} + \{Q\} \; . \tag{18}$$

If the procedure of dynamic reduction of the system as applied to the dynamic model (18), we obtain the following transformation of generalized coordinates of the system

$$\{q_2\} = -L\{q_1\} \tag{19}$$

where the matrix $[L_1]$ is given in the form

$$[L'] = \begin{bmatrix} L_1 \\ \hline L_2 \end{bmatrix} , \qquad (20)$$

while matrix $[L']$ represents a solution of the Riccati matrix algebraic equation

$$[A_{21}] - [A_{22}][L'] - [L'][A_{12}][L'] = 0 \qquad (21)$$

where the system matrix and the external forces matrix are rearranged in the following manner

$$\left[\begin{array}{c|c} 0 & -E^{-1}K \\ \hline I & 0 \end{array}\right] \leftrightarrow \left[\begin{array}{c|c} 0 & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$$

$$\left[\begin{array}{c} E^{-1} \\ \hline 0 \end{array}\right] \leftrightarrow \left[\begin{array}{c} B_1 \\ \hline B_2 \end{array}\right] .$$

The reduced dynamic model of the system given by the dynamic model (18) may also be written in the form

$$\{\ddot{q}_1\} = -[A_{12}][L']\{\dot{q}_1\} + [B_1](\{Q_0\} + \{Q\}) . \qquad (22)$$

The dynamic model given by expression (22) is the reduced dynamic model of an elastic structure under the action of an external reduced load, where the total damping of the system is equal to zero. Consequently, the integral of the system which represents its motion may be found in the form of sum of the particular and general integral of the system which satisfy the given initial conditions.

Let us go back to the system given by expression (17). The above reduced dynamic model (22) is the first approximation to the integral of system (17). In this paper we are not going to spend time discussing the problem of convergence of this procedure, considering that in practice, as well as in reference [2], it may be achieved provided that the given convergence conditions are satisfied in the reduction of the system, which may be done by a procedure of scaling the system matrices.

IV. Approximation of the System by the Method of Small Parametar

We must first seek the approximate integral of system (17), knowing the integral of system (18), i.e. of the unreduced linearized model of system (17).

Let us assume that the particular integral, which is a result of the action of external volume and surface forces, remains approximately unchanged due to the effects of geometric nonlinearities. This assumption suits the case of external loading of small intensity, but with such initial conditions that produce large oscillations of the structure.

If the above conditions are not satisfied, it is then possible to expand the particular integral of the system into a Fourier series. When looking for the integral of system (17), the particular integral may be determined by the variation of amplitudes and phases of its modes.

Let us suppose that the solution of the systems (17) and (18) is given in the form

$$\{q(t)\} = [A_0][\Omega(t)]\{f(t)\} \quad \{q(t)\} = [A_0]\{\Omega(t)\} . \qquad (23)$$

Thaking that the matrices $[\Omega(t)]$ and $[\Theta(t)]$ are diagonal matrices of system modes, that is a time derivative of the system modes, given in the form

$$\begin{aligned} [\Omega(t)] &= [\sin(\omega_i t + a_i(t)] \\ [\Theta(t)] &= [\cos(\omega_i t + a_i(t)] \end{aligned} \qquad (24)$$

and at the same time retaining the equivalent form of the solution of system (17) as well as in the case of system (18), we obtain a conditional matrix differential equation for the variation of amplitudes and phases in the form

$$[\Theta(t)][f(t)]\{\dot{a}(t)\} + [\Omega(t)]\{\dot{f}(t)\} = 0 \qquad (25)$$

where, by substituting the second of relations (23) in system (17), we obtain the second matrix differential equation for the variation of system amplitudes and phases in the form

$$\begin{aligned} &[M(q)][A_0][\omega]((-[\omega][\Omega(t)] - [\dot{a}_i][\Omega(t)])\{f(t)\} + \\ &+ [\Theta(t)]\{f(t)\}) + [P(q,\omega)][A_0][\Theta(t)][\omega]\{f(t)\} + \\ &+ [K(q,\omega,\dot{\omega},V,\phi,R)][A_0][\Omega(t)]\{f(t)\} = \{\overline{Q}\} \end{aligned} \qquad (26)$$

where

$$\begin{aligned} [M(q)] &= -[E] + [D_5(q)] \\ [P(q,\omega)] &= -[E(\omega)] + [D_4(q)] \\ [K(q,\omega,\dot{\omega},V,\phi,R)] &= [K] - [E(\omega,\dot{\omega})] - [Q(\phi)] - \\ &- [D_1(R)] + [D_2(q)] + [D_3(V,\omega)] \end{aligned} \qquad (27)$$

$$\{\overline{Q}\} = \{Q_0\} + \{Q\} + \{T(V,\dot{V},\omega,\dot{\omega})\} . \quad (27)$$

System (26) may be written out in its final form as follows

$$\begin{aligned} &\{\dot{f}\} = [\Theta(t)]\{Q_u\} \\ &\{\dot{a}(t)\} = -[f(t)]^{-1}[\Omega(t)]\{Q_u\} \\ &\{Q_u\} = ([M(q)][A_0][\omega])^{-1} \cdot (\{\overline{Q}\} - \\ &-[K(q,\omega,\dot{\omega},V,\phi,R)][A_0][\Omega(t)]\{f(t)\} - \\ &-[P(q,\omega)][A_0][\Theta(t)][\omega]\{f(t)\}) + \\ &+ [\omega][\Omega(t)]\{f(t)\} . \end{aligned} \quad (28)$$

An integration of system (28) gives the values of the amplitudes and phases of the system modes, which are variable with time.

Since the functional matrices of system (28) are of an oscillatory type, at first sight the integration of the system appears to be extremely difficult, considering the small periods of the system's high-frequency modes. However, it may be shown practically that the integration process converges even with a multiplied step of integration, considering that the corresponding integrals outside the diagonals decrease with an increase in the difference of frequencies of the relevant modes, and the terms around the diagonals remain as predominant values. The same conclusion may be reached by considering the coefficients of matrix $[A_0]$, which corresponds to the eigen-vectors of system (18). It is easy to find out that the integration step which corresponds to the period of the slowest mode of the system is a satisfactory step of integration for the system (28).

The existence of convergence in the integration of system (28) may also be mathematically shown and proved, which cannot be done in this paper, considering the character of the problem discussed here and the allowable space for the presentation.

In the case of interaction between the aileron and the wing for an aileron deflection $\delta(t)$, corresponding to the axis of rotation which is defined by an ort $\vec{S}$, the angular velocity vector $\{\omega\}$ must be replace by the vector

$$\{\omega\} \to \{\omega\} + \dot{\delta}(t)\ \{S\} .$$

The boundary conditions must be also satisfied at all aileron supports. The angle of rotation $\delta(t)$ is defined as the angle of rotation between the body axis coordinate system of the aileron and the appropriate coordinate system of the underformed wing.

V. Conclusion

The described procedure of synthesis of the nonlinear dynamic model of an elastic structure system, as well as its approximation as a slightly nonlinear system, make it possible to give a good description of the structural dynamics of wings having a great aspect ratio, for which the linear theory does not always offer satisfactory solutions. On the other hand, the complexity of the given procedure, compared with the procedures that are generally used in linear theories, restricts the possibility of its application to the systems whose behaviour really corresponds to slightly nonlinear systems. In the opposite case, the integration of system (28) is not practically possible since the solutions obtained do not converge. In this case it is not possible to apply the procedure described here.

References

[1] Janković, J.: Reduced Nonlinear Flight Dynamic Model of Elastic Structure Aircraft. ICAS Proceedings 1982, Seattle, USA.

[2] Janković, J.: Anti-flutter Control Concept Using a Reduced Non-linear Dynamic Model of Elastic Structure, Proceedings of 24-th SDM Conference, 1983.

[3] Bibikov, Y.N.: Local Theory of Nonlinear Analytic Ordinary Differential Equations, Springer Verlag, 1979.

[4] Recent Advances in Nonlinear Computational Mechanics. Editors E. Hinton, D.R.J. Owen, C. Taylor; pages 1-118, Pineridge Press Limited, Swansea, U.K., 1982.