THREE DIMENSIONAL MATHEMATICAL MODEL OF THE CONTROL SYSTEM OF A HOMING MISSILE AND MISSILE-TARGET INTERSECTION EQUATIONS

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Abstract

On the first part of the paper the author has established the sight-line rotating rate equations by means of Euler angles. The sight-line rotating rate vector is projected first to the messile-body coordinate system,then to the antenna coordinate system.The antenna coordinate system is obtained by rotating three Euler angles of the missile body coordinate system.The gyro preccession rate vector is projected to the antenna coordinate-system too.The total error equations and the error equation of two channels of a homing head are established by comparison between the sight line rotating rate with the gyro preccession rate in the antenna coordinate.The phase of the target image in the antenna coordinates decides the distribu tions of the error signals in the angle-tracking loop and in two channels of the control system. This mathematical model futher resolved the channel coupling problem caused by the refrence coordinate torsion.The phase problem of the control signals in the case of deflection or no deflection of the coordinator.

In the second part of the paper the author has established missile-target equations.Under some suppositions the intersection direction of the missile to the target may be decided.
In the third part of the paper two simplified mathematical models are established under the supposition that the target maneuvered in one plane. As an example, using one of the simplified models, the au thor analysed the guidance accuracy of the homing missile and made some conclusions.

These mathematical models may be used to analyse the dynamic errors of the control system in the case of an arbitrary maneuver of the target, the influences of the missile rotation, channel coupling and unsymmetry of the parameters in two channels.Using these models the trajectory of parameters may be given out to analyse the kill-probability during space intercection.

1.Space sight-line rate vector $\vec{\dot{q}}$

The solution of the rotating rate of a line may be transfered in solution of the instantaneous rate of rigid body.When this line is perpendicularto the instantaneousrotating axis of the rigidbody,the both rotating angular rates are equal. The rotating rate of the rigid body may be presented by three Euler angles of a Cartesian coordinate system with respect to an inertial coordinate system. From this point of view we may establish a sight-line coordinate system $OX_sY_sZ_s$. OX_s is the sight direction. OY_s is the direction of the derivative of a unit vector on the sight-line. OZ_s , OX_s, OY_s are according to right hand rule. Three Euler angles of the sight -line coordinate-system with respect to the inertial coordinate system are

(Fig 1)

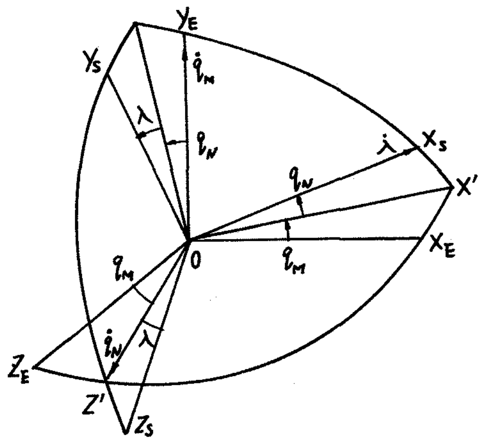

Fig 1. The sight-line coordinate system

Because of the properties of the established coordinate system, the rate vector of the sight-line is on OZ_s direction, and its components on the OX_s OY_s directions are 0.

From the Euler equations we have

$$\vec{\dot q} = \vec{\dot q}_M + \vec{\dot q}_N + \vec{\dot\lambda} \qquad (1)$$

The components of the space sight-line rate on three axes of the sight-line coordinates are

$$\dot q_{xs} = \dot\lambda + \dot q_M \sin q_N \qquad (2)$$

$$\dot q_{ys} = \dot q_M \cos q_N \cos\lambda + \dot q_N \sin\lambda \qquad (3)$$

$$\dot q_{zs} = -\dot q_M \cos q_N \sin\lambda + \dot q_N \cos\lambda \qquad (4)$$

$$\dot q_{xs} = 0$$

$$\dot\lambda = -\dot q_M \sin q_N \qquad (5)$$

Thecomponets of $\vec{\dot q}$ on three axes of the inertial coordinate system may be obtained by projecting $\dot q_{zs}$ tothe inertial coordinate system.They are

$$\begin{bmatrix} \dot q_{xE} \\ \dot q_{yE} \\ \dot q_{zE} \end{bmatrix} = [A] \begin{bmatrix} 0 \\ 0 \\ \dot q_{zs} \end{bmatrix}$$

[A] is the transformation matrix consisted of direction consines between two coordinate systems.

The both sides of the equation (4) multiplied by $\cos\lambda$ and using equation (3) $\dot q_{ys}=0$ will result:

$$\dot q_{zs} \cos\lambda = \dot q_N \qquad (6)$$

The both sides of the equation (4) multiplied by $\sin\lambda$,using (3) $\dot q_{ys}=0$ will result:

$$\dot q_{zs} \sin\lambda = -\dot q_M \cos q_N \qquad (7)$$

The equation of space $\vec{\dot q}$ in the inertial system may be obtained, using equations (6) and (7).

$$\vec{\dot q} = (-\cos q_M \cos q_N \sin q_N \dot q_M + \sin q_M \dot q_N)\vec{i} + \cos^2 q_N \dot q_M \vec{j} + (\cos q_M \dot q_N + \sin q_M \sin q_N \cos q_N \dot q_M)\vec{k} \qquad (8)$$

2. The sight-line rate vector in the missile-body coordinate system

The missile-body coordinate system with respect to the inertial coordinate system can be specified by three Euler angles ψ, ϑ, γ .Using the direction consines of the two coordinate systems may obtain three components of the rotating rate $\dot q_{x1}, \dot q_{y1}, \dot q_{z1}$ inthe missile-body coordinate system.

$$\begin{bmatrix} \dot q_{x1} \\ \dot q_{y1} \\ \dot q_{z1} \end{bmatrix} = [B] \begin{bmatrix} \dot q_{xE} \\ \dot q_{yE} \\ \dot q_{zE} \end{bmatrix}$$

[B] is the transformation matrix consisted of direction consines between two coordinate systems.

3. The sight-line rate in the antenna coordinate system

The antenna coordinate system is the refrence system of tracking error angles, with the center of the gimbals at the origine. The missile-body coordinate system is parallelly moved here. Suppose when antenna gimbal angle $\varphi=0$, antenna coordinate system will be in coincidence with the missile-body coordinate system. When $\varphi\neq0$, the antenna coordinate system can be obtained by rotating the missile-body coordinate system through three Euler angles $\varphi_2, \varphi_1, \lambda_1$ The antenna coordinate system is represented by $OX_A Y_A Z_A$.

For no self rotating antenna, the antenna coordinate system is fixed on the antenna. The rotating antenna may be considered as non-rotating, but antenna has its angular momentum. OX_A will be the electre-axis of the antenna, the relative motion between the antenna coordinate system and the missile-body coordinate systems only will result in angle φ, therefore vector $\vec{\dot{\varphi}}$ is the rotating rate of the antenna coordinate system with respect to the missile body coordinate system. $\vec{\dot{\varphi}}$ must be perpendicular to OX_A, That's why the projection of $\vec{\dot{\varphi}}$ to OX_A direction will be 0. deduce

$$\dot{\lambda}_1 = -\dot{\varphi}_2 \sin\varphi_1 \tag{9}$$

Using the direction cosines of these two coordinate system, the components of the sight-line rotaing rate on three axes of the antenna coordinate system may be obtained.

$$\begin{bmatrix} \dot{q}_{xs.A} \\ \dot{q}_{ys.A} \\ \dot{q}_{zs.A} \end{bmatrix} = [C] \begin{bmatrix} \dot{q}_{x1} \\ \dot{q}_{y1} \\ \dot{q}_{z1} \end{bmatrix}$$

$[C]$ is the transformation matrix consisted of direction cosines between two coordinate systems.

$$\varphi = \sin^{-1}\sqrt{\sin^2\varphi_1 + \cos^2\varphi_1 \sin^2\varphi_2} \tag{10}$$

4. The gyro preccession rate in the antenna coordinate system

It may be proved (see Fig.2), that the gyro preccession is on the OY_1Z_1 plane of the missile-body coordinate system.

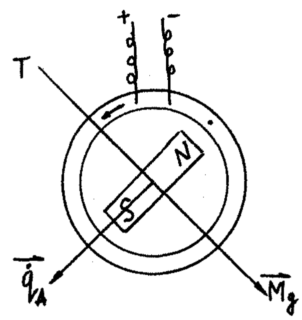

Fig. 2. the relationship between $\vec{M_g}$ and $\vec{\dot{q}_A}$

The angular momentum of the magnet is pointing at the opposite direction of OX_A. $\vec{M_g}$ is the external moment of gyro preccession, $\vec{\dot{q}_A}$ is the rate vector of gyro preccession. T presents the target, at this time the preccession current reaches its positive max. Using the left hand rule, the force undertaken by the wire is on the longitudinal plane across the OX_1.
The external preccession moment caused by the opposite force of the magnet must be on OY_1Z_1 plane across the point O and perpendicular to this longitudinal. Relationships between the external moment and the preccession angular rate, anguar momentum are as follows:

$$\vec{M_g} = \vec{\dot{q}_A} \times \vec{H}$$

$$\vec{H} \times \vec{M_g} = \vec{H} \times (\vec{\dot{q}_A} \times \vec{H})$$

We may deduce

$$\dot{\vec{\varphi}}_A = \frac{1}{H^2}(\vec{H}\times\vec{M_g})$$

projected this vector to the antenna coordinate system.

$$\dot{\varphi}_{yA} = \frac{1}{H} M_{zg} \qquad (11)$$

$$\dot{\varphi}_{zA} = -\frac{1}{H} M_{yg} \qquad (12)$$

M_{YA} M_{ZA} may be obtained through two components M_{Y1} and M_{Z1} of the external moment $\vec{M_g}$ inthe missile body coordinate system. $[C]$ is the transformation matrix and the M_{Y1} M_{Z1} are obtained from the preccession current.

5. The total error angle

Consider angle between the sight-line and the antenna electric-axis as the total error angle. (Fig. 3)

$$\varepsilon = \sqrt{\varepsilon_1^2+\varepsilon_2^2}$$

Because ε_1 , ε_2 are very small.

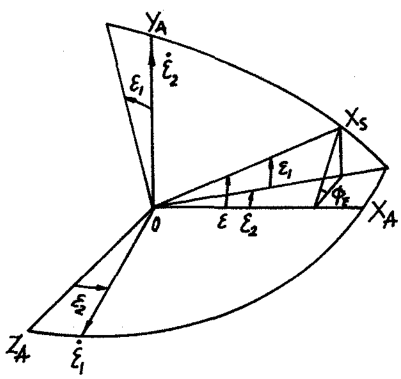

Fig.3. relationships between ε and ε_1 , ε_2

ε_1 and ε_2 are the error angles with respect to two channels.

$$\varepsilon_1 = \int_0^t \dot{\varepsilon}_1 dt = \int_0^t [(\dot{\varphi}_{zsA}-\dot{\varphi}_{zA})\cos\varepsilon_2 + (\dot{\varphi}_{xsA}-\dot{\varphi}_{xA})\sin\varepsilon_2] dt$$

$$\doteq \varphi_{zsA}-\varphi_{zsA} \qquad (13)$$

$$\varepsilon_2 = \varphi_{ysA}-\varphi_{yA} \qquad (14)$$

6. The phase problem in the control signal

When $\varphi=0$, the antenna coordinate system is in coincidence with the missile body coordinate system, the position of the target in the antenna coordinate system determined the phase in dis tributing the control signal to two control channels. From Fig.3 we may deduce

$$\phi_E = tg^{-1}\frac{\varepsilon_1}{\varepsilon_2}$$

Consider a four-quadrant problem, we should add $n\pi$ to the main magnitude ϕ_ε

$$n = \begin{cases} 0 & \varepsilon_2 \geq 0 \\ 1 & \varepsilon_2 < 0 \end{cases}$$

The total preccession current $I_0 = f(\varepsilon)$ is given by test, and the preccession current to two channels I_{YA} I_{ZA} may be written.

$$I_{yA} = -I_0 \sin(n\pi+\phi_E) \qquad (15)$$

$$I_{zA} = I_0 \cos(n\pi+\phi_E) \qquad (16)$$

The components of the preccession moment on the missile-body coordinates OY_1 , OZ_1 are

$$M_{y1} = KI_{yA}$$

$$M_{z1} = KI_{zA}$$

When $\varphi=0$ the components of the **external moment** in the missile -body coordinate system and in the antenna coordinate system are equal.

$$\dot{\varphi}_{zA} = \frac{K}{H} I_0 \sin(n\pi+\phi_E) \qquad (17)$$

$$\dot{\varphi}_{yA} = \frac{K}{H} I_0 \cos(n\pi+\phi_E) \qquad (18)$$

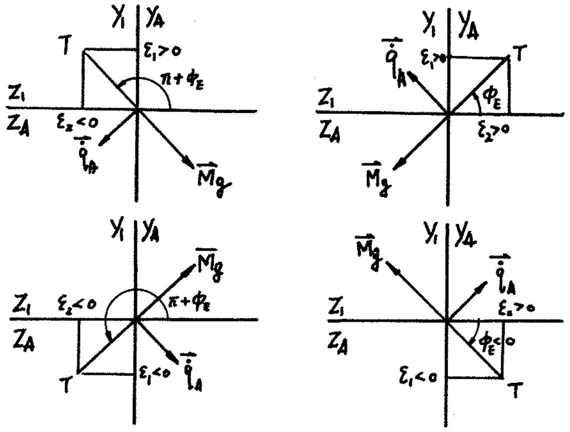

Fig. 4. A four-quadrant phase problem

Using equations (17) and (18) from the Fig.4. you can see that the calculated results will be true.

When $\varphi \neq 0$ in this case the antenna coordinate system is not in coincidence with the missile-body coordinate system.

Suppose a plane consisted of instantaneuos sight-line OX_S and missile axis OX_1 will across with the plane OY_1Z_1 with the line OP. The antenna on the OP OX_1 plane will result in angle φ because Of its target tracking. At that time the antenna coordinate system $OX_A Y_A Z_A$ will rotate around the instantaneuos axis $\vec{\dot{q}}_A$ OZ_A moves around $\vec{\dot{q}}_A$ and consists of a conic and angle with $\vec{\dot{q}}_A$ is the same α' The angle between ϕ_E and generatingline of the conic (i.e. the phase angle) is also not variable. When OX_A makes preccession on a curved surface the antenna coordinates also may transfer from $\varphi=0$ condition to the condition with a no variable ϕ_E .Only the axis of the conic is not the instantaneos preccession rate vector.

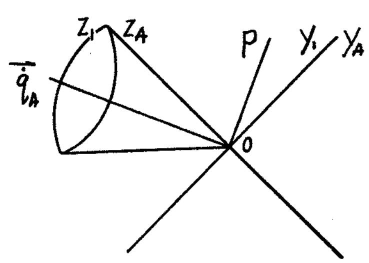

Fig.5.

7. <u>Refrence coordinate torsion</u>

The torsion of the refrence coordinates in decomposing error signals will result in the channel coupling. If the torsion angle of the refrence coordinates is the currents of the control signal to the control loops may be written:

$$I_{11}=I_0\sin(n\pi+\phi_E-\beta) \qquad (19)$$

$$I_{12}=I_0\cos(n\pi+\phi_E-\beta) \qquad (20)$$

Angle may be positive and negative, actually β is small, but through this angle the influnce of the channel coupling may be calculated.

8. <u>The feedback problem of the angle φ</u>

The angle φ is the angle between the antenna axis and the missile axis. In order to find φ_1, φ_2 you should find the expressions of $\dot{\varphi}_1, \dot{\varphi}_2$.

However $\dot{\varphi}_1$ and $\dot{\varphi}_2$ may be obtained by projecting the space rotating rate $\vec{\dot{Q}}$ of the missile axis and the space rotating rate $\vec{\dot{q}}_A$ of the antenna to $\vec{\dot{\varphi}}_2$ $\vec{\dot{\varphi}}_1$ direction.

Just as to deduce the expression of the space $\vec{\dot{q}}$, we may write out the values of $\dot{Q}_{Y1}$ and $\dot{Q}_{Z1}$ in the iner tial coordinate system and in the missile body coordinate system.

Using the direction cosines of the antenna coordinates and the missile-body coordinates,the three compenets $\dot{q}_{XA1}$ $\dot{q}_{YA1}$ $\dot{q}_{ZA1}$ of gyro preccession rate in the missile-body coordinate system may be obtained.In the end we will get

$$\dot{\varphi}_2=\dot{q}_{yA1}-\dot{Q}_{y1} \qquad (21)$$

$$\dot{\varphi}_1=\dot{q}_{zA1}\cos\varphi_2+\dot{q}_{xA1}\sin\varphi_2-\dot{Q}_{z1}\cos\varphi_2 \qquad (22)$$

9. The space kinetic equations

According to the formula on velocity of a point of the composite motion we may obtain;

$$\vec{V}_T = \vec{V}_M + \vec{\omega}\times\vec{D} + \frac{\tilde{d}\vec{D}}{dt} \qquad (23\text{-a})$$

Were $\vec{V}_T$ is the target velocity vector, $\vec{V}_M$ is the missile velocity vector, $\vec{\omega}$ is the rotating rate of the moving coordinates $OX'Y_EZ'$ (Fig 1) with the respect to the inertial coordinates and is equal to $\dot{q}_M$. $\vec{D}$ is the missile-target distance. Project the relation (23-a) to the selected coordinates. We may write out:

$$\dot{D} = [V_T\cos\theta_T\cos(\Psi_T - q_M) - V_M\cdot\cos\theta_M\cos(\Psi_M - q_M) + D\sin\dot{q}_N q_N]/\cos q_N \qquad (23)$$

$$\dot{q}_N = 57.3[V_T\sin\theta_T - V_M\sin\theta_M - \dot{D}\sin q_N]/D\cos q_N \qquad (24)$$

$$\dot{q}_M = 57.3[V_T\cos\theta_T\sin(\Psi_T - q_M) - V_M\cos\theta_M\sin(\Psi_M - q_M)]/D\cos q_N$$

10. The space intersection equation

The space intersection angle is the angle between the missile velocty vector and the target velocity vector at the instant of missile-target collision.

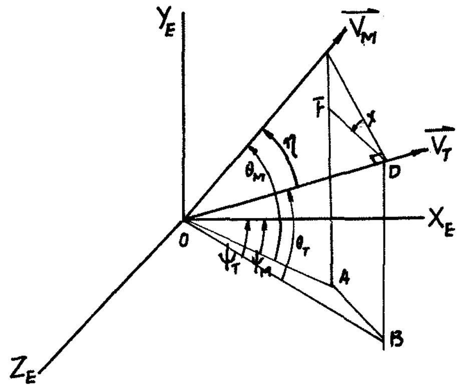

Fig. 6. the relationship in the case of space intersection.

From Fig. 6. we may deduce:

$$\sin\eta\cos[\sin^{-1}(\sin\theta_M - \cos\eta\sin\theta_T)/\sin\eta] = [(\cos^2\eta\cos^2\theta_T + \cos^2\theta_M - 2\cos\eta\cos\theta_M\cos(\Psi_T - \Psi_M))]^{\frac{1}{2}} \qquad (26)$$

Through iteration we may find space intersection angle η .

11. The determination of the missile-target aspect at the instant of intersection.

In order to study the kill probability of the fuze-warhead system in the condition of space intersection we should know the azimuth and pitch angles of the missile when it closes to the target. (observe from the target coordinate system)

Suppose the target velocity vector is in coincidence with its axis, the motion equation of the target center of gravity may be written:

$$V_T = const \qquad (27)$$

$$\dot{\theta}_T = 57.3 n_{yT}\cdot g/V_T \qquad (28)$$

$$\dot{\Psi}_T = -57.3 n_{zT} g/V_T\cos\theta_T \qquad (29)$$

While the relative velocity

$$V_R=(V_M^2+V_T^2-2V_MV_T\cos\eta)^{\frac{1}{2}} \qquad (30)$$

If the target load factors in the vertical plane n_{YT} and in the horizontal plane n_{ZT} are given, the target may maneuver arbitrarily in space.

$$r_T=tg^{-1}\frac{n_{yT}}{n_{zT}} \qquad (31)$$

Three Euler angles $\psi_T, \theta_T, \gamma_T$ of the target coordinates with respect to the inertialcoordinates. The relationship between the relative velocity vector and the inertial coordinates may use the equation (26) to obtain ψ_R, θ_R (see Fig.7.)

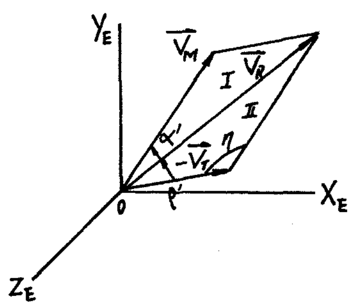

Fig.7.

In the triangle I

$$V_M^2+V_R^2-2V_MV_R\cos\alpha'=V_T^2$$

$$\alpha'=\cos^{-1}(V_M^2+V_R^2-V_T^2)/2V_M V_R$$

$$\beta'=\pi-\eta-\alpha'$$

If V_M V_R V_T η aregiven α' β' may be obtained. If α' β' are known, using equation (26) we may solve the simultaneaus equations and obtain

$$\sin\alpha'\cos[\sin^{-1}(\sin\theta_M-\cos\alpha'\sin\theta_R)/\sin\alpha')]=[(\cos^2\alpha'\cos^2\theta_R+\cos^2\theta_M-2\cos\alpha'\cos\theta_M\cos(\Psi_R-\Psi_M)]^{\frac{1}{2}}$$

$$\sin\beta'\cos[\sin^{-1}(\sin\theta_R-\cos\beta'\sin\theta_T)/\sin\beta']=[(\cos^2\beta'\cos^2\theta_T+\cos^2\theta_R-2\cos\beta'\cos\theta_R\cos(\Psi_T-\Psi_R)]^{\frac{1}{2}} \qquad (32)$$

Two unknown factors ψ_R, θ_R may be solved with two equations. Then we may determine these components of the relative velocity in the inertial coordinate system. Using three Euler angles of the target coordinate system with respect to the inertial coordinate system three components of the relative velocity in the target coordinate system V_{RX} V_{RY} V_{RZ} may be determined. Then at the instant of intersection the pitch angle and the azimuth angle are:

$$q_{NT}=\sin^{-1}(V_{Ry}/V_R) \qquad (33)$$

$$q_{MT}=tg^{-1}(V_{RZ}/V_{RX}) \qquad (34)$$

When $V_{RY}>0$, the missile intersects below the target, When $V_{RZ}>0$,the missile intersects from the right side of the target, when $V_{RX}>0$ the missile intersects from the rear part of the target.

12. Two simplified models and some calculated results.

When the target maneuvers in one plane and the control system worksin three channels, two simplified models may be obtained.

The first simplified model:

The first simplified model is established on the fact that the projection of the vector $\vec{\dot{q}}$ to the missile-body coordinate system is the same as to the antenna coordinate system. Because in the case of plane motion, for an example, on the vertical plane, then the directions of $\vec{\dot{q}}$ and $\vec{\dot{q}}_A$ are the same.As the phase problem proved before (see Fig.5) in the course of target tracking by the antenna,as α' is not variable, the projections of $\vec{\dot{q}}_A$ $\vec{\dot{q}}$ to the missile-body coordinate system and the antenna coordinate system are the same.

The second simplified model:

This simplified model is used to solve

three Euler angles $\varepsilon_2, \varepsilon_1, \lambda_2$ between sight-line coordinate system and the antenna coordinate system.

Because of the plane motion, $\vec{\dot{q}}$ is in coincidence with $\vec{\dot{q}}_A$. The difference between them is the relative rate of the two coordinate system.

$$\vec{\dot{\varepsilon}} = \vec{\dot{\varepsilon}}_2 + \vec{\dot{\varepsilon}}_1 + \vec{\dot{\lambda}}_2$$

$\vec{\dot{\varepsilon}}$ is the same as $\vec{\dot{q}}$ and has its value only on OZ_s direction. The following expressions may be deduced:

$$\left.\begin{array}{l} \dot{\varepsilon} = \dot{\psi} - \dot{\psi}_A \\ \dot{\psi}_A = K\varepsilon \\ \dot{\varepsilon}_1 = \dot{\varepsilon}\cos\lambda_2 \\ \dot{\varepsilon}_2\cos\varepsilon_1 = -\dot{\varepsilon}\sin\lambda_2 \\ \dot{\lambda}_2 = -\dot{\varepsilon}_2\sin\varepsilon_1 \end{array}\right\} \quad (36)$$

Solve this simultaneous equations we may obtain $\varepsilon, \psi_A, \varepsilon_1, \varepsilon_2, \lambda_2$. $\dot{q}$ may be obtained using equation (4).Using the first simplified model a calculation was made for " a certain missile. " The dynamic error influences were considered with different time costant, constant roll, channel coupling, unsymmetary of K etc.

The influences of angle tracking amplification, time constant and the roll rate to the miss distance are only given here (Fig.8,9)

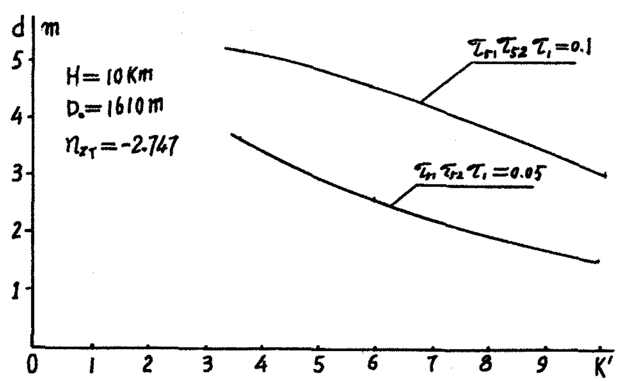

Fig. 8. Relationships between miss-distance and angle-tracking amplication

It is obvious that enhancement of the system rapidity will reduce the dynamic error (of course the enhancement of amplification is limited because of noise).

The relationship between miss distance and constant roll rate is: ifthe roll rate is not large, it will have small effect to the dynamic error, and if the roll rate approaches the certain degree the increase of roll rate will strongly increase the miss distance. At the same time we may come to the conclusion that the enhancement of the system rapidity will permit to have more roll rate.

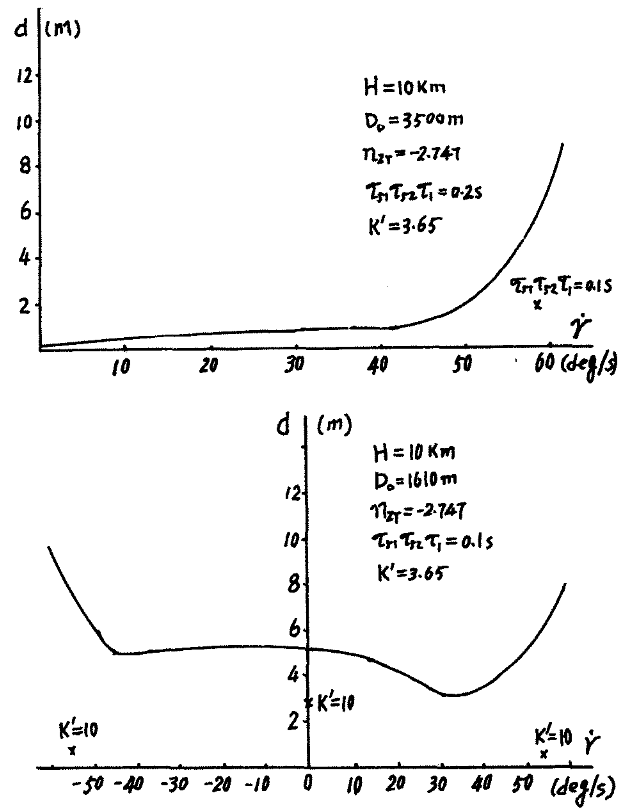

Fig.9. The relationship between miss distance and missile roll motion

Reference:

"Flight —— Dynamics Of Pilotless Aircraft "

by A.A Leberjeiv and R.C Chilnorberelovejin.