A NONLINEAR, DISCRETE-VORTEX-PERTURBATION METHOD FOR UNSTEADY LIFTING-SURFACE PROBLEMS WITH EDGE SEPARATIONS†

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Abstract

The Nonlinear-Discrete Vortex method is coupled with a perturbation method to solve the problem of a rectangular wing with small oscillation about high angles of attack. The solution of the problem is based on decoupling the steady and unsteady effects. The steady part of the problem is a nonlinear one and is solved by the Nonlinear-Discrete Vortex method. The unsteady part of the problem is a linear one and is solved directly without any iteration. So far, the developed method is restricted to flat rectangular surfaces with pitching oscillations. Total and distributed loads of several rectangular wings are presented as numerical results.

I. Introduction

Progress in the nonlinear, unsteady, lifting-surface problem up to this point has followed two distinct lines of approach. In one approach, Green's theorem is used to recast the problem in an integral formulation[1,2]. The problem is then solved by assuming certain distributions with undetermined coefficients for the unknowns. These coefficients are determined by satisfying the boundary conditions at certain control points. In the other approach, fundamental solutions to Laplace's equation is superimposed (e.g. nonlinear, unsteady, discrete-vortex methods[3,4]). The unknown strengths of the singularities and the shape of the wake issued from the edges of separation are found by satisfying the boundary conditions at certain control points. In both approaches, the unsteadiness has been handled by shedding vorticity at discrete time steps, and resolving the problem at each time step. These methods provide a powerful tool for the determination of the transient solution of the general unsteady flow problem, but they are inadequate for determining the steady-state behavior of oscillating wings. This is due to the long computational time required to solve such problems.

The present method does not suffer from this drawback. Here, we present a coupled numerical-perturbation method to solve the problem. The unsteady problem is decoupled into a time-independent part and a time-dependent part. The former part is solved by a nonlinear, steady, discrete-vortex method[5-7] while the latter part is solved by a modified, discrete vortex method in which we account for vortex shedding and convection.

II. Formulation of the Problem

We consider a rigid rectangular lifting surface undergoing small harmonic oscillations about a mean position which makes an angle of attack α_0 with the undisturbed, time independent, uniform flow $\bar{e}_\infty$. The angle of attack α_0 need not be small. The fluid is assumed to be ideal and the flow outside the wing and its wake is irrotational. Thus, the disturbance potential $\phi(\bar{r},t)$ of the flow is governed by Laplace's equation

$$\nabla^2\phi = 0 \quad \text{in } R \tag{1}$$

where R is the infinite region outside the wing and its wake. The velocity potential must satisfy the following conditions on the boundary ∂R:

(a) The flow-tangency condition on the wing surface

$$\frac{\partial F}{\partial t} + (\bar{e}_\infty + \nabla\phi) \cdot \nabla F = 0 \quad \text{on } F(\bar{r},t) = 0 \tag{2}$$

where F is the wing surface.

(b) The kinematical and dynamical conditions on the wake surface

$$\frac{\partial W}{\partial t} + (\bar{e}_\infty + \nabla\phi) \cdot \nabla W = 0 \quad \text{on } W(\bar{r},t) = 0 \tag{3}$$

$$\Delta C_p(\bar{r},t) = -(\nabla\phi_1 - \nabla\phi_2) \cdot (2\bar{e}_\infty + \nabla\phi_1 + \nabla\phi_2) - 2\frac{\partial}{\partial t}(\phi_1 - \phi_2) = 0 \quad \text{on } W(\bar{r},t) = 0 \tag{4}$$

respectively. The subscripts 1 and 2 refer to the upper and lower surfaces of the wake $W(\bar{r},t) = 0$.

(c) The infinity condition

$$\nabla\phi \to 0 \text{ far from } F(\bar{r},t) = 0 \text{ and } W(\bar{r},t) = 0 \tag{5}$$

(d) Kutta condition along the edges of separation

$$\frac{\partial F}{\partial t} + (\bar{e}_\infty + \nabla\phi) \cdot \nabla F = 0 \quad \text{on } F(\bar{r},t)\Big|_{\bar{r} = \bar{r}_{T.E.},\ \bar{r} = \bar{r}_{S.E.}} = 0 \tag{6}$$

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$$\Delta C_p(\bar{r},t) = -(\nabla\phi_1 - \nabla\phi_2) \cdot (2\bar{e} + \nabla\phi_1 + \nabla\phi_2) - 2\frac{\partial}{\partial t}(\phi_1 - \phi_2) = 0 \quad \text{on } W(F,t)| = 0 \quad \bar{r} = \bar{r}_{T.E.},\ \bar{r} = \bar{r}_{S.E.} \tag{7}$$

Figure 1 shows the problem under consideration. The complexity of the problem stems from the boundary conditions of equations (3) and (4). The wake surface $W(\bar{r},t)$ is an unknown of the problem in addition to the disturbance potential or its gradient. In fact, both $W(\bar{r},t)$ and $\phi(\bar{r},t)$ are dependent upon each other and hence the boundary condition of equation (3) is nonlinear. Obviously, the boundary condition of equation (4) is also a nonlinear one. Nevertheless, the unsteadiness adds to the difficulty of the problem.

One step toward the simplification of the problem can be achieved by considering the disturbances due to the unsteadiness of the flow to be small. The problem therefore may be divided, as it will be shown later, into a steady problem which is in general a nonlinear one and an unsteady problem which is a linear one. The solution of the nonlinear, steady problem is obtained following Kandil (1974) while the solution of the unsteady problem follows Kandil (1978). The latter reference represents the most general form of this problem. It includes the symmetric and asymmetric problems with wing deformations.

III. Approximation with Small, Harmonic Oscillations

The problem being considered here is that of a wing undergoing small harmonic oscillations about a mean position which has an angle of attack α_0. Accordingly, the wing surface can be written as

$$F(\bar{r},t) = F_s(\bar{r}) + F_u(\bar{r})\exp(i\omega t) \tag{8}$$

where $F_u(\bar{r})$ is real and small compared with the mean value $F_s(\bar{r})$. We now assume the following forms for the solutions of the velocity potential, pressure and wake surface:

$$\phi(\bar{r},t) = \phi_s(\bar{r}) + \phi_u(\bar{r})\exp(i\omega t) \tag{9}$$

$$C_p(\bar{r},t) = C_{ps}(\bar{r}) + C_{pu}(\bar{r})\exp(i\omega t) \tag{10}$$

$$W(\bar{r},t) = W_s(\bar{r}) + W_u(\bar{r})\exp(i\omega t) \tag{11}$$

where ϕ_u, C_{pu} and W_u are complex and small compared with the mean values ϕ_s, C_{ps} and W_s. Substituting equations (8) - (11) into equations (1) - (7), neglecting terms containing higher powers of the small quantities (e.g. $\nabla\phi_u \cdot \nabla F_u$, ∇W_u,...etc), and equating terms of the same order, we obtain the following two parts of the problem:

(a) The Steady-Flow Part of the Problem

$$\nabla^2\phi_s = 0 \quad \text{in } R \tag{12}$$

on ∂R, we have the following boundary conditions

$$(\bar{e}_\infty + \nabla\phi_s) \cdot \nabla F = 0 \quad \text{on } F_s(\bar{r}) = 0 \tag{13}$$

$$(\bar{e}_\infty + \nabla\phi_s) \cdot \nabla W = 0 \quad \text{on } W_s(\bar{r}) = 0 \tag{14}$$

$$\Delta C_{p_s}(\bar{r}) = -(\nabla\phi_{s1} - \nabla\phi_{s2}) \cdot (2\bar{e}_\infty + \nabla\phi_{s1} + \nabla\phi_{s2}) = 0 \quad \text{on } W_s(\bar{r}) = 0 \tag{15}$$

$$\nabla\phi_s \to 0 \text{ far from } F_s(\bar{r}) = 0 \text{ and } W_s(r) = 0 \tag{16}$$

$$(\bar{e}_\infty + \nabla\phi_s) \cdot \nabla F_s = 0 \quad \text{on } F_s(\bar{r})| = 0 \quad \bar{r} = \bar{r}_{T.E.},\ \bar{r} = \bar{r}_{S.E.} \tag{17}$$

$$\Delta C_{p_s}(\bar{r}) = -(\nabla\phi_{s_1} - \nabla\phi_{s_2}) \cdot (2\bar{e}_\infty + \nabla\phi_{s_1} + \nabla\phi_{s_2}) = 0 \quad \text{on } F_s(\bar{r})| = 0 \quad \bar{r} = \bar{r}_{T.E.},\ \bar{r} = \bar{r}_{S.E.} \tag{18}$$

(b) The Unsteady-Flow Part of the Problem

$$\nabla^2\phi_u = 0 \quad \text{in } R \tag{19}$$

$$i\omega F_u + (\bar{e}_\infty + \nabla\phi_s) \cdot \nabla F_u + \nabla\phi_u \cdot \nabla F_s = 0 \quad \text{on } F_s(\bar{r}) = 0 \tag{20}$$

$$i\omega W_u + (\bar{e}_\infty + \nabla\phi_s) \cdot \nabla W_u + \nabla\phi_u \cdot \nabla W_s = 0 \quad \text{on } W_s(\bar{r}) = 0 \tag{21}$$

$$2i\omega(\phi_{1u} - \phi_{2u}) + (\nabla\phi_{1u} - \nabla\phi_{2u}) \cdot (2\bar{e}_\infty + \nabla\phi_{1s} + \nabla\phi_{2s}) + (\nabla\phi_{1s} - \nabla\phi_{2s}) \cdot (\nabla\phi_{1u} + \nabla\phi_{2u}) = 0 \quad \text{on } W_s(\bar{r}) = 0 \tag{22}$$

$$i\omega F_u + (\bar{e}_\infty + \nabla\phi_s) \cdot \nabla F_u + \nabla\phi_u \cdot \nabla F_s = 0 \quad \text{on } F_s(\bar{r})| = 0 \quad \bar{r} = \bar{r}_{T.E.},\ \bar{r} = \bar{r}_{S.E.} \tag{23}$$

$$i\omega W_u + (\bar{e}_\infty + \nabla\phi_s) \cdot \nabla W_u + \nabla\phi_u \cdot \nabla W_s = 0 \quad \text{on } F_s(\bar{r})| = 0 \quad \bar{r} = \bar{r}_{T.E.},\ \bar{r} = \bar{r}_{S.E.} \tag{24}$$

IV. Method of Solution

1. Steady-Flow Part of the Problem

The solution of this problem is obtained following Kandil (1974). For the sake of completion, the procedure is outlined here.

It is well known that a bound circulation exists around a lifting surface at an angle of attack in a steady-flow. This fact can be established in the mathematical model of the lifting problem by introducing a type of mathematical singularity which exhibits this circulation. One of the simplest singular solutions of Laplace's equation, equation (12), which enjoys such a property is that of a potential vortex.

In addition, the assumption of irrotational flow in R and the infinity condition, equation (16), tell us that the vorticity distribution must be on the inner surface of R. Thus, the lifting surface F_s may be replaced by a bound-vortex sheet of continuous vorticity distribution. Furthermore, we approximate this vortex sheet by a bound-vortex lattice.

On the other hand, Kelvin's theorem of the spatial conservation of circulation tells us that a vortex line cannot simply end in the fluid. Hence the ends of the bound-vortex lattice along the edges of separation must be connected to vortex lines which extend downstream to infinity. These lines are called free-vortex lines. They represent the wake surface W_s and hence their shape and positions are unknowns of the problem. We discretize each free-vortex line into small, straight finite segments (near-wake region) and one semi-infinite line (far-wake region). This discretization process gives the potential to the vortex lines to model the force-free wake surface upon imposing the boundary conditions given by equations (14) and (15).

Kutta condition is enforced at the edges of separation by not placing any bound-vortex line along these edges. In addition, when the solution of the problem is obtained, the wake surface is rendered as a force-free surface and as a stream-surface. Hence, in the limit as we approach the edges of separation from the wake surface, the pressure will be automatically continuous at these edges. So far, the model satisfies equations (12) and (16) and partially satisfies Kutta condition.

To complete the model for the exact solution of the problem, we need to enforce the boundary conditions of equations (13) - (15). This is achieved by defining certain points on F_s and W_s where the boundary conditions are satisfied. These points are called control points. On the wing surface, a control point is the average point of the four corner points of the quadrilateral vortex element. On the wake surface, a control point is the upstream end of a vortex segment.

The solution is effected through an iterative method to simultaneously satisfy the boundary conditions of equations (13) - (15). To accomplish this task, we need to express $\nabla\phi_s$ as a function of the circulation distribution $\Gamma_s(\bar{r})$. This is achieved by Biot-Savart's Law for a straight vortex line. Equation (13) is then satisfied at the control points of the wing surface. Here, we should recall that the wake surface $W_s(\bar{r})$ is not known yet and hence we initially need to assume it (assume the position and shape of the free-vortex lines). The solution of this equation provides an initial distribution of the circulation.

Next, the kinematical and dynamical boundary conditions on the wake surface, equations (14) and (15), are satisfied. We recall that the wake surface $W_s(\bar{r})$ was replaced by a number of vortex lines and each one of them was discretized into a series of small, straight segments (near wake region) and one semi-infinite segment (far-wake region). Now, we can combine the two conditions on the wake into a very simplified condition in which we require that each vortex segment in the wake must be aligned with the local velocity at its upstream end. This means that the vortex segment would be a segment of a streamline [this satisfies the kinematical boundary condition, equation (14)], and it also means that the force on the vortex segment is zero according to Kutta-Joukowski theorem [this satisfies the dynamical boundary condition, equation (15)]. Obviously, this is in an advantage of the discrete vortex approach in solving the nonlinear, steady lifting surface problem. Once the boundary conditions on the wake surface are satisfied, we satisfy the boundary condition on the wing surface with the new wake surface. The solution moves back and forth from the wing surface to the wake surface where the corresponding boundary conditions are satisfied by the most updated circulations and wake surface until convergence of the free-vortex lines is achieved. Once this step is completed successfully, we obtain the circulation distribution $\Gamma_s(\bar{r})$ and the wake surface $W_s(\bar{r})$ as well.

2. Unsteady-Flow Part of the Problem

The solution of this problem is obtained following Kandil (1978). For the sake of completion, the procedure is outlined here.

The smooth flow at the edges of separation is continuously disturbed by the harmonic oscillation of the wing. Accordingly, the bound circulation around the wing changes continuously and this is accompanied by a continuous process of formation and shedding of vortexes from the edges of separation to restore the smooth flow at these edges. Within any infinitesimal time step, the change in the bound circulation around the wing is met by the formation of an infinitesimal vortex strip emanating from the edges of separation which has a strength of equal and opposite sense to the change of the bound circulation. In the next time step this vortex strip is convected downstream and a new infinitesimal vortex strip emanates from the edges of separation. The newly formed vortex strip is smoothly connected to the preceding vortex strip. Thus, a vortex sheet is continuously growing downstream as the harmonic oscillation continues. If the continuous motion of the wing is discritized into a series of impulsive motions, the continuously growing vortex sheet can be replaced by a growing vortex lattice in the wake. Therefore, the discrete vortex model which is used for the solution of this part of the problem has to account for this shedding process at least in the near wake region. This is achieved by employing a discrete vortex model similar to that of the steady flow problem but with the exception of modeling the wake by a vortex lattice in the vicinity of the edges of separation. The circulation distribution of the vortex lattice in the wake is related to the circulation distribution of the bound lattice on the wing. The solution of this part of the problem is effected by a different approach from the one used for the steady part of the problem. Here, the basic unknowns are the velocity potential $\phi_u(\bar{r})$ and the wake surface W_u (r). The former unknown appears in the equations of the boundary conditions, equations (20) - (22) in the form of $\nabla\phi_u$ and $\phi_{u1} - \phi_{u2}$. These two terms can be explicitly expressed in terms of circulation distribution $\Gamma_u(\bar{r})$. For $\nabla\phi_u$, this is achieved by Biot-Savart's law while for $\phi_{u1} - \phi_{u2}$ it is achieved by the fact that the jump of the velocity potential across a vortex sheet at a certain location is equal to the strength of the sheet at this location.

To show the relationship between $\nabla\phi_u$ and Γ_u, we consider a short, straight vortex segment of length $\bar{\ell}$ and strength Γ, Figure 2 . The induced velocity $\bar{V}(\bar{r},t)$ at a field point p is found by Biot-Sacart's Law.

$$\bar{V}(\bar{r},t) = (\Gamma/4\pi h^2)(\cos\theta_1 - \cos\theta_2)\bar{h} \tag{25}$$

where

$$\cos\theta_1 = \bar{\ell} \cdot \bar{r}_1/\ell r_1,\ \cos\theta_2 = \bar{\ell} \cdot \bar{r}_2/\ell r_2 \text{ and } \bar{h} = \bar{\ell} \times \bar{r}_1/\ell \tag{26}$$

If the vortex segment is undergoing a small harmonic oscillation at a certain frequency ω, then $\bar{h}$, $\bar{\ell}$, $\bar{r}_1$ and $\bar{r}_2$ are time dependent. This can be due to the oscillation of the wing if the vortex segment is on its surface or due to the oscillation of the wake if the wake segment is on its surface. In each case, the strength of the vortex segment is also time dependent. Now, according to our assumption of small harmonic oscillation, we write each of $\bar{h}$, $\bar{\ell}$, $\bar{r}_1$, $\bar{r}_2$ and Γ as the sum of a time-dependent part and a small time-dependent part as follows:

$$\left.\begin{aligned}
\bar{h} &= \bar{h}_s + \tilde{\bar{h}}_u = \bar{h}_s + \bar{h}_u\exp(i\omega t)\\
\bar{\ell} &= \bar{\ell}_s + \tilde{\bar{\ell}}_u = \bar{\ell}_s + \bar{\ell}_u\exp(i\omega t) = \bar{\ell}_s\\
&\quad +(\Delta\bar{\ell}_2 - \Delta\bar{\ell}_1)\exp(i\omega t)\\
\bar{r}_1 &= \bar{r}_{1s} + \tilde{\bar{r}}_{1u} = \bar{r}_{1s} + \bar{r}_{1u}\exp(i\omega t) = \bar{r}_{1s}\\
&\quad - \Delta\bar{\ell}_1\exp(i\omega t)\\
\bar{r}_2 &= \bar{r}_{2s} + \tilde{\bar{r}}_{2u} = \bar{r}_{2s} + \bar{r}_{2u}\exp(i\omega t) = \bar{r}_{2s}\\
&\quad - \Delta\bar{\ell}_2\exp(i\omega t)\\
\Gamma &= \Gamma_s + \tilde{\Gamma}_u = \Gamma_s + \Gamma_u\exp(i\omega t)
\end{aligned}\right\} \tag{27}$$

Moreover, we assume that the vortex segment is rigid, i.e.

$$|\bar{\ell}| = |\bar{\ell}_s| \quad \text{or } \bar{\ell}_s \cdot \Delta\bar{\ell}_1 - \bar{\ell}_s \cdot \Delta\bar{\ell}_2 = 0 \tag{28}$$

Substituting equation (26) into equation (25), using equations (27) and (28), and neglecting terms which contian powers of the small time-dependent quantities higher than one, we obtain

$$\bar{V}(\bar{r},t) = \bar{V}_s(\bar{r}) + \bar{V}_u(\bar{r})\exp(i\omega t) \tag{29}$$

where

$$\bar{V}_s(\bar{r}) = (G\Gamma_s/4\pi\ell h_s^2)\bar{h}_s \tag{30}$$

$$\begin{aligned}
\bar{V}(\bar{r}) &= (\bar{h}_s/4\pi h_s^2)[(\Gamma_u - 2\Gamma_s\bar{h}_s \cdot \bar{h}_u/h_s^2)G\\
&+ \Gamma_s(\Delta\bar{\ell}_2 \cdot \Delta\bar{\ell}_1) \cdot (\bar{r}_{1s}/r_{1s} - \bar{r}_{2s}/r_{2s})\\
&+ \Gamma_s(\Delta\bar{\ell}_1 \cdot \bar{r}_{1s}/r_{1s}^2)(\bar{\ell}_s \cdot \bar{r}_{1s}/r_{1s})\\
&- \Gamma_s(\Delta\bar{\ell}_2 \cdot \bar{r}_{2s}/r_{2s}^2)(\bar{\ell}_s \cdot \bar{r}_{2s})]\\
&+ (G\Gamma_s/4\pi\ell h_s^2)\bar{h}_u
\end{aligned} \tag{31}$$

$$G = \bar{\ell}_s \cdot \bar{r}_{1s}/r_{1s} - \bar{\ell}_s \cdot \bar{r}_{2s}/r_{2s} \tag{32}$$

$$\bar{h}_s = \bar{r}_{1s} \times \bar{r}_{2s}/\ell \tag{33}$$

$$\bar{h}_u = (\Delta\bar{\ell}_2 \times \bar{r}_{1s} - \Delta\bar{\ell}_1 \times \bar{r}_{2s})/\ell \tag{34}$$

Equation (32) gives the relationship between $\bar{V}_u$ and Γ_u for a single vortex segment. For the bound-vortex lattice which represents the wing and the free-vortex lines which represent the wake, the relationship between $\nabla\phi_u$ at any field point and Γ_u distribution is simply obtained as the sum of $\bar{V}_u$, as given by equation (31), over all the vortex segments of the bound-vortex lattice and the free-vortex lines. In this relationship, we notice that $\Delta\ell$(i.e. $\Delta\bar{\ell}_1$ or $\Delta\bar{\ell}_2$) is completely known for any vortex segment on the wing surface because F_u is known but they are unknown for any segment on the wake surface because W_u is unknown with the exception of the upstream ends of the vortex segments emanating from the edges of separation of the wing. Along these edges, $\Delta\bar{\ell}$ is completely known because the wing surface F_u is known.

Next, we show the relationships between W_u or ∇W_u and $\Delta\bar{\ell}$. This is achieved by considering a small quadrilateral nonplanar element whose corners are labeled a, b, c and d, figure 3 . The position vector of the average point m on the element is given by

$$\bar{r}_m = (\bar{r}_a + \bar{r}_b + \bar{r}_c + \bar{r}_d)/4 \tag{35}$$

The nonplanar element is approximated by a planar element which passes through point m perpendicular to the unit normal which is defined by

$$\bar{n} = (\bar{r}_c - \bar{r}_a) \times (\bar{r}_b - \bar{r}_d)/ (\bar{r}_c - \bar{r}_a) \times (\bar{r}_b - \bar{r}_d) \tag{36}$$

The equation of the plane is given by

$$W(\bar{r},t) = (\bar{r} - \bar{r}_m) \cdot \bar{n} = 0 \tag{37}$$

Now, we write the vectors $\bar{r}_a$, $\bar{r}_b$, $\bar{r}_c$ and $\bar{r}_d$ in the same form as that of equation (27), i.e.

$$\left.\begin{aligned}
\bar{r}_a &= \bar{r}_{as} + \tilde{\bar{r}}_{au} = \bar{r}_{as} + \bar{r}_{au}\exp(i\omega t) = \bar{r}_{as} +\\
&\quad \Delta\bar{\ell}_a\exp(i\omega t)\\
\bar{r}_b &= \bar{r}_{bs} + \tilde{\bar{r}}_{bu} = \bar{r}_{bs} + \bar{r}_{bu}\exp(i\omega t) = \bar{r}_{bs} +\\
&\quad \Delta\bar{\ell}_b\exp(i\omega t)\\
\bar{r}_c &= \bar{r}_{cs} + \tilde{\bar{r}}_{cu} = \bar{r}_{cs} + \bar{r}_{cu}\exp(i\omega t) = \bar{r}_{cs} +\\
&\quad \Delta\bar{\ell}_c\exp(i\omega t)\\
\bar{r}_d &= \bar{r}_{ds} + \tilde{\bar{r}}_{du} = \bar{r}_{ds} + \bar{r}_{du}\exp(i\omega t) = \bar{r}_{ds} +\\
&\quad \Delta\bar{\ell}_d\exp(i\omega t)
\end{aligned}\right\} \tag{38}$$

Substituting equations (35) and (36) into equation (37), using equation (38), and neglecting terms which contain powers of the small time-dependent quantities higher than one, we obtain

$$W(\overline{r},t) = W_s(\overline{r}) + W_u(\overline{r})\exp(i\omega t) \qquad (39)$$

where

$$W_s(\overline{r}) = (\overline{r} - \overline{r}_{ms}) \cdot \overline{n}_s \qquad (40)$$

$$W_u(\overline{r}) = (\overline{r} - \overline{r}_{ms}) \cdot \overline{n}_u - \overline{r}_{mu} \cdot \overline{n}_s \qquad (41)$$

$$\overline{r}_{ms} = (\overline{r}_{as} + \overline{r}_{bs} + \overline{r}_{cs} + \overline{r}_{ds})/4 \qquad (42)$$

$$\overline{n}_s = (\overline{r}_{cs} - \overline{r}_{as}) \times (\overline{r}_{bs} - \overline{r}_{ds}) / \left| (\overline{r}_{cs} - \overline{r}_{as}) \times (\overline{r}_{bs} - \overline{r}_{ds}) \right| \qquad (43)$$

$$\overline{r}_{mu} = (\Delta\overline{\ell}_a + \Delta\overline{\ell}_b + \Delta\overline{\ell}_c + \Delta\overline{\ell}_d)/4 \qquad (44)$$

$$\overline{n}_u = [\overline{N}_u - (\overline{n}_s \cdot \overline{N}_u)\overline{n}_s] / \left| (\overline{r}_{cs} - \overline{r}_{as}) \times (\overline{r}_{bs} - \overline{r}_{ds}) \right| \qquad (45)$$

$$\overline{N}_u = (\overline{r}_{ds} - \overline{r}_{bs}) \times \Delta\overline{\ell}_a + (\overline{r}_{as} - \overline{r}_{cs}) \times \Delta\overline{\ell}_b + (\overline{r}_{bs} - \overline{r}_{ds}) \times \Delta\overline{\ell}_c + (\overline{r}_{cs} - \overline{r}_{as}) \times \Delta\overline{\ell}_d \qquad (46)$$

The gradient of $W_u(\overline{r})$ is found from equation (41) by

$$\nabla W_u(\overline{r}) = \overline{n}_u \qquad (47)$$

Equation (41) - (47) give the required expressions of W_u and ∇W_u in terms of $\Delta\overline{X}$.

Thus, the boundary conditions of equations (20) - (22) may be rewritten in terms of Γ_u and $\Delta\overline{X}$ as the basic unknowns of the problem. These equations and equation (28) as well are solved simultaneously for the circulation distribution Γ_u of the wing and the wake and the position of the wake $\Delta\overline{X}$.

More details of the method of solution and its implementation can be found in reference 8.

V. Numerical Examples

A computer program is developed to implement the present discrete-vortex-perturbation method to rectangular wings through a CDC-6600 computer of NASA-Langley Research Center, Hampton, Va. The program solves the steady flow part of the problem (or it may read it if it is available) then it proceeds to the solution of the unsteady flow part of the problem. The output includes a detailed solution for the steady and unsteady parts of the wake shape, circulation distribution (Γ_s, Γ_u) on both the bound and free vortex lattices, pressure distribution (ΔC_{p_s}, ΔC_{p_u}), total-load coefficients (C_{ns}, C_{ms}, C_{nu}, C_{mu},---etc) and total-load derivative coefficients [$(C_{nu})_{\alpha_u}$, $(C_{mu})_{\alpha_u}$,---etc.)].

The following numerical examples include rectangular wings undergoing small pitching and plunging oscillations at different mean angles of attack and with different axes of rotation.

Figure 4 depicts a typical solution of the steady wake surface for a rectangular wing at 15° angle of attack. The views given in the figure show a side view, a top view, and a three-dimensional view. The different parameters on the following diagrams are defined as follows:

AR = aspect ratio
b = half wing span
c_r = root chord
Δc_r = segment length in the chordwise direction
$C_{mu} = \frac{F_{nu}}{\frac{1}{2}\rho_\infty U_\infty^2 S_p c_r}$ = amplitude of unsteady pitching-moment coefficient calculated about the axis of rotation
$(C_{mr})_\alpha = \frac{\partial C_{mu}}{\partial\alpha}$ = amplitude of unsteady pitching-moment derivative coefficient
$C_{nu} = \frac{F_{nu}}{\frac{1}{2}\rho_\infty U_\infty^2 S_p}$ = amplitude of unsteady normal-force coefficient
$(C_{nu})_\alpha = \frac{\partial C_{nu}}{\partial\alpha}$ = amplitude of unsteady normal-force derivative coefficient
R, I = refer to real and imaginary parts
S_p = area of wing planform
X_o/c_r = location of axis of rotation
Y/b = spanwise station
α_o = mean angle of attack
α_u = amplitude of pitching oscillation (positive with nose up)
ϕ_h = phase angle between pitching and plunging oscillations
$\omega = \frac{\omega^* \Delta c_r}{U_\infty}$ = reduced frequency
h_u = amplitude of plunging oscillation (positive downward)

Figures 5 and 6 show the amplitudes of real and imaginary parts of the unsteady normal-force and pitching-moment coefficients versus the reduced frequency for a rectangular wing of AR = 1.6 which undergoes a small plunging oscillation and a small pitching oscillation about an axis of rotation at the 1/4 chord length. For the low range of frequency 0-0.1, the curves show a linear behavior. For higher frequencies, the amplitudes are varying nonlinearly with the frequency. The effect of the mean angle of attack is obvious in the figures. The amplitudes increase as the mean angle of attack increases. This shows that the classical unsteady analysis in which the unsteady part of the problem is solved about a zero mean angle of attack cannot be extended to the present case.

Figures 7 and 8 show the amplitudes of the real and imaginary parts of the unsteady pressure coefficient for the same case at different spanwise stations (Y/b = 0.375, 0.875) and different frequencies (ω = 0.05, 0.2).

Figures 9 and 10 show polar diagrams of the complex amplitudes of the normal-force and pitching-moment derivative coefficients for a rectangular wing undergoing small plunging and pitching oscillations with different plunging amplitudes. The first and second quadrants of figure 9 and the third and fourth quadrants of figure 10 are the stable regions. The other quadrants in these figures are the unstable regions. For the cases considered, we see that for h_u = -0.0001 (opposite phase to the pitching motion) and for the range of reduced frequency ω = 0-1.2, the wing extracts energy from the air and there is a tendency to aerodynamic flutter.

The execution time on the CDC-6600 computer for a 4 x 4 bound lattice and one chord length of the free-vortex lattice behind the trailing edge was 167 CP seconds. This computation time includes the solution of the steady and unsteady parts of the problem where the iterative methods of solutions are used for both [8]. The case used 120,500 B of memory storage.

VI. Concluding Remarks

A numerical-perturbation method is developed to solve the nonlinear problem of unsteady flow about finite rectangular wings which undergo small amplitude oscillations about large mean angles of attack. The problem is decoupled into a time-independent part and a time dependent part. The former part is solved by the nonlinear-discrete vortex method while the latter part is linearized about the former nonlinear part. Furthermore, time is factored out from the latter part and the resulting equations may be solved simultaneously without iteration. To reduce the required memory storage on the computer, the latter part is solved by iteration[8].

The present method is superior to the step-by-step time approach[4] from the computation time point of view.

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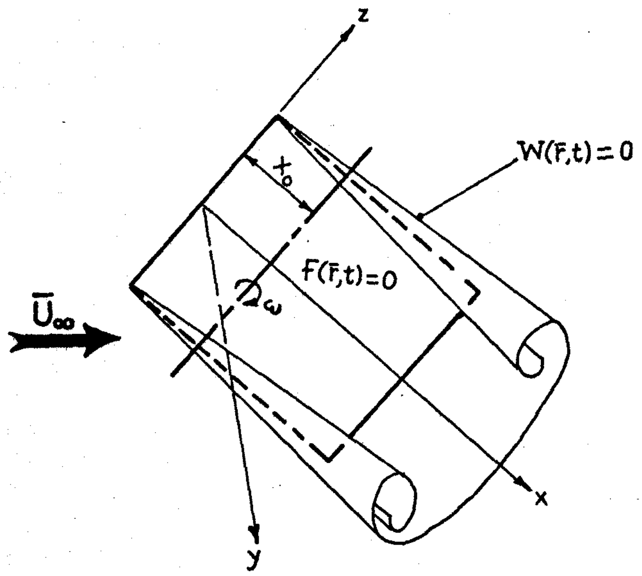

Figure 1. Wing surface, wake surface, and space-fixed frame of reference.

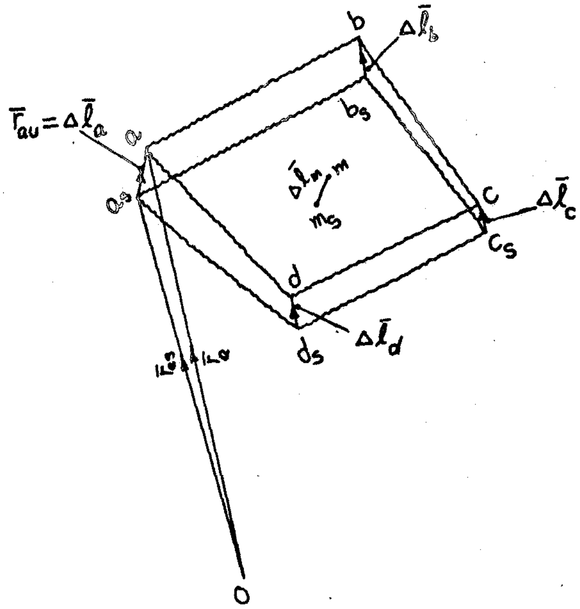

Figure 3. A quadrilateral vortex segment undergoing a small harmonic oscillation.

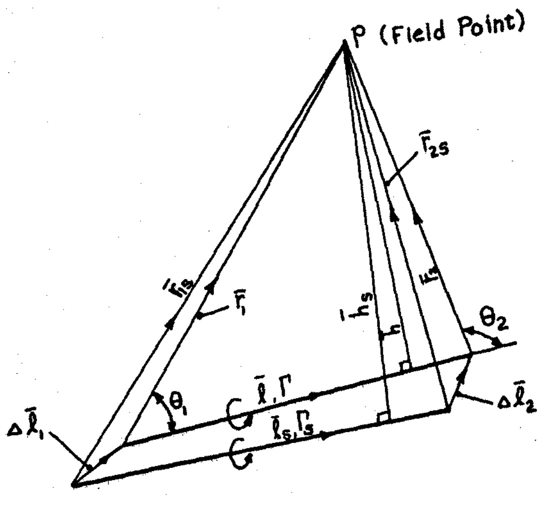

Figure 2. Vortex segment undergoing a small harmonic oscillation

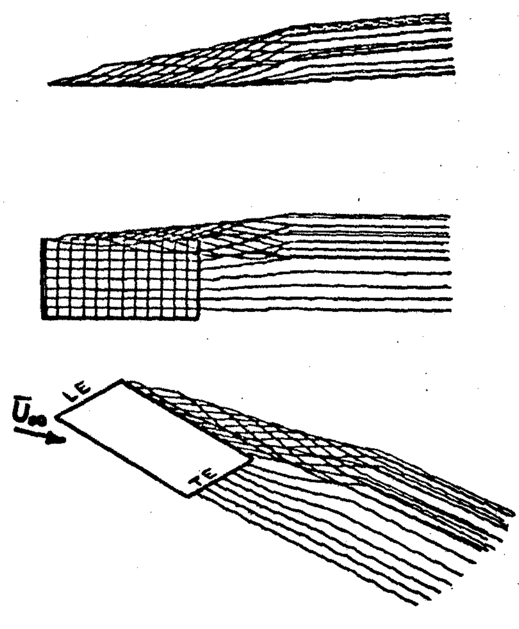

Figure 4. Solution of the steady wake surface $W_s(\bar{r}) = 0$, for a rectangular wing, AR = 1, α_0 = 15°, 12 x 8 lattice.

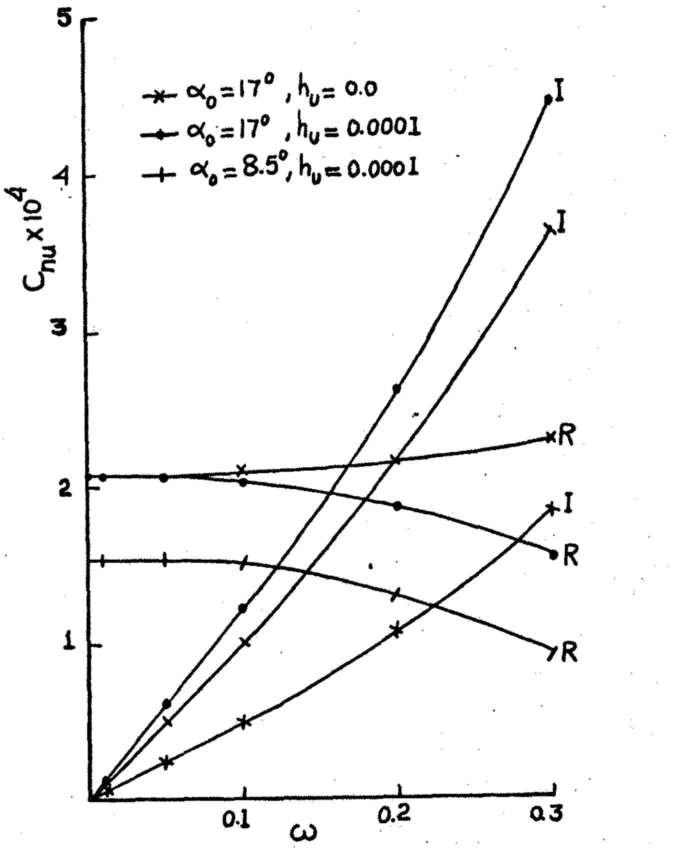

Figure 5. Amplitudes of real and imaginary parts of the normal-force coefficient versus the reduced frequency for a rectangular wing; AR = 1.6, α_u = 0.0017°, x_0/c_r = 0.25, 4 x 4 lattice.

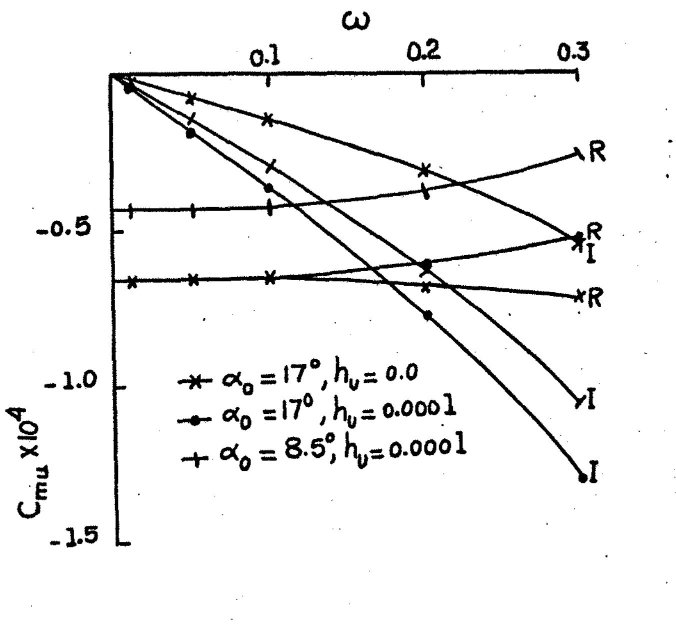

Figure 6. Amplitudes of real and imaginary parts of the pitching-moment coefficient about the axis of rotation versus the reduced frequency for a rectangular wing; AR = 1.6, α_u = 0.0017°, x_0/c_r = 0.25, 4 x 4 lattice.

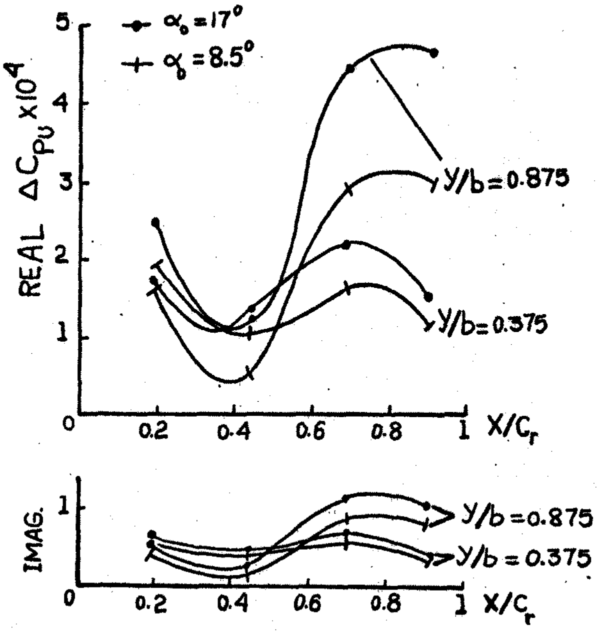

Figure 7. Chordwise variation of the amplitudes of real and imaginary parts of pressure-coefficient at different spanwise stations for a rectangular wing; AR = 1.6, α_u = 0.0017, ω = 0.05, h_u = 0.0001, 4 x 4 lattice.

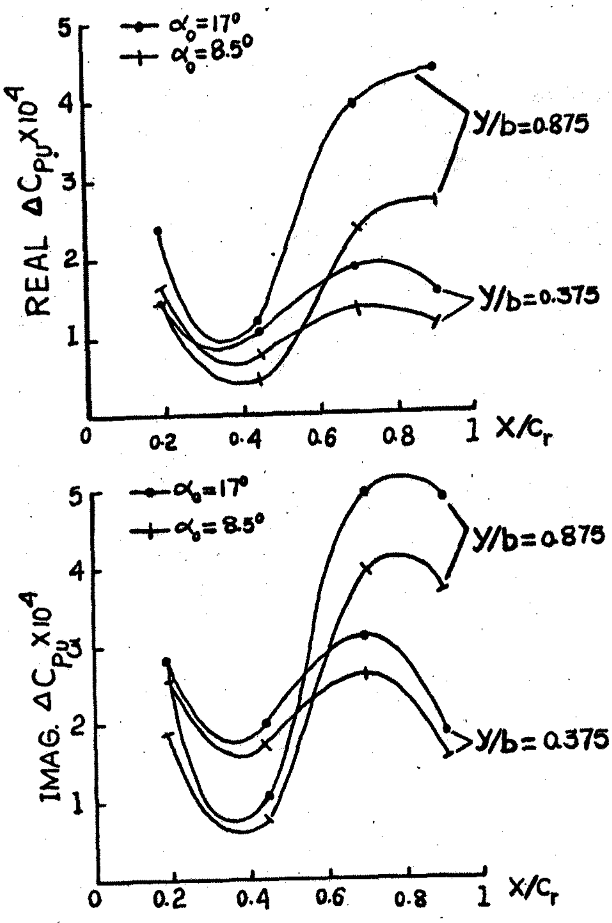

Figure 8. Chordwise variation of the amplitudes of real and imaginary parts of pressure-coefficient at different spanwise stations for a rectangular wing; AR = 1.6, α_u = 0.0017, ω = 0.2, h_u = 0.0001, 4 x 4 lattice.

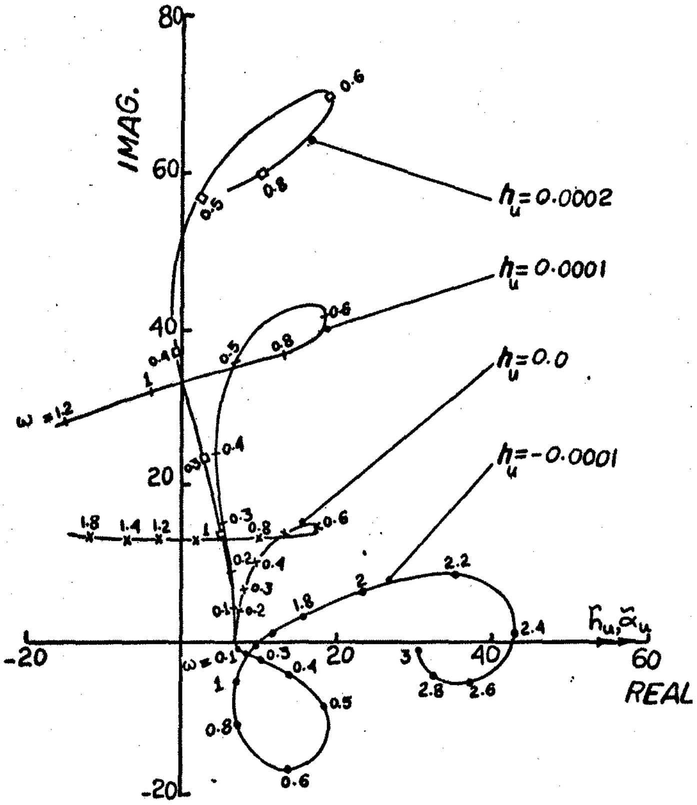

Figure 9. Polar diagram of the complex amplitudes of normal-force derivative $(C_{nu})_{\alpha}$ for a rectangular wing undergoing a pitching oscillation about the 1/4-chord and a vertical translation with different amplitudes; AR = 1.6, $\alpha_0 = 17°$, $\alpha_u = 0.0017°$, $x_0/c_r = 0.25$, $\phi_h = 0.0$, 4 x 4 lattice.

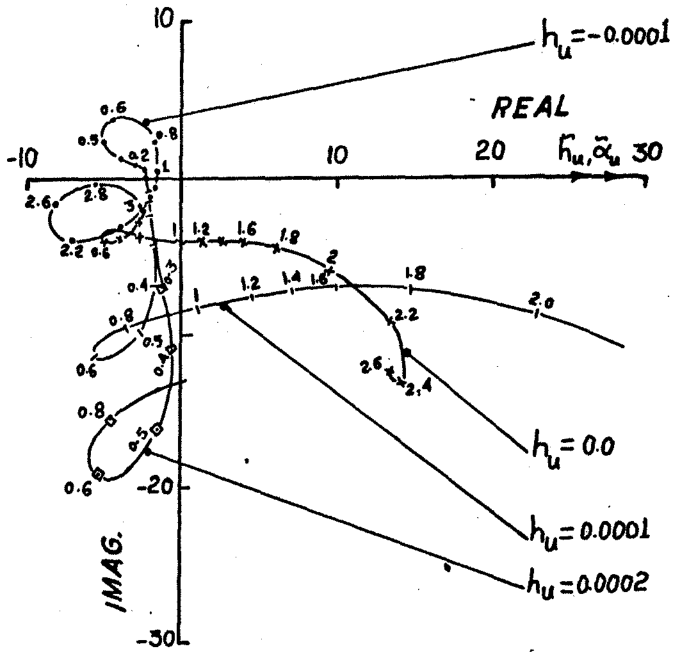

Figure 10. Polar diagram of the complex amplitudes of derivative $(C_{mu})_{\alpha}$ for a rectangular wing undergoing a pitching oscillation about the 1/4-chord and a vertical translation with different amplitudes; AR = 1.6, $\alpha_0 = 17°$, $\alpha_u = 0.0017$, $x_0/c_r = 0.25$, $\phi_h = 0.0$, 4 x 4 lattice.