ESTIMATING TIME TO LOSS OF SEPARATION WITH UNCERTAIN POSITION AND VELOCITY MEASUREMENTS

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Abstract

Maintaining adequate separation between aircraft is of paramount importance for safety of flight. One metric that can be used to measure the level of collision threat that two aircraft pose to each other is the time until adequate separation between the two aircraft will be lost. This time to loss of separation depends on the relative position and velocity between the two aircraft. Unfortunately, there is always some uncertainty in those parameters as well as uncertainty in whether or not the aircraft will maintain their current velocities. This paper examines the effects of uncertainty on estimates of the time to loss of separation and explores methodologies for incorporating that uncertainty into the estimation process. Effective algorithms for time to loss of separation and also probability of loss of separation are developed and their effectiveness demonstrated.

1 Introduction

Maintaining adequate separation between aircraft operating in controlled airspace is a vital safety function of air traffic control (ATC). However, controllers often must operate using uncertain data about aircraft positions and velocities. While the most heavily trafficked areas have highly accurate radar systems and future aircraft will be required to broadcast precise position information using the ADS-B system [1], ATC systems and onboard traffic avoidance systems will continue to have to deal with imprecise information from noncooperative traffic (aircraft without a transponder), aircraft out of radar or ADS-B coverage, and aircraft that are moving erratically.

The estimated time when two aircraft will be within a specified distance of each other is a metric that can be used to evaluate whether or not those two aircraft pose a serious threat to each other. That metric will be called the time to loss of separation in this paper.

What constitutes adequate separation varies depending on location and other factors. Typical minimum separation distances are 3 miles when close to a radar installation and 5 miles when further away [2]. Smaller separation distances may be used in certain circumstances and larger distances may be used where radar is not available. For this paper, a 3 mile minimum separation distance will be used.

2 Time to Loss of Separation

The figure below illustrates an encounter between two airplanes: one at position $\vec{p}_1$ and traveling with velocity $\vec{v}_1$, and the second at position $\vec{p}_2$, and traveling with velocity $\vec{v}_2$. If their velocity vectors converge, then at some later time, denoted by t_{ls}, the two airplanes will be at a distance, s_{min}, that represents the minimum separation required by FAA guidelines [2].

The condition for the two aircraft to be the minimum separation distance apart can be written

$$|\Delta\vec{p} + t_{ls}\Delta\vec{v}|^2 = s_{min}^2 \tag{1}$$

where the delta terms are the differences between the positions and velocities:

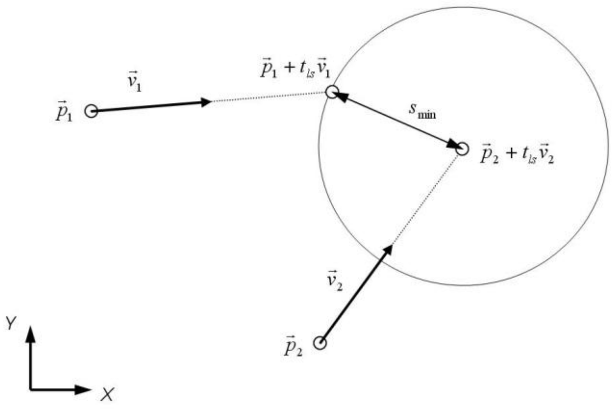

Figure 1. Geometry of encounter between two aircraft.

$$\begin{aligned}\Delta\vec{p} &= \vec{p}_1 - \vec{p}_2 \\ \Delta\vec{v} &= \vec{v}_1 - \vec{v}_2\end{aligned} \tag{2}$$

Equation (1) is a quadratic equation for the time to loss of separation, t_{ls}, which can be solved to yield the following equation.

$$t_{ls} = \frac{-\Delta\vec{p}\cdot\Delta\vec{v}}{|\Delta\vec{v}|^2} \pm \sqrt{\left(\frac{\Delta\vec{p}\cdot\Delta\vec{v}}{|\Delta\vec{v}|^2}\right)^2 - \frac{|\Delta\vec{p}|^2 - s_{min}^2}{|\Delta\vec{v}|^2}} \tag{3}$$

Note that there will be two solutions to this equation and these solutions may be complex depending on the sign of the quantity under the square root. A complex solution indicates that the two aircraft will never reach the specified minimum separation distance and are thus not a collision threat. If the solutions are real and both are positive, then the airplanes will converge at some point in the future and the smaller value is the time to convergence. If the solutions are real and both negative, then the airplanes are diverging and are not a collision threat. If the two solutions are real and are of opposite sign, then the two aircraft are already within the minimum separation distance.

3 Modeling Uncertainty

To determine the effect of position and velocity uncertainty on the estimated time to loss of separation, the position and velocity estimates can be treated as normally distributed random variables. In this analysis, we will assume that the mean values of the position and velocities are the estimated values and the covariance of the random variables reflects the uncertainty in the estimates. Thus for the first aircraft we can write its position and velocity as

$$\begin{Bmatrix} p_{1_x} \\ p_{1_y} \\ v_{1_x} \\ v_{1_y} \end{Bmatrix} \sim N(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1) \tag{4}$$

where $\boldsymbol{\mu}_1$ represents the mean/estimated values of the position and velocities:

$$\boldsymbol{\mu}_1 = \begin{Bmatrix} \bar{p}_{1_x} \\ \bar{p}_{1_y} \\ \bar{v}_{1_x} \\ \bar{v}_{1_y} \end{Bmatrix} \tag{5}$$

and $\boldsymbol{\Sigma}_1$ represents the covariance matrix for the random variables:

$$\boldsymbol{\Sigma}_1 = \begin{Bmatrix} \sigma_{p_{1x}}^2 & \sigma_{p_{1x}p_{1y}} & \sigma_{p_{1x}v_{1x}} & \sigma_{p_{1x}v_{1y}} \\ \sigma_{p_{1x}p_{1y}} & \sigma_{p_{1y}}^2 & \sigma_{p_{1y}v_{1x}} & \sigma_{p_{1y}v_{1y}} \\ \sigma_{p_{1x}v_{1x}} & \sigma_{p_{1y}v_{1x}} & \sigma_{v_{1x}}^2 & \sigma_{v_{1x}v_{1y}} \\ \sigma_{p_{1x}v_{1y}} & \sigma_{p_{1y}v_{1y}} & \sigma_{v_{1x}v_{1y}} & \sigma_{v_{1y}}^2 \end{Bmatrix} \tag{6}$$

Similar definitions will be applied for the second aircraft's position and velocity and furthermore it will be assumed that the random variables for the two aircraft are uncorrelated. Note that for this paper two dimensional Cartesian coordinates have been used for simplicity.

3.1 Taylor Series Expansion

With these assumptions, the distribution of the time to loss of separation can be estimated using a series expansion technique. As shown by Papoulis [3], the expected value of a function, $f(x,y)$ of two random variables, x and y, can be written

$$\begin{aligned} E\{f(x,y)\} = f(\eta_x,\eta_y) \\ + \frac{1}{2}\left(\sigma_x^2\frac{\partial^2 f}{\partial x^2} + 2\sigma_{xy}\frac{\partial^2 f}{\partial x\partial y} \right. \\ \left. + \sigma_y^2\frac{\partial^2 f}{\partial y^2}\right) \end{aligned} \tag{7}$$

Using the same series expansion technique the variance of the function can be written

$$\sigma_{f(x,y)}^2 = \left(\frac{\partial f}{\partial x}\right)^2 \sigma_x^2 + 2\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\sigma_{xy} + \left(\frac{\partial f}{\partial y}\right)^2 \sigma_y^2 \tag{8}$$

Applying these equations to the present problem and using matrix notation, the expected time to loss of separation and the variance of time to loss of separation can be estimated using the equations:

$$E\{t_{ls}\} = t_{ls}(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2) + \frac{1}{2}\mathrm{trace}\left(\boldsymbol{H}(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2)\begin{pmatrix}\boldsymbol{\Sigma}_1 & 0 \\ 0 & \boldsymbol{\Sigma}_2\end{pmatrix}\right) \quad (9)$$
$$\sigma_{t_{ls}}^2 = \boldsymbol{J}(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2)^T\begin{pmatrix}\boldsymbol{\Sigma}_1 & 0 \\ 0 & \boldsymbol{\Sigma}_2\end{pmatrix}\boldsymbol{J}(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2)$$

where $\boldsymbol{H}$ and $\boldsymbol{J}$ are the Hessian and Jacobian matrices respectively for the time to loss of separation function. The Hessian and Jacobian matrices were derived using the symbolic math capabilities of the Matlab software package. They have not been included here due to their length.

Unfortunately, this approach has some limitations. In particular, if two aircraft are traveling so that the measured positions and velocities indicate that they will just barely maintain separation, Equation (9) will not provide any indication that there is a possibility that the two aircraft could encroach on each other's airspace.

3.2 Unscented Transform Algorithm

A second methodology for estimating the expected time to loss of separation and its variance is a technique known as the Unscented Transform [4][5][6]. The Unscented Transform does not use analytical derivatives to propagate the error through the nonlinear function, but rather uses a numerical sampling technique. As described in [7], a set of data points known as sigma points is constructed based on the square root of the covariance matrix of the random variables. For a function of n random variables, a set of $2n+1$ sigma points is required. These sigma points are propagated through the nonlinear function and the mean and covariance of the transformed points are calculated. The distribution of the sigma points is designed so that the errors in the calculated mean and variance estimates are of fourth order. This approach has been applied quite successfully in nonlinear Kalman filtering [6].

The algorithm proceeds as follows for some function, $\boldsymbol{y} = f(\boldsymbol{x})$, where $\boldsymbol{x}$ is a vector of random variables of length n with mean $\bar{\boldsymbol{x}}$ and covariance matrix $\boldsymbol{P}_{\boldsymbol{xx}}$.

1. Calculate n sigma vectors, $\boldsymbol{\sigma_i}$, as the rows or columns of the square root of the matrix $\sqrt{(n+\kappa)\boldsymbol{P}_{\boldsymbol{xx}}}$.
2. Calculate $2n+1$ sigma points as
$$\begin{aligned}\boldsymbol{\chi_0} &= \bar{\boldsymbol{x}} \\ \boldsymbol{\chi_i} &= \bar{\boldsymbol{x}} + \boldsymbol{\sigma_i} \\ \boldsymbol{\chi_{n+i}} &= \bar{\boldsymbol{x}} - \boldsymbol{\sigma_i}\end{aligned} \quad (10)$$
3. Transform the sigma points using the function
$$\boldsymbol{\mathcal{Y}_i} = f(\chi_i) \quad (11)$$
4. Calculate the estimated mean as
$$\bar{\boldsymbol{y}} = \frac{1}{n+\kappa}\left\{\kappa\boldsymbol{\mathcal{Y}_0} + \frac{1}{2}\sum_{i=1}^{2n}\boldsymbol{\mathcal{Y}_i}\right\} \quad (12)$$
5. Calculate the estimated covariance as
$$\boldsymbol{P}_{\boldsymbol{xx}} = \frac{1}{n+\kappa}\left\{\kappa[\boldsymbol{\mathcal{Y}_0} - \bar{\boldsymbol{y}}][\boldsymbol{\mathcal{Y}_0} - \bar{\boldsymbol{y}}]^T + \sum_{i=1}^{2n}[\boldsymbol{\mathcal{Y}_i} - \bar{\boldsymbol{y}}][\boldsymbol{\mathcal{Y}_i} - \bar{\boldsymbol{y}}]^T\right\} \quad (13)$$

3.3 Modified Series Expansion Algorithm

As will be discussed later, initial Monte Carlo simulation results showed that the Taylor series expansion formulation was not particularly effective and was outperformed in many instances by the naïve approach of ignoring uncertainty altogether. Moreover, the standard deviation of the time to loss of separation estimates showed excessive error at large miss distances. To try and address some of these shortcomings, a modified algorithm was explored.

Since the naïve approach of ignoring uncertainty worked reasonably well for small miss distances, Equation (3) was used directly to estimate the time to loss of separation when the estimated miss distance was less than the minimum separation distance. For larger estimated miss distances, the time to closest approach was used. The time to closest approach is the first term in Equation (3) and can be written:

$$t_{ca} = \frac{-\Delta\vec{p}\cdot\Delta\vec{v}}{|\Delta\vec{v}|^2} \quad (14)$$

This approximation is justified by noting that as the estimated miss distance approaches the minimum separation distance, the time to loss of separation approaches the time to closest approach. Consequently, the time to closest approach is a reasonable approximation to the time to loss of separation when the estimated miss distance is close to the minimum separation distance.

The relationship between time to loss of separation and time to closest approach was also exploited to arrive at an improved algorithm for estimating the variance in the time to loss of separation. The time to closest approach equation, Equation (14), is considerably simpler than the time to loss of separation equation, Equation (3). However, because of their geometric relationship it was reasoned that the behavior of the variance of the two equations may be similar. For the modified algorithm, a Taylor expansion of Equation (14) was used to estimate the variance of the time to loss of separation.

4 Monte Carlo Simulation

To provide a ground truth for evaluating the various algorithms for calculating time to loss of separation, Monte Carlo simulation was used to estimate both the time to loss of separation as well as the probability of loss of separation. For each scenario evaluated, the nominal positions and velocities of two aircraft were assumed as well as the uncertainties associated with those positions and velocities. The assumed positions were treated as the means of normal distributions and the uncertainties were specified as the variances of those normal distributions. Monte Carlo simulation proceeded by adding normally distributed pseudo-random variables with the specified variances to the mean positions and velocities and then calculating the time to loss of separation for that combination of perturbed aircraft positions and velocities. The statistics associated with the time to loss of separation were then compiled over many iterations of this process. This process was implemented using a program written for the Matlab environment and the Matlab-provided function, mvnrnd, was used for producing the pseudo-random perturbations.

The encounter geometry used for the simulations is illustrated below in Figure 2. Two aircraft were simulated on converging courses. Both slow aircraft, traveling at 120 kts which would be typical of a light general aviation airplane, and fast aircraft, traveling at 240 kts which would be typical of a jet airplane operating below 10,000 ft were simulated. Cases were run with two slow aircraft, with one slow and one fast aircraft, and finally with two fast aircraft. In each case, the geometry was setup so that the point of closest approach was 5 minutes from the starting position. For the slow aircraft, this placed them 10 nmi from the intercept point, for the fast aircraft, the starting position was 20 nmi from the intercept point. The half angle between the two tracks, ψ, was varied from 15 deg to 90 deg. In addition, the nominal miss distance between the two aircraft was varied. This was achieved by offsetting the starting point of one of the aircraft perpendicular to its course until the point of closest approach between the two tracks was the desired value. The nominal miss distance was varied from zero to 30,000 ft.

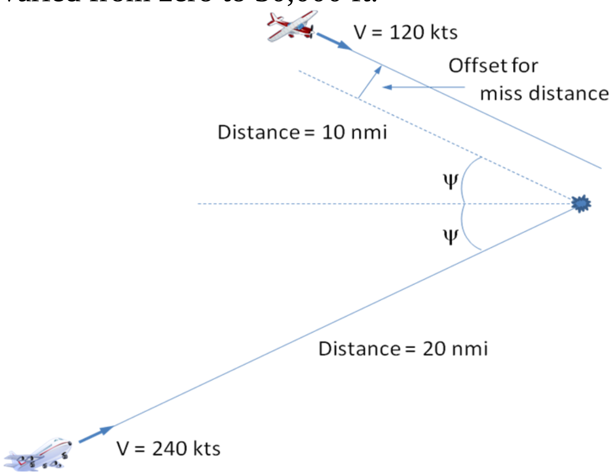

Figure 2. Geometry of encounters used for Monte Carlo simulation.

To determine the number of Monte Carlo iterations required to produce reasonable results, one of the scenarios was run out to 1,000,000 iterations a total of 10 different times and the time to loss of separation statistics (mean, standard deviation, skewness, and kurtosis) were compiled over the course of the simulations. For each trial, the scenario was held constant, but the seed used for the pseudo-random number generator was varied. These data are shown in Figure 3, Figure 4, and Figure 5 below. The independent variable in each of the figures is the number of iterations of the Monte Carlo simulation completed. Each of the curves presented represents a single run of the Monte Carlo simulation.

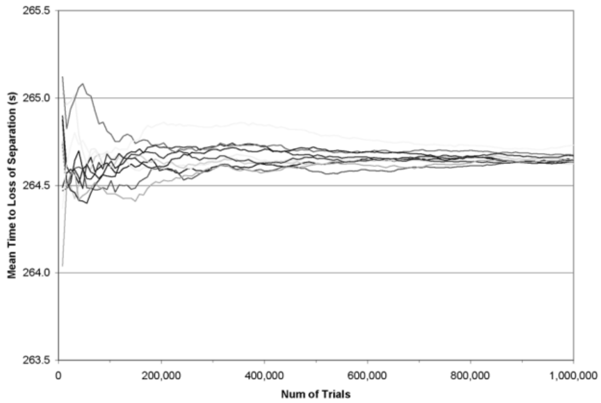

Figure 3. Effect of number of iterations on estimate of mean time to loss of separation.

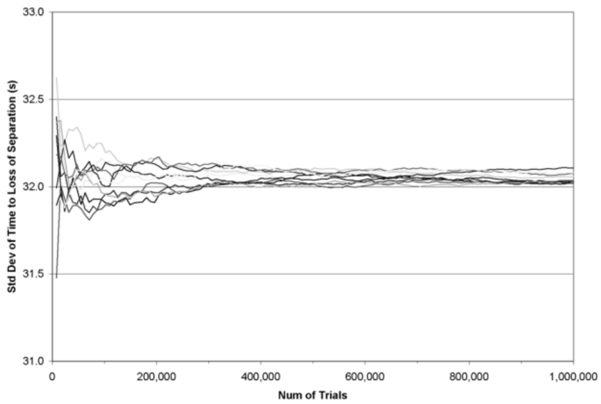

Figure 4. Effect of number of iterations on estimate of standard deviation of time to loss of separation.

These figures indicate that the scatter in the calculated statistics between the 10 trials is significantly reduced after 100,000 iterations. At 500,000 iterations, the scatter in the mean estimate between the 10 trials is less than 0.1% (measured as the difference between the maximum estimate and the minimum estimate, divided by the average estimate). The scatter in the standard deviation estimate is 0.3% and the scatter in the skewness is just over 3%. These uncertainties in the statistics are felt to be reasonable and so 500,000 iterations were used for all of the remaining Monte Carlo simulations.

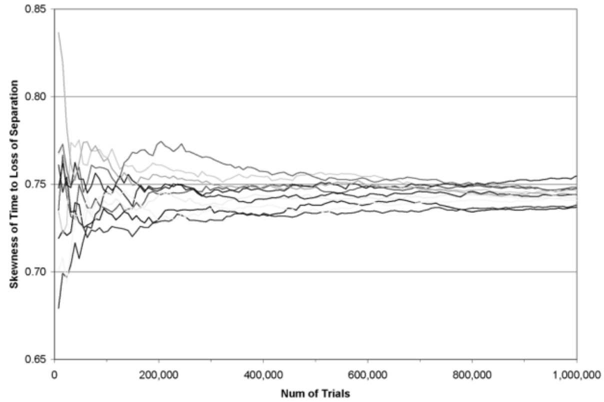

Figure 5. Effect of number of iterations on estimate of skewness of time to loss of separation.

5 Time to Loss of Separation Results

Table 1 presents the results from estimating the time to loss of separation using the four different algorithms described above. The table presents the percent error between each of the estimates and the Monte Carlo simulation results serving as the "truth." For each algorithm the results are presented at combinations of intercept angles from 15 – 90 degrees and nominal miss distances from zero to 30,000 ft. The first set of results is for the naïve approach of ignoring uncertainty altogether. Here the measured values of position and velocity are assumed to be correct and fed into Equation (3) to get the time to loss of separation. At low intercept angles and small miss distances this approach moderately under predicts the time until loss of separation. As miss distance increases, the error is significantly reduced, but then it increases again as the nominal miss distance approaches the minimum separation distance (3 nmi or 18,228 ft). The most significant problem however, is that at nominal miss distances above the minimum separation distance (24,000 ft and 30,000 ft in the table), the naïve approach ignores the probability that the aircraft tracks may indeed result in a loss of separation, and gives no results for those cases.

Table 1. Percent error in estimate of mean time to loss of separation, low speed / low speed encounter.

Intercept Angle (deg)	0	6,000	12,000	18,000	24,000	30,000
Naïve approach						
	Nominal Miss Distance (ft)					
15	-8.25	-4.71	-0.46	9.00		
30	-3.68	-2.62	0.20	8.88		
45	-2.47	-1.83	0.33	7.47		
60	-2.01	-1.47	0.33	6.51		
75	-1.76	-1.30	0.33	6.02		
90	-1.69	-1.27	0.32	5.96		
Taylor series						
	Nominal Miss Distance (ft)					
15	3.77	14.93	48.82	6521.0		
30	1.97	4.75	16.02	1872.4		
45	1.38	2.93	10.08	1108.5		
60	1.08	2.27	7.84	838.0		
75	0.98	1.99	6.85	720.6		
90	0.95	1.87	6.50	677.4		
Unscented Transform						
	Nominal Miss Distance (ft)					
15	5.43	9.48	14.37	31.29		
30	-0.36	1.43	2.30	6.94	-5.91	-6.59
45	-0.83	0.36	1.03	4.48	-4.71	-4.68
60	-0.91	0.10	0.66	3.54	-3.98	-3.54
75	-0.88	0.02	0.52	3.12	-3.55	-3.01
90	-0.87	-0.03	0.45	3.02	-3.49	-2.81
Modified series expansion						
	Nominal Miss Distance (ft)					
15	-8.25	-4.71	-0.46	9.00	8.59	6.40
30	-3.68	-2.62	0.20	8.88	9.03	6.91
45	-2.47	-1.83	0.33	7.47	7.99	6.36
60	-2.01	-1.47	0.33	6.51	7.06	5.80
75	-1.76	-1.30	0.33	6.02	6.67	5.46
90	-1.69	-1.27	0.32	5.96	6.62	5.39

The second set of results in the table used the Taylor series expansion method to predict the effects of the measurement uncertainty. At very low nominal miss distances, this approach works quite well, giving estimates within 5% of the Monte Carlo results. However, as the miss distance increases and approaches the minimum separation distance, the algorithm wildly over predicts the time to loss of separation. Moreover this approach has the same problem as the naïve approach in that it provides no answer when the miss distance is greater than the minimum separation distance.

The unscented transform method provides good results across the range tested except for the shallowest intercept angle. At the 15 degrees intercept angle, the error grows dramatically as the nominal miss distance increases and at high miss distances the algorithm fails to provide any result. The cases where the unscented transform fails are typically cases where the sampling approach fails. For example at higher nominal miss distances, the unscented transform will fail if all of the sigma points represent scenarios where the two aircraft do not lose separation. Moreover for estimating the variance, the algorithm will fail unless at least two of the sigma points represent scenarios where the two aircraft lose separation. This occurs in the results presented in Table 2.

The modified series expansion method shows reasonable accuracy across the range of miss distance and intercept angle. At nominal miss distances less than the minimum separation standard, the results are identical to the naïve approach. At larger nominal miss distances, the method uses the time to closest approach as a surrogate for the time to loss of separation. The maximum error is 9% across the range tested.

Table 2 presents the error in the estimate of standard deviation in time to loss of separation. The standard deviation of time to loss of separation is useful as a measure of how precisely time to loss of separation is known. The cases presented in the table correspond to those in Table 1. However, no results are presented for the naïve approach since it does not provide any information about variance or standard deviation.

The Taylor series method provides accurate estimates of standard deviation at small nominal miss distances and steep intercept angles. However, at shallow angles the error goes up considerably and at nominal miss distances approaching the minimum separation distance, 18,228 ft, the standard deviation estimates become completely unusable with errors greater than 500%. Moreover, as was seen in Table 1, the Taylor series method does not provide an answer when nominal miss distance exceeds the minimum separation distance even though there is still a significant probability of loss of separation.

The unscented transform has similar issues as the Taylor series method. As miss distance increases so does the error though not to the same extent as the Taylor series results. At nominal miss distances above the minimum separation distance the unscented transform is unable to estimate standard deviation because not enough sample points fall within the minimum separation distance region.

Table 2. Percent error in estimate of standard deviation of time to loss of separation, low speed / low speed encounter.

Taylor series						
Intercept Angle (deg)	Nominal Miss Distance (ft) 0	6,000	12,000	18,000	24,000	30,000
15	-34.10	-18.08	8.57	437.96		
30	-11.44	-4.40	24.17	546.17		
45	-8.37	-1.87	25.70	545.24		
60	-7.22	-0.95	26.56	539.08		
75	-6.65	-0.49	26.03	533.88		
90	-6.47	-0.77	26.33	535.40		
Unscented Transform						
Intercept Angle (deg)	Nominal Miss Distance (ft) 0	6,000	12,000	18,000	24,000	30,000
15	-9.62	6.24	12.30	23.32		
30	-3.70	-0.43	1.07	17.49		
45	-4.44	-2.09	-1.20	18.31		
60	-4.59	-2.52	-1.62	18.33		
75	-4.52	-2.61	-2.43	18.26		
90	-4.49	-3.10	-2.25	19.38		
Modified series expansion						
Intercept Angle (deg)	Nominal Miss Distance (ft) 0	6,000	12,000	18,000	24,000	30,000
15	7.12	3.11	-3.15	-11.92	-13.12	-10.15
30	10.60	8.83	7.10	5.45	3.92	2.53
45	6.66	6.27	5.18	4.54	3.87	2.57
60	4.85	4.74	4.17	3.09	2.28	1.86
75	4.10	4.07	2.93	2.05	1.31	1.53
90	3.88	3.49	3.02	2.28	1.51	1.03

The modified series expansion method produces good results across the entire set of cases. The largest error occurs at shallow intercept angles, but stays under 15%.

The calculations shown in Table 1 and Table 2 were repeated for cases where a low speed airplane met a high speed airplane and cases where two high speed aircraft encountered one another. Similar trends were observed for these two other encounters. Table 3 and Table 4 summarize the results for all three of the different encounter types. For mean time to loss of separation in Table 3 the naïve approach shows the lowest average error over the three encounter types, but as described earlier it does not provide an answer for the higher nominal miss distances. The unscented transform approach and modified series expansion method provide similar levels of accuracy, but the modified series expansion method provided a valid answer for all cases where the unscented transform broke down for a few of the cases.

Table 3. Percent error in estimate of mean time to loss of separation, summary.

	Low speed / low speed	Low speed / high speed	High speed / high speed
Naïve approach	2.80	1.81	1.48
Taylor series	290.22	101.69	113.73
Unscented transform	3.75	2.24	1.26
Modified series expansion	4.43	3.25	2.57

Table 4 summarizes the errors seen in the estimates of standard deviation. The Taylor series method shows very poor results due to the huge errors seen at large miss distances. The unscented transform does much better but still shows greater than 30% average error for the low/high speed cases. Both the Taylor series and unscented transform methods were unable to provide results at nominal miss distances greater than the minimum separation distance. In contrast, the modified series expansion method demonstrated good accuracy for all of the cases and was robust in providing results for all of the cases.

Table 4. Percent error in estimate of standard deviation of time to loss of separation, summary.

	Low speed / low speed	Low speed / high speed	High speed / high speed
Taylor series	92.36	93.08	92.51
Unscented transform	6.10	32.23	4.62
Modified series expansion	4.54	2.37	1.87

6 Probability of Loss of Separation

Knowing the most likely time when two aircraft will lose separation is not the whole story. If the aircraft are on estimated courses with a closest approach greater than the minimum separation distance, they may or may not be a threat to each other depending on the uncertainty in their trajectories. Calculating the probability that their current velocities will result in a loss of separation gives another indication of the threat that they pose to each other. In order to estimate this probability, one first must be able to estimate the miss distance and the variance of the miss distance. Table 5 presents the performance of three different methods for calculating the expected miss distance. The first method is the naïve approach of ignoring uncertainty and just plugging in the estimated positions and velocities into Equation (1). This approach does quite well at large miss distances and steep intercept angles, but is much less accurate at shallow intercept angles and close encounters. At a nominal miss distance of zero, an encounter where the measurements indicate the two aircraft would hit, the naïve approach calculates an expected miss distance of zero. This results in a -100% error because the true expected value of miss distance will always be greater than zero. In real terms however, the dimensional error for these cases is small.

The second algorithm is the Taylor series approximation. This approach shows excellent agreement with the data for all cases. The maximum error noted was just over 1%.

The unscented transform approach also does reasonably well. The largest percent errors occur at zero nominal miss distance. However, as noted above the true expected value of miss distance is very small so even the 23% error is a small number in dimensional terms.

To find the probability of loss of separation, we need the standard deviation of the miss distance as well as the expected value. Table 6 presents the error in estimates of the standard deviation. The naïve approach does not provide a standard deviation estimate so it is not included in this table.

Table 5. Percent error in estimate of mean miss distance, low speed / low speed encounter.

Intercept Angle (deg)	Nominal Miss Distance (ft) 0	6,000	12,000	18,000	24,000	30,000
Naïve approach						
15	-100.0	-61.86	-45.45	-37.17	-32.19	-28.99
30	-100.0	-51.33	-24.88	-12.34	-6.31	-3.40
45	-100.0	-48.65	-19.91	-7.60	-2.74	-0.77
60	-100.0	-47.76	-18.14	-6.13	-1.71	-0.26
75	-100.0	-47.32	-17.28	-5.33	-1.31	-0.11
90	-100.0	-46.76	-16.42	-4.74	-0.99	-0.05
Taylor series						
15	0.12	-0.27	-0.38	-0.82	-0.99	-1.19
30	0.07	-0.19	-0.33	-0.41	-0.37	-0.44
45	0.07	0.07	-0.12	-0.24	-0.23	-0.11
60	-0.01	-0.13	-0.14	-0.26	-0.08	0.00
75	-0.01	-0.31	-0.24	-0.19	-0.06	0.01
90	0.15	0.11	0.24	0.27	0.45	0.46
Unscented Transform						
15	-22.72	-7.72	-1.66	0.35	1.33	1.73
30	-18.27	0.22	5.51	3.97	0.89	-2.30
45	-17.35	2.05	6.07	1.97	-3.21	-1.24
60	-17.11	2.40	5.98	0.71	-2.03	-0.58
75	-16.98	2.47	5.77	0.02	-1.56	-0.37
90	-16.81	2.94	5.88	-0.50	-1.23	-0.29

The Taylor series algorithm shows excellent agreement with the Monte Carlo results for all of the cases. The maximum error is just over 2% and the average error across all cases is 0.5%. The unscented transform performs considerably worse showing a maximum error of 65% and an average error of 14%.

As with the time to loss of separation analysis, cases were run for the low speed/high speed encounter and for the high speed/high speed encounter and these results followed the same trends noted in Table 5 and Table 6.

Table 6. Percent error in estimate of standard deviation of miss distance, low speed / low speed encounter.

Taylor series						
Intercept Angle (deg)	Nominal Miss Distance (ft) 0	6,000	12,000	18,000	24,000	30,000
15	0.18	0.83	1.42	1.62	2.02	2.32
30	0.03	0.07	0.58	0.79	1.21	1.25
45	0.01	0.39	0.40	0.55	0.53	0.36
60	0.07	-0.06	0.41	0.09	0.24	0.24
75	-0.08	0.01	0.02	0.08	0.44	0.12
90	0.10	0.09	0.42	0.20	0.42	0.12
Unscented Transform						
Intercept Angle (deg)	Nominal Miss Distance (ft) 0	6,000	12,000	18,000	24,000	30,000
15	52.91	18.72	3.60	-3.88	-7.70	-9.93
30	62.04	16.22	-6.60	-10.24	-4.59	3.94
45	63.93	15.79	-8.00	-5.65	8.14	3.64
60	64.68	14.87	-8.21	-2.93	5.99	2.17
75	64.69	14.67	-8.66	-0.93	5.26	1.46
90	65.05	14.34	-8.54	0.61	4.45	1.00

Since the goal is to estimate the probability of loss of separation, the type of distribution is needed as well as the mean and variance. The probability of loss of separation is the integral of the probability density function for miss distance from zero to the minimum separation distance. Figure 6, shows the distribution of actual miss distance for a scenario with a 45 degree intercept half angle and an 18,000 ft nominal miss distance. The distribution is clearly not a normal distribution. The left hand side is much higher than the right and stops at zero since a negative miss distance is not possible. Based on the appearance, a folded normal distribution was hypothesized and used to model the miss distance. The solid line on the graph shows the results of the analysis and demonstrates excellent agreement with the Monte Carlo results.

The probability of loss of separation corresponds to the cumulative probability distribution for miss distance up to the minimum separation distance. Table 7 presents this statistic from Monte Carlo simulation and estimated using the Taylor series method and an assumed folded normal distribution for miss distance. The probability varied from 80% at zero nominal miss distance to 18% at 30,000 ft nominal miss distance. For each run case, both the Monte Carlo (truth) value and the estimate from the Taylor series method are shown. The maximum discrepancy between the two values is 1% and in most cases is only a few tenths of a percent.

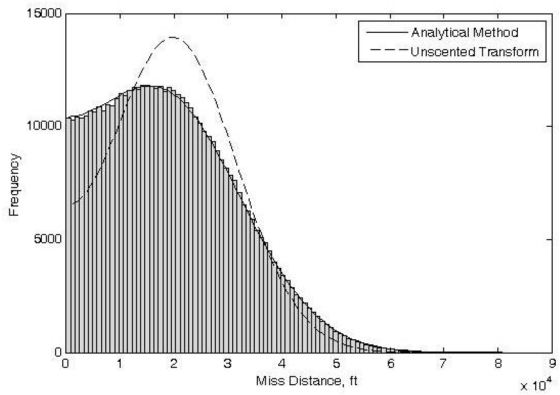

Figure 6. Distribution of miss distance, $\psi = 45^{\circ}$, low speed / low speed encounter, nominal miss distance = 18,000 ft.

Table 7. Estimated probability of loss of separation from Monte Carlo simulation and from Taylor series method, low speed / low speed encounter.

Intercept Angle (deg)	Nominal Miss Distance (ft) 0	6,000	12,000	18,000	24,000	30,000
15	84.8 / 84.7	64.3 / 64.6	48.5 / 49.1	37.8 / 38.7	30.6 / 31.6	25.5 / 26.6
30	84.8 / 84.7	76.1 / 76.2	62.7 / 63.0	48.7 / 49.2	36.4 / 37.1	27.0 / 27.5
45	84.7 / 84.7	78.6 / 78.6	65.7 / 65.9	50.0 / 50.2	34.8 / 35.2	22.7 / 22.9
60	84.7 / 84.7	79.4 / 79.5	66.9 / 67.0	50.3 / 50.5	33.8 / 34.0	20.4 / 20.4
75	84.7 / 84.7	79.8 / 80.0	67.4 / 67.5	50.4 / 50.6	33.1 / 33.2	19.0 / 19.0
90	84.8 / 84.7	80.3 / 80.2	67.9 / 67.8	50.5 / 50.4	32.6 / 32.3	17.9 / 17.6

Note: Data is presented in the form A / B, where A is the probability of loss of separation from Monte Carlo simulation and B is the estimate from the Taylor series method.

7 Conclusion

With improvements in sensors and new technologies for navigation/air traffic control such as ADS-B coming online, air traffic control has more precise position and velocity information than ever before. However, these estimates still have errors and uncertainties and by incorporating measure of these errors and uncertainties into calculations one can estimate how accurate those calculations really are. In this paper we have evaluated several methods for incorporating uncertainty into calculations of time to loss of separation and probability of loss of separation. The methods evaluated included a naïve approach where the uncertainty was ignored altogether, a standard Taylor series expansion technique, an unscented transform approach, and a modified series expansion technique for the time to loss of separation.

For the time to loss of separation calculation, the modified series expansion technique was clearly the best approach. It was the only algorithm that provided results across the entire problem space and it provided good accuracy for both the expected value and standard deviation with an average error less than 5%. The modified series expansion method avoided using the series expansion of the time to loss of separation equation because it is not well behaved. Instead the time to loss of separation equation was used when the nominal miss distance was with the minimum separation distance and the time to closest approach equation for larger miss distances. For the standard deviation, a Taylor series expansion technique is used, but instead of expanding the time to loss of separation equation, the time to closest approach equation is again used.

For calculating the probability of loss of separation, a standard Taylor series expansion provided to be extremely effective along with the assumption of a folded normal distribution for the miss distance. The assumption of a folded normal distribution is based on the form of the miss distance equation and the shapes of the distributions from the Monte Carlo simulations. The probability of loss of separation was estimated within 1% for all of the cases examined.

These two recommended algorithms for time to loss of separation and probability of loss of separation are straight forward and computationally efficient. Developing separation algorithms based on these sorts of probabilistic algorithms may provide improvements in safety of flight and efficient utilization of airspace. These algorithms may be particularly applicable in areas where surveillance radar is unavailable or when dealing with air traffic that maneuvers unexpectedly.

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