A PROBABILITY ASSESSMENT OF THE FLIGHT SAFETY RELEVANT TO COLLISION AVOIDANCE

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Abstract

A key aspect of Air Traffic Management in the future is to determine (i) the technical requirements to (ii) ensure safety with (iii) increased capacity. The methods to calculate collision probabilities have been applied to reduced vertical separation minima, to lateral separation, to crossing aircraft to free flight and to flight in terminal areas.

In the present paper the cumulative probability of coincidence (CPC) is calculated for comparison with the ICAO Alternative Target Level of Safety (ATLS) probability of collision per nautical mile. The comparison of the CPC with the ATLS is made for four typical cruise flight conditions.

1 Introduction

The steady growth of air traffic at a rate of 3-7% per year over several decades has placed increasing demands on capacity that must be met with undiminished safety (Vismari & Júnior, 2011). The trend is in fact to improve safety, while meeting more stringent requirements for environment impact, efficiency and cost. The traditional method of safety assurance in Air Traffic Management (ATM) is the setting of separation rules (Houck & Powell, 2001). The separation distances are determined by: (i) wake vortex effects on approach to land and take-off queues at runways at airports (FAA, 2011; International Civil Aviation Organization [ICAO], 2007); (ii) collision probabilities for the in-flight phases of aircraft operations (Campos & Marques, 2002; Reich, 1966; Yuling & Songchen, 2010). Only the latter aspect is considered in the present paper.

A key aspect of ATM in the future (Eurocontrol, 1998) is to determine (i) the technical requirements to (ii) ensure safety with (iii) increased capacity. The concepts of 'capacity', 'safety' and 'technology' can be given a precise meaning (Eurocontrol, 2000) in the case of airways with aircraft flying on parallel paths with fixed lateral/vertical, or longitudinal separation: (i) the 'capacity' increases for smaller separation L; (ii) navigation and flight 'technology' should provide a reduced r.m.s. position error σ; (iii) the combination of L and σ should be such that the probability of collision (ICAO, 2006) does not exceed ICAO Target Level of Safety (TLS) of 5×10^{-9} per hour (ICAO, 2005). Thus the key issue is to determine the relation between aircraft separation L and position accuracy σ, which ensures that the ICAO TLS is met. Then the technically achievable position accuracy σ specifies L, viz. the safe separation distance (SSD).

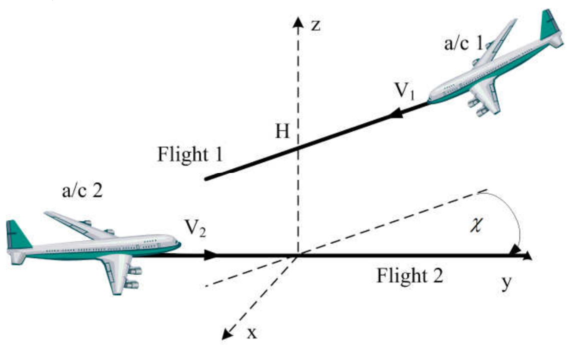

Fig. 1. Geometry of crossing aircraft.

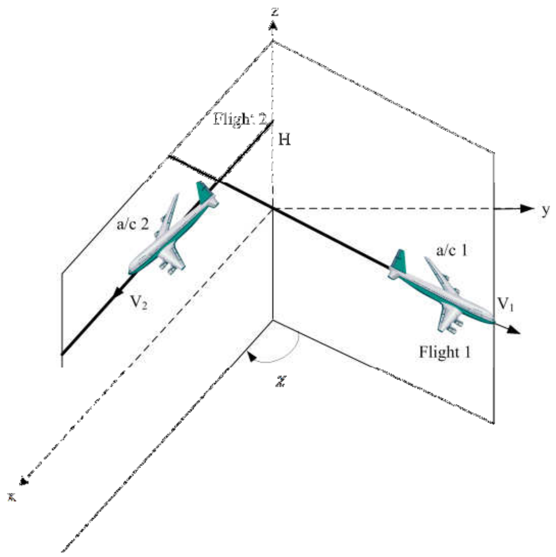

Fig. 2. Geometry of climbing/descending aircraft.

The two main ATM flight scenarios are: (i) parallel paths with fixed separations in flight corridors typical of transoceanic flight (Bousson, 2008); (ii) crossing (Figure 1) and climbing/descending (Figure 2) flight paths typical of terminal flight operations (Shortle at al., 2010; Zhang & Shortle, 2010). Since aircraft collisions are rare, two-aircraft events are more likely and this the case considered in the present paper.

The methods to calculate collision probabilities (Reich, 1966) have been applied to Reduced Vertical Separation Minima (RSVM), to lateral separation (Campos, 2001; Campos & Marques, 2002), to crossing aircraft (Campos & Marques, 2007, 2011), to free flight (Barnett, 2000) and to flight in terminal areas (Shortle et al., 2004). The fundamental input to the models of collision probabilities, is the probability distribution (Johnson & Balakrisshann, 1995; Mises, 1960) of flight path deviations; since it is known that the Gaussian distribution underestimates collision probabilities, and the Laplace distribution though better (Reich, 1966) is not too accurate, the generalized error distribution (Campos & Marques, 2002; Eurocontrol, 1988), and extensions or combinations have been proposed (Campos & Marques, 2004a). It can be shown (Campos & Marques, 2002) that for aircraft on parallel flight corridors an upper bound to the probability of collision is the probability of coincidence (PC). Its integration along the line joining the two aircraft leads to the cumulative probability of coincidence (CPC); the latter has the dimensions of inverse length, and multiplied by the airspeed, gains the dimensions of inverse time, i.e., can be compared to the ICAO TLS. Alternatively the ICAO TLS can be converted to collision per unit distance, which is directly comparable to the CPC. Since most commercial aircraft fly no faster than $V_0 = 625kt$, the ICAO TLS of $P_0 \leq 5\times 10^{-9}/h$, is met by $Q_0 = P_0 / V_0 \leq 8\times 10^{-12}/nm$. The latter can thus be used as an Alternate Target Level of Safety (ATLS).

In the present paper the CPC is calculated for comparison with the ICAO ATLS of $8\times 10^{-12}/nm$ probability of collision per nautical mile; three probability distributions are compared and discussed in detail: the Gaussian; the Laplace; a generalized error distribution, which is less simple but more accurate, viz. it has been shown to fit aircraft flight path deviations measured from radar tracks (Campos & Marques, 2002, 2004a; Eurocontrol, 1988). The comparison of the CPC with the ATLS, is made for four typical cruise flight conditions: (i/ii) lateral separation La =50nm in uncontrolled (e.g. oceanic) airspace (Section 3.1) and Lb = 5nm in controlled airspace (Section 3.2); (iii/iv) standard altitude separation Lc = 2000 ft used worldwide and RVSM Ld = 1000 ft introduced by Eurocontrol (1988) to increase capacity at higher flight levels (FL290 to FL410). Longitudinal separation along the same flight path could be considered to the limit of wake vortex effects (Campos & Marques, 2004b; Spalart, 1998). In each of the four cases: (i) the CPC is calculated for several position accuracies, to determine the minimum which meets the safety (ATLS) standard; (ii) the Gauss, Laplace and generalized distributions are compared for the collision probabilities of two aircraft with similar position errors; (iii) the case of aircraft with dissimilar position errors σ_1 and σ_2 is considered from the beginning, and analysed in detail for the most accurate probability distribution. The discussion summarizes the conclusions concerning airways capacity versus position accuracy, for an undiminished safety.

2. Comparison of probability distributions for aircraft flight path

An upper bound for the probability of collision of aircraft on parallel flight tracks (Section 2.1) is calculated using Laplace (Section 2.2), Gaussian (Section 2.3) and generalized (Section 2.4) probability distributions, for aircraft with generally dissimilar r.m.s. position errors.

2.1 Comparison of three probability distributions for flight path deviations

Consider two aircraft flying at: (i) either constant lateral or altitude separation L in parallel flight paths, (ii) or at constant longitudinal separation L on the same flight path . In the case of vertical separation there may be an asymmetry in the probability distributions, which has been treated elsewhere (Campos & Marques, 2007); in the case of longitudinal separation wake effects need to be considered as well (Campos & Marques, 2004b; Spalart, 1998). Apart from these effects, a class of probability distributions (Johnson & Balakshishnan, 1995; Mises, 1960) relevant to large aircraft flight deviations (Campos & Marques, 2002; Eurocontrol, 1998), which are rare events (Reiss & Thomas, 2001; Nassar et al., 2011), is the generalized error distribution (Campos & Marques, 2004a), viz.:

$$F_k(x;\sigma) = A \exp\left(-a|x|^k\right), \tag{1}$$

where k is the weight. The constant a is determined by the condition of unit total probability:

$$A = k\, a^{1/k} / [2\Gamma(1/k)], \tag{2a}$$

where $\Gamma(...)$ is the Gamma function. The constant a can be related by:

$$a^{2/k} = \sigma^{-2}[\Gamma(3/k)/\Gamma(1/k)], \tag{2b}$$

to the r.m.s. position error σ or variance σ^2. The case of weight unity in (2a,b), viz.:

$$k=1: \quad a = \sqrt{2}/\sigma, \qquad A = 1/\left(\sigma\sqrt{2}\right), \tag{3a,b}$$

corresponds by (1) to the Laplace probability distribution:

$$F_1(x;\sigma) = \left[1/\left(\sigma\sqrt{2}\right)\right] \exp\left\{-\sqrt{2}|x|/\sigma\right\}; \tag{4}$$

the case of weight two in (2a,b), viz.:

$$k=2: \quad a = 1/\left(2\sigma^2\right), \quad A = 2/\left(\sigma\sqrt{2\pi}\right), \tag{5a,b}$$

Leads by (1) to the Gaussian probability distribution:

$$F_2(x;\sigma) = \left[1/\left(\sigma\sqrt{2\pi}\right)\right] \exp\left\{-x^2/\left(2\sigma^2\right)\right\}; \tag{6}$$

the best approximation to large aircraft flight path deviations (Campos & Marques, 2002, 2007) corresponds approximately to weight one-half, so that (2a,b):

$$k=1/2: \quad a^4 = 120/\sigma^2, \quad A = \sqrt{15/2}\,\sigma, \tag{7a,b}$$

substituted in (1) leads to:

$$F_{1/2}(x;\sigma) = \left(\sqrt{15/2}/\sigma\right) \exp\left\{-\sqrt[4]{120}\,|x/\sigma|^{1/2}\right\}, \tag{8}$$

which may be designated for brevity the 'generalized' distribution. For any probability distribution, it can be shown (Campos & Marques, 2002) that an upper bound for the probability of collision is the probability of coincidence, which: implies (i) a deviation for the first aircraft, with r.m.s. position error σ_1; (ii) a deviation L-x for the second aircraft error σ_2. For statistically independent aircraft deviations, the probability of coincidence at position x the product:

$$P_k(x;L,\sigma_1,\sigma_2) = F_k(x;\sigma_1)\, F_k(L-x;\sigma_2). \tag{9}$$

Its integral over all positions along the line joining the two aircraft is the commutative probability of coincidence (CPC), viz.:

$$Q_k(L;\sigma_1,\sigma_2) = \int_{-\infty}^{+\infty} P_k(x;L,\sigma_1,\sigma_2)\, dx = \int_{-\infty}^{+\infty} F_k(x;\sigma_1)\, F_k(L-x;\sigma_2)\, dx, \tag{10}$$

and, in particular, for aircraft with equal r.m.s. position errors:

$$\sigma \equiv \sigma_1 = \sigma_2: \quad Q_k(L;\sigma) \equiv Q_k(L;\sigma,\sigma) = \int_{-\infty}^{+\infty} F_k(x;\sigma)\, F_k(L-x;\sigma)\, dx. \tag{11}$$

The CPC has the dimensions of inverse length. The ICAO TLS of 5×10^{-9}/h (12a) can be converted for a maximum airspeed $V_0 = 625\,kt$ in (12b) to a alternative target level of safety (ATLS) given

$$\bar{Q}_0 = 5\times10^{-9}\quad hour^{-1}, \qquad V_0 \le 625\,kt,$$
$$Q \le Q_0 = \bar{Q}_0 / V_0 \le 8\times10^{-12}\ nm^{-1}, \tag{12a-c}$$

which is an upper bound for the CPC. The safety criterion (12c) is applied next to the Laplace (§2.2), Gaussian (§2.3) and generalized (§2.4) probability distributions.

2.2 Laplace distributions for the dissimilar aircraft

The ATLS (12c) is the upper bound for the CPC (10) calculated for aircraft whose position errors follow the Laplace probability distribution (4), with dissimilar r.m.s. position errors for the two aircraft:

$$Q_0 \ge Q_1(L;\sigma_1,\sigma_2) = \left[1/(2\sigma_1\sigma_2)\right] \times \int_{-\infty}^{+\infty} \exp\left\{-\sqrt{2}\left[|x|/\sigma_1 + |L-x|/\sigma_2\right]\right\} dx. \tag{13}$$

The appearance of modulus in the argument of the exponential in (13), requires that the range of integration $-\infty,+\infty$ be split in three parts. The first part corresponds to coincidence at $0 \le x \le L$ between the flight paths of the two aircraft:

$$2\sigma_1\sigma_2 Q_{11} = \int_0^L \exp\left\{-\sqrt{2}\left[x/\sigma_1 + (L-x)/\sigma_2\right]\right\} dx = \exp\left(-\sqrt{2}L/\sigma_2\right)\int_0^L \exp\left[-\sqrt{2}\,x\left(1/\sigma_1 - 1/\sigma_2\right)\right] dx, \tag{14}$$

and involves an elementary integration:

$$2\sigma_1\sigma_2 Q_{11} = \exp\left(-\sqrt{2}\,L/\sigma_2\right) \times\left\{1-\exp\left[-\sqrt{2}\,L\left(1/\sigma_1 - 1/\sigma_2\right)\right]\right\} \times\left[\sqrt{2}\left(1/\sigma_1 - 1/\sigma_2\right)\right]^{-1}, \tag{15}$$

and simplifies to:

$$Q_{11} = \left[2\sqrt{2}\left(\sigma_2 - \sigma_1\right)\right]^{-1} \times\left\{\exp\left(-\sqrt{2}\,L/\sigma_2\right) - \exp\left(-\sqrt{2}\,L/\sigma_1\right)\right\}, \tag{16}$$

and should be the main contribution (i) to (13). To evaluate (13) exactly, the remaining contributions, besides (i), are also considered: (ii) the coincidence to the outside the path of second aircraft:

$$2\sigma_1\sigma_2 Q_{12} = \int_L^\infty \exp\left\{-\sqrt{2}\left[x/\sigma_1 + (x-L)/\sigma_2\right]\right\}, \tag{17}$$

leads to an elementary integral:

$$2\sigma_1\sigma_2\ Q_{12} = \exp\left(\sqrt{2}\,L/\sigma_2\right) \times\int_L^\infty \exp\left\{-\sqrt{2}\,x\left(1/\sigma_1 + 1/\sigma_2\right)\right\} dx = \exp\left(\sqrt{2}\,L/\sigma_2\right)\left[\sqrt{2}\left(1/\sigma_1 + 1/\sigma_2\right)\right]^{-1}, \times\exp\left[-\sqrt{2}L\left(1/\sigma_1 + 1/\sigma_2\right)\ \right], \tag{18}$$

which simplifies to:

$$Q_{12} = \left[2\sqrt{2}\left(\sigma_1 + \sigma_2\right)\right]^{-1} \exp\left(-\sqrt{2}\,L/\sigma_1\right); \tag{19}$$

(iii) the coincidence $-\infty < x < 0$ outside the flight path of the first aircraft:

$$2\sigma_1\sigma_2\,Q_{13} = \int_{-\infty}^0 \exp\left\{\sqrt{2}\left[x/\sigma_1 - (L-x)/\sigma_2\right]\right\} dx = \exp\left(-\sqrt{2}\,L/\sigma_2\right)\int_0^\infty \exp\left\{-\sqrt{2}\,x\left(1/\sigma_1 + 1/\sigma_2\right)\right\} dx, \tag{20}$$

is again an elementary integral:

$$Q_{13} = \left[2\sqrt{2}\left(\sigma_2 + \sigma_1\right)\right]^{-1} \exp\left(-\sqrt{2}\,L/\sigma_2\right). \tag{21}$$

The sum of (21), (19) and (16) specifies the CPC where:

$$Q_1(L;\sigma_1,\sigma_2) = \left[2\sqrt{2}\left(\sigma_2 - \sigma_1\right)\right]^{-1} \times\left[\exp\left(-\sqrt{2}L/\sigma_2\right) - \exp\left(-\sqrt{2}L/\sigma_1\right)\right] +\left[2\sqrt{2}\left(\sigma_2 + \sigma_1\right)\right]^{-1} \times\left[\exp\left(-\sqrt{2}\,L/\sigma_1\right) + \exp\left(-\sqrt{2}\,L/\sigma_2\right)\right], \tag{22a}$$

for the Laplace distribution:

$$Q_1(L;\sigma_1,\sigma_2) = Q_{11} + Q_{12} + Q_{13} \le Q_0 = 8\times10^{-12}\ nm^{-1}, \tag{22b}$$

and hence (12c) the safety criterion. Of the preceding expressions, only (16) breaks down for $\sigma_2 - \sigma_1 = 0$, i.e. aircraft with the same r.m.s. position error $\sigma_1 = \sigma_2 \equiv \sigma$. In this case the probability of coincidence is given: (i) between the flight paths of the two aircraft, instead of (14-16) by:

$$\sigma_1 = \sigma_2 \equiv \sigma: \qquad \bar{Q}_{11} = \left(2\sigma^2\right)^{-1} \times\int_0^L \exp\left(-\sqrt{2}\,L/\sigma\right) dx = \left(L/2\sigma^2\right)\exp\left(-\sqrt{2}\,L/\sigma\right); \tag{23}$$

(ii) outside the flight path of the second aircraft (17-19) is replaced by:

$$\sigma_1 = \sigma_2 \equiv \sigma : \qquad \bar{Q}_{12} = \left(2\sigma^2\right)^{-1} \exp\left(\sqrt{2}\, L/\sigma\right) \int_L^\infty \exp\left(-2\sqrt{2}\, x/\sigma\right) dx = \left(4\sqrt{2}\,\sigma\right)^{-1} \exp\left(-\sqrt{2}\, L/\sigma\right); \tag{24}$$

(iii) outside the flight path of the second aircraft (20-22) is replaced by:

$$\sigma_1 = \sigma_2 \equiv \sigma : \qquad \bar{Q}_{13} = \left(2\sigma^2\right)^{-1} \exp\left(-\sqrt{2} L/\sigma\right) \times \int_{-\infty}^0 \exp\left(2\sqrt{2}\, x/\sigma\right) dx = \left(4\sqrt{2}\sigma\right)^{-1} \exp\left(-\sqrt{2}\, L/\sigma\right). \tag{25}$$

The sum of (23), (24) and (25) specifies:

$$Q_1 = \exp\left(-\sqrt{2}\, L/\sigma\right)\left(2\sigma\right)^{-1}\left(L/\sigma + 1/\sqrt{2}\right), \tag{26a}$$

as the safety criterion

$$\sigma_1 = \sigma_2 \equiv \sigma : \qquad Q_1\left(L;\sigma\right) = \bar{Q}_{11} + \bar{Q}_{12} + Q_{13} \le Q_0 = 8\times 10^{-12}\, nm^{-1}, \tag{26b}$$

for Laplace probabilities with equal r.m.s. position errors for both aircraft.

2.3 Gaussian distribution with distinct variances

The ATLS (12c) is the upper bound for the CPC (10) calculated next for aircraft whose flight path deviations satisfy the Gaussian probability distribution (6) for aircraft with dissimilar variances of position errors:

$$Q_0 \ge Q_2\left(L;\sigma_1,\sigma_2\right) = \left(2\pi\sigma_1\sigma_2\right)^{-1} \times \int_{-\infty}^{+\infty} \exp\left\{-\left[\left(x/\sigma_1\right)^2 + \left(\left(L-x\right)/\sigma_2\right)^2\right]/2\right\} dx. \tag{27}$$

The integral in (27) does not need splitting to be evaluated, e.g. in the case of equal variances:

$$\sigma_1 = \sigma_2 \equiv \sigma : \qquad Q_0 \ge Q_2\left(L;\sigma\right) = \left(2\pi\sigma^2\right)^{-1} \times \int_{-\infty}^{+\infty} \exp\left\{-\left[x^2 + \left(L-x\right)^2\right]/\left(2\sigma^2\right)\right\} dx, = \left(2\pi\sigma^2\right)^{-1} \exp\left[-L^2/2\sigma^2\right] \times \int_{-\infty}^{+\infty} \exp\left\{-\left(x^2 - xL\right)/\sigma^2\right\} dx, \tag{28}$$

the change of variable (29a):

$$y = \left(x - L/2\right)/\sigma, \qquad \int_{-\infty}^{+\infty} \exp\left(-y^2\right) dy = \sqrt{\pi}, \tag{29a,b}$$

leads to a Gaussian integral (29b), viz.:

$$Q_2\left(L;\sigma\right) = \left(2\pi\sigma^2\right)^{-1} \exp\left[-L^2/2\sigma^2\right] \times \int_{-\infty}^{+\infty} \exp\left(-y^2 + L^2/4\sigma^2\right) dy; \tag{30}$$

using (29b) in (30) leads to:

$$Q_2\left(L;\sigma\right) = \left(2\sqrt{\pi}\sigma^2\right)^{-1} \times \exp\left[-\left(L/2\sigma\right)^2\right] \le Q_0 = 8\times 10^{-12}\, nm^{-1}, \tag{31}$$

as the safety criterion. In the more general case (27) of aircraft with dissimilar r.m.s. position errors:

$$Q_2 = \left(2\pi\sigma_1\sigma_2\right)^{-1} \exp\left(-L^2\sigma_2^{-2}/2\right) \times \int_{-\infty}^{+\infty} \exp\left\{-\left[\left(x^2/2\right)\left(\sigma_1^{-2} + \sigma_2^{-2}\right) - xL\sigma_2^{-2}\right]\right\} dx, \tag{32}$$

the change of variable:

$$y = \left[x\sqrt{\sigma_1^{-2} + \sigma_2^{-2}} - L\sigma_2^{-2}/\sqrt{\sigma_1^{-2} + \sigma_2^{-2}}\right]/\sqrt{2}, \tag{33}$$

leads again to a Gaussian integral (29b), viz.:

$$Q_2 = \left(2\pi\sigma_1\sigma_2\right)^{-1} \exp\left(-L^2\sigma_2^{-2}/2\right) \times \exp\left\{-L^2\sigma_2^{-4}/\left[2\left(\sigma_1^{-2} + \sigma_2^{-2}\right)\right]\right\} \int_{-\infty}^{+\infty} \exp\left(-y^2\right) dy, \tag{34}$$

which simplifies the safety condition to:

$$Q_0 \ge Q_2\left(L;\sigma_1,\sigma_2\right) = \left(2\sqrt{\pi}\sigma_1\sigma_2\right)^{-1} \times \exp\left\{-\left(L^2/2\right)/\left[\left(\sigma_1\right)^2 + \left(\sigma_2\right)^2\right]\right\}. \tag{35}$$

This reduces to (31) in the case of equal r.m.s. position errors.

2.4 Generalized error or Gaussian distribution

The safety condition (12c) for (10) the more accurate (8) generalized probability distribution:

$$c \equiv 4\sqrt{120} \equiv 3.310: \quad Q_0 \geq Q_3\left(L;\sigma_1,\sigma_2\right)$$
$$= \left[15/\left(2\sigma_1\sigma_2\right)\right]$$
$$\times \int_{-\infty}^{+\infty} \exp\left\{-c\left[\left|x/\sigma_1\right|^{1/2} + \left|\left(L-x\right)/\sigma_2\right|^{1/2}\right]\right\} dx, \tag{36}$$

requires again a split in the region of integration as for the Laplace distribution (§2.2), with the difference that the evaluation of integrals is not elementary. The contribution to the cumulative probability of coincidence of the position between the flight paths of the two aircraft is:

$$Q_{31} = \left[\left(15/2\sigma_1\sigma_2\right)\right]\exp\left\{-c\left[\sqrt{x/\sigma_1} + \sqrt{\left(L-x\right)/\sigma_2}\right]\right\}, \tag{37a}$$

$$= \frac{15}{2\sigma_1,\sigma_2}\sum_{n=0}^{\infty}\frac{(-)^n c^n}{n!}\int_0^L \left\{\sqrt{x/\sigma_1} + \sqrt{\left(L-x\right)/\sigma_2}\right\}^n dx, \tag{37b}$$

where the exponential was expanded in power series, and binomial theorem:

$$\left[\sqrt{x/\sigma_1} + \sqrt{\left(L-x\right)/\sigma_2}\right]^n =$$
$$\sum_{m=0}^{n}\left\{n!/\left[m!\left(n-m\right)!\right]\right\}\left(x/\sigma_1\right)^{m/2}\left[\left(L-x\right)/\sigma_2\right]^{(n-m)/2}, \tag{38}$$

can also be used:

$$Q_{31} = \frac{15}{2\sigma_1\sigma_2} \tag{39a}$$
$$\times\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-)^m c^m}{m!\left(n-m\right)!}\sigma_1^{-m/2}\sigma_2^{-(n-m)/2}\, I_{nm},$$

and $I_{n,m}$ denotes the integral.

$$I_{nm} \equiv \int_0^L x^{m/2}\left(L-x\right)^{(n-m)/2} dx, \tag{39b}$$

which can be reduced to an Euler's Beta function. The beta function is defined (Whittaker & Watson, 1927) by:

$$B(\alpha,\beta) \equiv \int_0^1 y^{\alpha-1}(1-y)^{\beta-1}\, dy = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta), \tag{40a,b}$$

and can be evaluated in terms of Gamma functions. The integrals (39b) are evaluated in terms of the Beta function via a change of variable.

$$y \equiv x/L: \qquad L^{-1-n/2}\, I_{nm} = \int_0^1 y^{m/2}\left(1-y\right)^{(n-m)/2} dy$$
$$= B\left(1+m/2, 1+\left(n-m\right)/2\right)$$
$$= \Gamma\left(1+m/2\right)\Gamma\left(1+\left(n-m\right)/2\right)/\Gamma\left(2+n/2\right). \tag{41a-c}$$

Substitution of (41c) in (39a) yields:

$$Q_{31} = \frac{15L}{2\sigma_1\sigma_2}\sum_{n=0}^{\infty}\frac{(-)^m c^m}{m!\left(n-m\right)!}\left(\frac{L}{\sigma_1}\right)^{m/2}\left(\frac{L}{\sigma_2}\right)^{(n-m)/2}$$
$$\times\frac{\Gamma\left(1+m/2\right)\Gamma\left(1+n/2-m/2\right)}{\Gamma\left(2+n/2\right)}, \tag{42}$$

as the first contribution to (36).

It may be expected that (42) is the main contribution to (36), and thus we seek upper bounds for the two remaining contributions are sought next. The second contribution to (36) concerns coincidence outside the path of the second aircraft:

$$Q_{32} = \left[15/\left(2\sigma_1\sigma_2\right)\right]$$
$$\times\int_L^{\infty}\exp\left\{-c\left[\sqrt{x/\sigma_1} + \sqrt{\left(x-L\right)/\sigma_2}\right]\right\} dx; \tag{43a}$$

an upper bound is obtained by replacing $x \geq L$ by L in the first exponential:

$$Q_{32} \leq \left[15/\left(2\sigma_1\sigma_2\right)\right]\exp\left(-c\sqrt{L/\sigma_1}\right)$$
$$\times\int_L^{\infty}\exp\left[-c\sqrt{\left(x-L\right)/\sigma_2}\right] dx, \tag{43b}$$

the change of variable (44a) leads:

$$y = c\sqrt{\left(x-L\right)/\sigma_2},$$
$$Q_{32} \leq \frac{15}{\sigma_1 c^2}\exp\left(-c\sqrt{L/\sigma_1}\right)\int_0^{\infty} e^{-y}\, y\, dy, \tag{44a,b}$$

to an integral (44b) which is evaluated in terms (Whittaker & Watson, 1927; Goursat, 1950) of the Gamma function:

$$\int_0^{\infty} e^{-y} y^m dy = \Gamma\left(1+m\right) \equiv m!; \tag{45a}$$

using (45a) in (44b) leads to the upper bound for the second contribution to ((36), viz.:

$$Q_{32} \leq \left[15/\left(\sigma_1 c^2\right)\right]\exp\left(-c\sqrt{L/\sigma_1}\right). \tag{45b}$$

The third contribution to (36) corresponds to coincidence outside the flight path of the first aircraft:

$$Q_{33}=\left[15/\left(2\sigma_1\sigma_2\right)\right]\times\int_{-\infty}^{0}\exp\left\{-c\left[\sqrt{-x/\sigma_1}+\sqrt{\left(L-x\right)/\sigma_2}\right]\right\}dx, \quad (46a)$$

$$=\left[15/\left(2\sigma_1\sigma_2\right)\right]\int_0^\infty \exp\left(-c\sqrt{x/\sigma_1}\right)\times\exp\left(-c\sqrt{\left(L+x\right)}/\sigma_2\right)dx; \quad (46b)$$

an upper bound is obtained by replacing in the second exponential $L+x\geq L$ by L:

$$Q_{33}\leq\left[15/\left(2\sigma_1\sigma_2\right)\right]\exp\left(-c\sqrt{L/\sigma_2}\right)\times\int_0^\infty \exp\left(-c\sqrt{x/\sigma_1}\right)dx. \quad (46c)$$

The last integral is evaluated via a change of variable:

$$y=c\sqrt{x/\sigma_1}: \qquad Q_{33}\leq\left[15/\left(\sigma_2c^2\right)\right]\times\exp\left(-c\sqrt{L/\sigma_2}\right)\int_0^\infty e^{-y}y\,dy, \quad (47a)$$

leading by (45a) to:

$$Q_{33}\leq\left[15/\left(\sigma_2c^2\right)\right]\exp\left(-c\sqrt{L/\sigma_2}\right). \quad (47b)$$

If the upper bounds (45b) and (47b) are small relative to the first contribution (42) to (36), viz.:

$$Q_{31}>>\left(15/c^2\right)\times\left[\sigma_1^{-1}\exp\left(-c\sqrt{L/\sigma_1}\right)+\sigma_2^{-1}\exp\left(-c\sqrt{L/\sigma_2}\right)\right]\geq Q_{32}+Q_{33}, \quad (48a)$$

then (46) alone can be used in the safety criterions (12c), viz.:

$$8\times10^{-12}\,\mathrm{nm}^{-1}=Q_0\geq Q_{31}, \quad (48b)$$

with an error whose upper bound is specified by the ratio of the r.h.s. to l.h.s. of (48a).

If the latter error is not acceptable, then (43a) and (46b) must be evaluated exactly. Concerning the second contribution (43a) to (36), the change of variable (49a):

$$x=L\cosh^2\alpha, \qquad x-L=L\sinh^2\alpha, \quad (49a,b)$$

implies (49b), and transforms (43a) to:

$$Q_{32}=\left[15\,L/\left(\sigma_1\sigma_2\right)\right]\times\int_0^\infty d\alpha\cosh\alpha\sinh\alpha\times\exp\left\{-c\sqrt{L}\left(\sigma_1^{-1/2}\cosh\alpha+\sigma_2^{-1/2}\sinh\alpha\right)\right\} \quad (49c)$$

Concerning the third contribution (46b) to (36) the change or variable (50a):

$$x=L\sinh^2\alpha, \qquad x+L=L\cosh^2\alpha, \quad (50a,b)$$

implies (50b), and leads to:

$$Q_{33}=\left[15L/\left(\sigma_1\sigma_2\right)\right]\int_0^\infty d\alpha\sinh\alpha\cosh\alpha\times\exp\left\{-c\sqrt{L}\left(\sigma_1^{-1/2}\sinh\alpha+\sigma_2^{-1/2}\cosh\alpha\right)\right\}, \quad (50c)$$

which is similar to (49c) interchanging σ_1 with σ_2. The integrals (49c) and (50c) can be evaluated numerically, and coincide in the case of equal r.m.s. position errors:

$$\sigma_1=\sigma_2\equiv\sigma: \qquad Q_{32}=Q_{33}=\frac{15L}{4\sigma^2}\int_0^\infty\exp\left(-c\sqrt{L/\sigma}\,e^\alpha\right)\left(e^{2\alpha}-e^{-2\alpha}\right)d\alpha. \quad (51a)$$

A further change of variable (51b) yields:

$$y=c\sqrt{L/\sigma}\,e^\alpha: \quad Q_{32}+Q_{33}=\frac{15L}{2\sigma^2}\int_{c\sqrt{L/\sigma}}^\infty e^{-y}\left\{\left[\sigma/\left(c^2L\right)\right]y-\left(c^2L/\sigma\right)y^{-3}\right]dy. \quad (51b)$$

The exponential integral of order *n* is defined (Abramowitz & Stegun, 1965) by:

$$E_n(z)=\int_z^\infty y^n e^{-y}dy, \quad (52a)$$

and allows evaluation of (51b), viz.:

$$Q_{32}+Q_{33}=\left[15L/\left(2\sigma^2\right)\right]\times\left\{\left[\sigma/\left(c^2L\right)\right]E_1\left(c\sqrt{L/\sigma}\right)-\left[\left(c^2L\right)/\sigma\right]E_{-3}\left(c\sqrt{L/\sigma}\right)\right\}. \quad (52b)$$

The sum of the three contributions (42) plus (49c) and (50c) or (52b), specifies:

$$8\times10^{-12}\,\mathrm{nm}^{-1}=Q_0\geq Q_3\left(L;\sigma_1,\sigma_2\right)=Q_{31}+Q_{32}+Q_{33}, \quad (52c)$$

as the safety condition.

3 Application to four ATM scenarios

The preceding safety-separation criteria are applied to the four major airway scenarios, viz. lateral separation in uncontrolled (§3.1) and controlled (§3.2) airspace and standard (§3.3) and reduced (§3.4) vertical separation.

3.1 Lateral separation in oceanic airspace

The lateral separation in oceanic airspace is (53a):

$$L_a = 50\,nm, \quad \sigma_a = 7, 6, 5, 4, 3, 2, 1\ nm, \qquad (53a,b)$$

and the r.m.s. position error is given the seven values (53b) in *Table 1*, where the CPC are indicated for the Laplace, Gaussian and generalized probabilities. The Table 1 concerns aircraft with similar r.m.s. position errors.

3.2 Lateral separation in controlled airspace

In controlled airspace the lateral separation (53a) is reduced to (54a):

$$L_b = 5\,nm, \sigma_b = 0, 7, 06, 05, 0.4, 0.3, 0.1\ nm, (54a,b)$$

and the r.m.s. position errors considered (54b) are correspondingly smaller than (53b). As shown in Table 2 the safety criterion is met.

3.3 – Standard vertical separation

The probabilities of vertical separation can be less upward than downward, due to gravity, proximity to the service ceiling, etc…; apart from this correction (Campos & Marques, 2007, 2011), the preceding theory can be used with the standard vertical separation (55a):

$$L_c = 2000\ \ ft, \sigma_c = 300, 200, 150, 100, 70, 50, 40\ ft, \qquad (55a,b)$$

and r.m.s. deviations (55b), showing in Table 3 that the safety criterion is met.

3.4 Reduced vertical separation

The RSVM introduced by Eurocontrol in upper European air space halves the vertical separation (56a) to (58a):

$$L_d = 1000\,ft, \quad \sigma_d = 150, 100, 75, 50, 40, 35, 25, 20\ ft, \qquad (56a,b)$$

and the r.m.s. position errors are correspondingly reduced from (56b) to (58b) in Table 4.

4 Discussion

The separation-position accuracy or technology-capacity trade-off was made for an air corridor ATM scenario with aircraft flying along the same flight path or on parallel flight paths with a constant separation. The generalized probability distribution leads to lower values of the r.m.s. deviation to meet the ICAO TLS, than the Laplace and Gaussian. Although the latter distributions are simpler, they underestimated the collision risk, and do not yield safe predictions. Using simultaneously lateral and vertical separations leads to much lower collision probabilities, and allows reducing each separation for the same overall safety. In the case of aircraft flying on parallel tracks, it is desirable to use alternate directions of flight, because: (i) adjacent flight paths correspond to aircraft flying in opposite directions, which spend less time close to each other, reducing the collision probability (Campos & Marques, 2002; Eurocontrol, 1988; Reich, 1966); (ii) the aircraft which spend more time 'close' by are on a parallel track at twice the separation 2L, thus allowing a larger r.m.s. position error for the same safety.

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Table 1. Lateral *a* CPC for the Laplace, Gaussian and generalized probabilities.

Probability distribution		Laplace	Gauss	Generalized
quantity	σ_a	Q1a	Q2a	Q3a
Unit	nm	-	-	-
10	nm	2,42E-04	5,45E-06	3,80E-04
5	nm	7,72E-07	1,57E-13	3,58E-05
4	nm	3,47E-08	1,91E-19	1,28E-05
3	nm	1,68E-10	2,17E-32	2,75E-06
2	nm	2,84E-15	9,77E-70	1,92E-07
1	nm	4,95E-30	1,04E-272	3,88E-10
0.5	nm	3,84E-60	0,00E-00	4,70E-14

Table 2. Lateral *b* CPC for the Laplace, Gaussian and generalized probabilities.

Probability distribution		Laplace	Gauss	Generalized
quantity	σ_b	Q1b	Q2b	Q3b
Unit	nm	-	-	-
1,0	nm	2,42E-03	5,45E-04	3,80E-03
0,5	nm	7,72E-06	1,57E-11	3,58E-04
0,4	nm	3,47E-07	1,91E-17	1,28E-04
0,3	nm	1,68E-09	2,17E-30	2,75E-05
0,2	nm	2,84E-14	9,77E-68	1,92E-06
0,1	nm	4,95E-29	1,04E-270	3,88E-09
0,05	nm	3,84E-59	0,00E-00	4,70E-13

Table 3. Vertical *a* CPC for the Laplace, Gaussian and generalized probabilities.

Probability distribution		Laplace	Gauss	Generalized
quantity	σ_c	Q1c	Q2c	Q3c
Unit	ft	-	-	-
300	ft	9,88E-07	4,68E-11	4,03E-06
200	ft	1,93E-08	9,79E-17	8,76E-07
100	ft	5,39E-14	1,05E-48	2,11E-08
50	ft	1,10E-25	2,16E-178	8,12E-11
40	ft	1,24E-31	6,49E-276	8,21E-12

Table 4. Vertical *b* CPC for the Laplace, Gaussian and generalized probabilities.

Probability distribution		Laplace	Gauss	Generalized
quantity	σ_d	Q1a	Q2a	Q3a
Unit	ft	-	-	-
150	ft	1,98E-06	1,87E-10	8,05E-06
100	ft	3,86E-08	3,92E-16	1,71E-06
50	ft	1,08E-13	4,20E-48	4,04E-08
15	ft	2,55E-41	0,00E-00	6,86E-13