ANALYSIS OF MULTIPLE SITE DAMAGE IN AIRCRAFT STRUCTURES BY USING WEIGHT FUNCTION METHOD IN FRACTURE MECHANICS

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Abstract

This paper presents a review of recent progress of the development of weight function method for analyzing multiple site damage in aircraft structural panels. Classical weight function formulations were used, for the first time, for solutions of the strip yield models for two special collinear crack configurations, three symmetric collinear cracks with plastic zones critical coalesced and two equal-length collinear cracks in an infinite sheet. The plastic zone sizes and crack opening displacements were extensively verified and validated by excellent agreement with available analytical and finite element method results. Furthermore, weight function method for general collinear cracks was developed and applied for fracture mechanics analyses of three collinear cracks in an infinite sheet. The key fracture mechanics parameters including stress intensity factors, crack opening displacements and strip yield plastic zone sizes for three equal-length collinear cracks are presented. The weight function method provides a versatile, accurate and very efficient approach to the analyses of multiple site damage in aircraft structures.

1 Introduction

Failure due to Multiple Site Damage (MSD) imposes great threat to the safe operation of aircraft structures. A historical milestone case of this type of failure was the in-flight disintegration of a 5.5m long piece of the pressure cabin skin of upper fuselage of Aloha Airlines Boeing 737-200 over Hawaii in 1988 [1]. After the Aloha accident, all the major commercial airplane manufacturers were required to evaluate their aircraft for MSD in the critical areas of the wing, empennage and pressure fuselage [2].

Various models and methods [3-5] have been developed to determine the stress intensity factors, plastic zone sizes and crack opening displacements for the prediction of the fatigue life and residual strength for structures with MSD. Most current approaches rely mainly on advanced numerical techniques, especially the Finite Element Method (FEM). Despite its powerfulness, reliable FEM solutions for MSD require great efforts, time and experience in modeling and computation. Classical analytical method, e.g. the complex variable method, is limited to idealized MSD configurations. The weight function method (WFM) is versatile and cost-effective for tackling crack problems, especially for complex load conditions [6-8]. However, to the authors' knowledge, the potential of WFM for MSD problems has not yet been exploited. Recent efforts by the present authors have shown that the WFM provides a valuable approach to the MSD analysis. The approach is outlined in this paper.

2 Weight function method and application to special collinear cracks

The weight function method was first proposed by Bueckner [9], and further advancements were made by many researchers. Weight functions for various two-dimensional crack problems were given in Ref.[10].

2.1 Basic principle

According to the weight function theory, for a crack subjected to an arbitrary pressure σ(*x*) distributed at the crack faces, the non-dimensional stress intensity factor *f* can be determined by a simple quadrature [10].

$$f = \int_0^a \frac{\sigma(x)}{\sigma} \cdot \frac{m(a,x)}{\sqrt{\pi a}} \mathrm{d}x \qquad (1a)$$

where $m(a, x)$ is the weight function for the crack body, and is given by

$$m(a,x) = \frac{E'}{f_r(a)\sigma\sqrt{\pi a}} \frac{\partial u_r(a,x)}{\partial a} \qquad (1b)$$

$E'=E$ for plane stress, $E'=E/(1-\nu^2)$ for plane strain, σ is a reference stress, *a* and *x* are the non-dimensional crack length and coordinate along the crack normalized by the characteristic length *W* (often taken as unity), here *W* refers to half plate width for the finite width panel containing a center crack. $f_r(a)$ and $u_r(a, x)$ are the stress intensity factor and crack opening displacement, repectively, for a reference load case.

It should be emphasized that, the σ(*x*) in Eq. (1a) refers to the stress distribution at the prospective crack line, and is determined from stress analysis for the same configuration but without crack. This fact greatly simplifies the crack problem analysis, avoiding all the complications related to the singularity behavior of the crack tip field and the repeated calculations at different crack lengths. The method separates the two parameters on which the stress intensity factor depends, load and geometry. Once the weight function is known for a given crack geometry, it can be used to solve linear elastic crack problems for any other load case for the same cracked body. This particular feature is especially useful for the strip yield model analysis of multiple site damage, in which uniform stress segments are distributed in each crack tip region.

Crack opening displacements can also be easily determined when the relevant weight function, $m(a, x)$, is available. From Eq. (1b), we have

$$u(a,x) = \sigma / E' \cdot \int_{a_0}^{a} [f(s)\sqrt{\pi a}] \cdot m(s,x)\mathrm{d}s \qquad (2)$$

Where the non-dimensional stress intensity $f(s)$ is obtained using Eq. (1a).

2.2 Weight function method for some special collinear crack configurations

Consider two multiple site damage cases: i) one large center crack formed by coalescing three un-equal length center cracks in a panel of finite width, with compressive yield stress σ_s uniformly acting along the un-cracked ligament and in the crack tip region, Fig.1a; and ii) Strip yield model for two equal-length collinear cracks in an infinite sheets shown in Fig.2.

These two cases can be treated as a single crack. For the first case in which the plastic zones are coalesced, the analysis is conducted by assuming the coalesced three un-equal length cracks as one single fictitious crack subjected to segment pressure distribution in plastic zones, in addition to the applied external load, Fig.1b.

Essentially, the Dugdale strip yield model is the superposition of two linear elastic solutions. One is for remote uniform tension stress, which is available in Ref. [11]. Another is for segment uniform compressive yield stress acting in the plastic zones. The stress intensity factor and crack opening displacement for this load case can be determined by using WFM, equations (1) and (2). The weight functions for a center crack in a finite sheet are given in Ref.[10].

For the three coalesced cracks, the critical applied stress and the fictitious crack length are determined based on the following conditions [4]:

i) Vanishing of the stress singularity at the fictitious crack tips shown in Fig.1b; and

ii) Zero of the minimum crack opening displacement at the ligament $[a_1, a_1+d]$.

For a given crack configuration where a_1=0.3, a_2=0.1 and *d*=0.2, the critical applied stress σ_c/σ_s and fictitious crack length *a* equal to 0.3777 and 0.7417 respectively. Figure 3 shows the corresponding crack opening displacements determined by WFM and FEM, and very good agreement is observed.

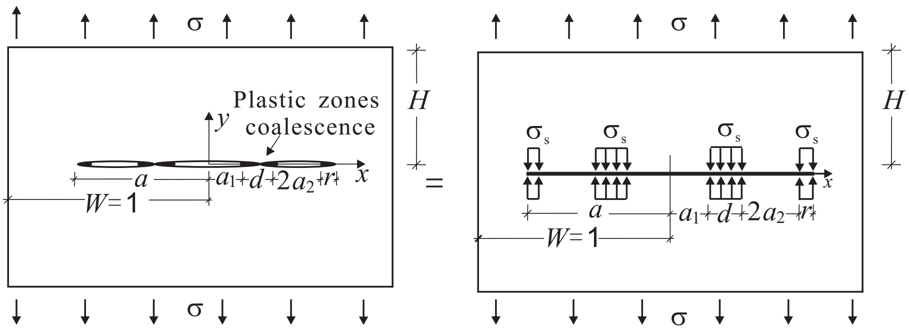


Fig.1 A coalesced center crack in a finite width panel containing three un-equal cracks, the total length of the fictitious crack includes all the strip yield zones.

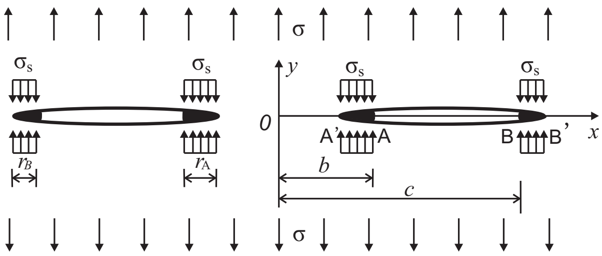


Fig.2 Strip yield model for two equal-length collinear cracks in an infinite sheet.

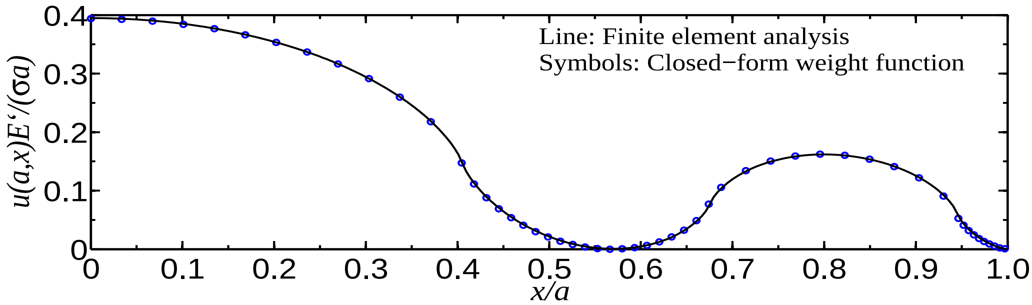


Fig.3. Crack opening profile for the fictitious crack of three un-equal length cracks with coalesced plastic zones. (in Fig.1, $a_1 = 0.3$, $d = 0.2$, $2a_2 = 0.2$, the half length of fictitious crack $a = 0.7417$).

For two equal-length collinear cracks in an infinite sheet, it is treated as a single crack due to the symmetry. The strip yield plastic zone sizes are determined by following the concept of Dugdale model[12], which is zero stress intensity factors of the fictitious crack tip A' and B'. That is,

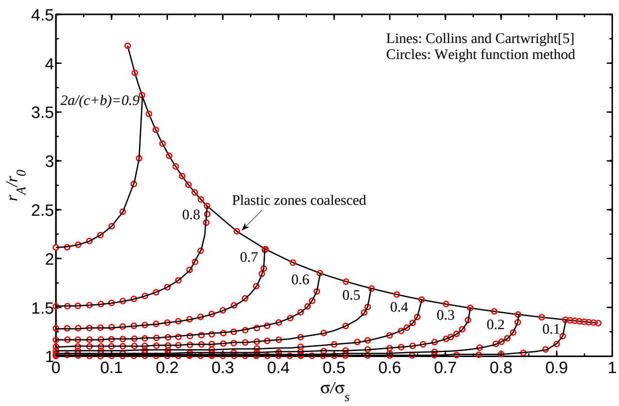
$$\begin{cases} K_{A'}^{s} + K_{A'} = 0 \\ K_{B'}^{s} + K_{B'} = 0 \end{cases} \tag{3}$$

where $K_{A'}^{s}$ ($K_{B'}^{s}$) and $K_{A'}$ ($K_{B'}$) represent the stress intensity factors of fictitious crack tip A' (B') due to the strip yield stress σ_s and the remote uniform stress σ, respectively. They are available in Ref.[13] and Eqs.(4) and (5), respectively.

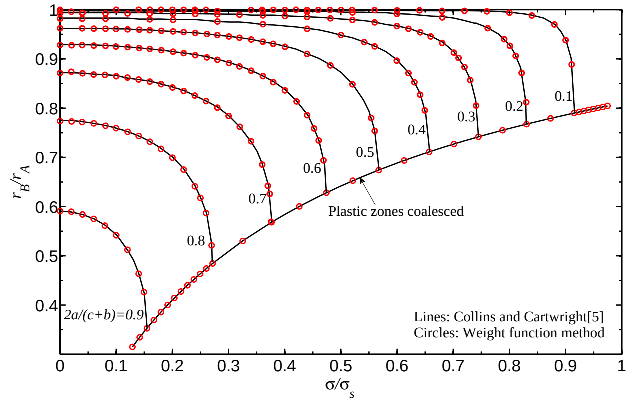
$$K_A = \int_0^{2a} \sigma(x)\cdot m_A(b,c,x)\mathrm{d}x \tag{4}$$

$$K_B = \int_0^{2a} \sigma(x)\cdot m_B(b,c,x)\mathrm{d}x \tag{5}$$

where $2a=c-b$, $m_A(b,c,x)$ and $m_B(b,c,x)$ are weight functions for the inner and outer crack tip shown in Fig.2. The expressions are given in Ref.[6].



(a) r_A/r_0



(b) r_B/r_A

Fig.4. Variation of the inner and outer plastic zone size with applied stress – comparison with Collins and Cartwright [5].

For a given uniform stress σ, the corresponding plastic zone sizes are determined by solving the above nonlinear equations. Figure 4a shows some typical results of the inner plastic zone sizes as a function of the applied stress. These results are normalized by the plastic zone size r_0 of a single Dugdale crack of the same length, $r_0=a[\sec(0.5\pi\sigma/\sigma_s)-1]$, $a=(c-b)/2$. In Fig.4b, the ratios between the outer and inner plastic zones size (r_B/r_A) are plotted against the applied stress. Also shown in these figures are the results for the plastic zones critical coalescence. To verify the solution accuracy of the present weight function approach, the results are compared to those given by Collins and Cartwright [5] by using complex stress function method. Figures 4 show the perfect agreement between the two approaches.

3 Weight function method for general collinear cracks

The WFM for single crack configuration is successfully extended to analyze some special multiple collinear crack problems. However, it was found that there is marked difference between the weight functions for single and multiple cracks [8].

3.1 Weight functions

The derivation of the weight function method for general collinear cracks is based on the reciprocity theorem and the superposition principle [8]. It was found that the weight functions for general collinear cracks are quite different from that for a single crack configuration. Take a typical MSD configuration - three collinear cracks in an infinite sheet shown in Fig.5 as an example. The weight functions for the crack tips A, B and C are given in Eqs.(6-8), respectively [8].

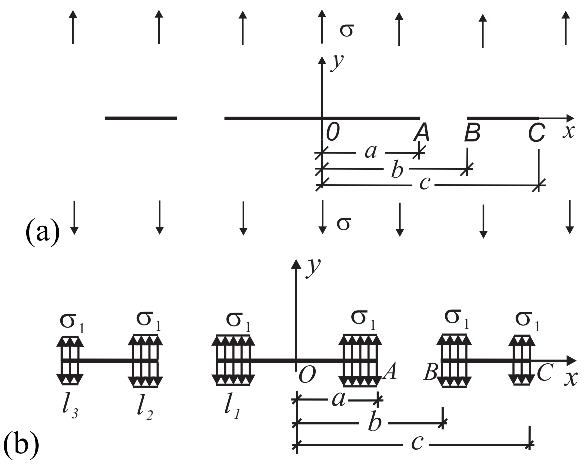


Fig.5. Three symmetric collinear cracks in an infinite sheet subjected to (a) remote uniform stress and (b) segment uniform pressure.

$$m_a(a,b,c,x)=\frac{E'}{f_{rA}(a,b,c)\cdot\sigma\sqrt{\pi a}}\cdot\begin{cases}\partial u_1^r(a,b,c,x)/\partial a;\ x\in[0,a]\\ \partial u_2^r(a,b,c,x)/\partial a\ ;\ x\in[b,c]\end{cases} \tag{6}$$

$$m_b(a,b,c,x)=\frac{-E'}{f_{rB}(a,b,c)\cdot\sigma\sqrt{\pi(c-b)/2}}\cdot\begin{cases}\partial u_1^r(a,b,c,x)/\partial b\ ;\ x\in[0,a]\\ \partial u_2^r(a,b,c,x)/\partial b\ ;x\in[b,c]\end{cases} \tag{7}$$

$$m_c(a,b,c,x)=\frac{E'}{f_{rC}(a,b,c)\cdot\sigma\sqrt{\pi(c-b)/2}}\cdot\begin{cases}\partial u_1^r(a,b,c,x)/\partial c;\ x\in[0,a]\\ \partial u_2^r(a,b,c,x)/\partial c;x\in[b,c]\end{cases} \tag{8}$$

where the non-dimensional stress intensity factors f_{rA} (a,b,c), f_{rB} (a,b,c) and f_{rC} (a,b,c) for remote uniform tension stress were given by Sih [13]. And, the corresponding crack opening displacements for center and side crack $u_1^r(a,b,c,x)$ and $u_2^r(a,b,c,x)$ are given in the following Eqs.(9a) and (9b), respectively.

$$u_1^r(a,b,c,x)=\frac{2\sigma}{E'}\int_{-a}^{x}\frac{x\left\{x^2-\left[\left(c^2-a^2\right)E(k)/K(k)+a^2\right]\right\}}{\sqrt{\left(a^2-x^2\right)\left(b^2-x^2\right)\left(c^2-x^2\right)}}\cdot\mathrm{d}x\ ;\ -a\le x\le a \tag{9a}$$

$$u_2^r(a,b,c,x)=\frac{2\sigma}{E'}\int_{b}^{x}\frac{x\left\{x^2-\left[\left(c^2-a^2\right)E(k)/K(k)+a^2\right]\right\}}{\sqrt{\left(x^2-a^2\right)\left(x^2-b^2\right)\left(c^2-x^2\right)}}\cdot\mathrm{d}x\ ;\ b\le x\le c \tag{9b}$$

For the general collinear cracks, the weight function for a considered tip is related to the crack opening displacement of *all* the cracks, as given by Eqs.(6-8), whereas for a single crack, the weight function is derived from its *own* crack opening displacement.

3.2 Stress intensity factor solutions

With above weight functions, the non-dimensional stress intensity factors for crack tips *A*, *B* and *C* under arbitrary loading are easily determined by the following equations [8], respectively.

$$f_A=\frac{1}{\sigma\sqrt{\pi a}}\int_L \sigma(x)\cdot m_a(a,b,c,x)\cdot\mathrm{d}x \tag{10}$$

$$f_B=\frac{1}{\sigma\sqrt{\pi(c-b)/2}}\int_L \sigma(x)\cdot m_b(a,b,c,x)\cdot\mathrm{d}x \tag{11}$$

$$f_C=\frac{1}{\sigma\sqrt{\pi(c-b)/2}}\int_L \sigma(x)\cdot m_c(a,b,c,x)\cdot\mathrm{d}x \tag{12}$$

where $L=[0,a]\cup[b,c]$ and $m_a(a,b,c,x)$, $m_b(a,b,c,x)$ and $m_c(a,b,c,x)$ are given in Eqs.(6-8), respectively. Herein these equations are used to calculate the stress intensity factors for the load case shown in Fig.5b as an example.

Stress intensity factors for crack tips *A*, *B* and *C* are determined by using Eqs.(10-12), respectively. Some typical results for $l_1=l_2=l_3=l$ and $2a=(c-b)$ denoted by f_{Aseg}, f_{Bseg} and f_{Cseg}, respectively, are shown in Fig.6. It is noted that the difference of stress intensity factors between cracks tips *A* and *B* is rather small. The stress intensity factor f_{seg} for a single center crack in an infinite sheet subjected to two segments of uniform stress acting in the immediate wake of each crack tip is [11]

$$f_{seg}=\left[1-2/\pi\cdot\arcsin\left(1-l/a\right)\right] \tag{13}$$

As expected, when these crack tips are far apart, ($2a/(a+b)=0.1$ in Fig.6), the stress intensity factors for crack tips *A*, *B* and *C* agree very well with a single center crack, Eq.(13). For the special case of $l/a=1$, i.e., the whole crack length is loaded by uniform pressure, the stress intensity factors obtained from Eqs.(10-12) should be equal to the corresponding results given in Ref.[13]. Figure 6 shows that the end points of the curves agree perfectly with the corresponding square symbols obtained from Ref.[13].

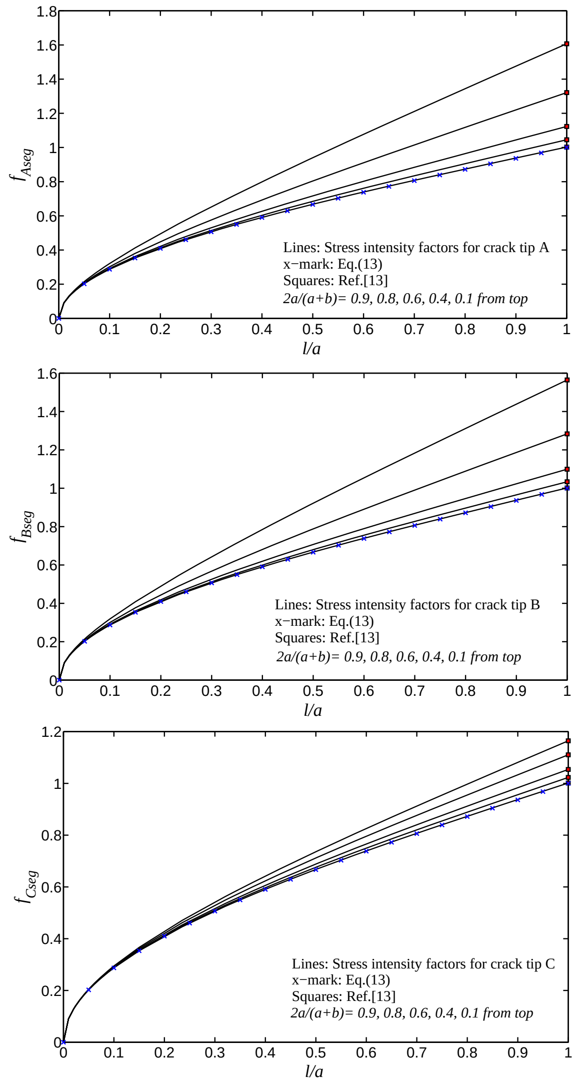


Fig.6. Non-dimensional stress intensity factors for three equal-length collinear cracks subjected to segment uniform pressure.

3.3 Crack opening displacement solutions

Crack opening displacements are often required for residual strength analysis and fatigue crack life predictions of structures with MSD. The WFM provides a powerful tool for determination of crack opening displacements for multiple collinear cracks. To illustrate the approach, the three collinear cracks subjected to segment uniform stress are taken as an example, Fig.5b. The crack opening displacements for the central and outer cracks are denoted by $u_1(a,b,c,x)$ and $u_2(a,b,c,x)$, respectively. Because weight function is a property of crack geometry, and is independent of the load case, the following relationship holds.

$$u_1(a,b,c,x)=\int_x^a\left[\frac{1}{\sigma\sqrt{\pi s}}\int_{L_1}\sigma(x)\cdot m_a(s,b,c,x)\mathrm{d}x\right]\cdot\frac{\partial u_1^r(s,b,c,x)/\partial s}{f_{rA}(s,b,c)\sigma\sqrt{\pi s}}\cdot\mathrm{d}s\ ;\ x\in[0,a] \tag{14}$$

$$u_2(a,b,c,x)=\int_b^x\left[\frac{1}{\sigma\sqrt{\pi(c-s)/2}}\int_{L_2}\sigma(x)\cdot m_b(a,s,c,x)\mathrm{d}x\right]\cdot\frac{\partial u_2^r(a,s,c,x)/\partial s}{f_{rB}(a,s,c)\sigma\sqrt{\pi(c-s)/2}}\cdot\mathrm{d}s\ ;\ x\in[b,c] \tag{15a}$$

or

$$u_2(a,b,c,x)=\int_x^c\left[\frac{1}{\sigma\sqrt{\pi(s-b)/2}}\int_{L_3}\sigma(x)\cdot m_c(a,b,s,x)\mathrm{d}x\right]\cdot\frac{\partial u_2^r(a,b,s,x)/\partial s}{f_{rC}(a,b,s)\sigma\sqrt{\pi(s-b)/2}}\cdot\mathrm{d}s\ ;\ x\in[b,c] \tag{15b}$$

where $m_a(s,b,c,x)$, $m_b(a,s,c,x)$ and $m_c(a,b,s,x)$ given in Eqs.(6-8), are the weight functions for crack tips *A*, *B* and *C*, respectively. *s* is integration variable. The integral intervals are: $L_1=[0, s]\cup[b, c]$, $L_2=[0, a]\cup[s, c]$, $L_3=[0, a]\cup[b, s]$. The stress distribution along the crack faces $\sigma(x)$ for the load case of Fig.5b is

$$\sigma(x)=\begin{cases}\sigma_1\ ;x\in\pm\left([a-l_1,a]\cup[b,b+l_2]\cup[c-l_3,c]\right)\\ 0\ \ ;\ \text{else}\end{cases} \tag{16}$$

3.4 Strip yield model solutions

Figure 7 shows a strip yield model for three collinear cracks in an infinite sheet. The crack tip plastic zone sizes (r_A, r_B and r_C) are determined by superposition of two linear elastic solutions. One is the stress intensity factors for the fictitious cracks subjected to remote uniform tension σ. The other is for compressive yield stress $-\sigma_s$ in the strip-yield zones. With the condition of vanishing stress singularities at the fictitious crack tips, the plastic zone sizes can be obtained by solving the following equations.

$$\begin{cases}f_{A'seg}-f_{rA'}=0\\ f_{B'seg}-f_{rB'}=0\\ f_{C'seg}-f_{rC'}=0\end{cases} \tag{17}$$

where, $f_{rA'}$, $f_{rB'}$ and $f_{rC'}$ are the non-dimensional stress intensity factors for the fictitious cracks under remote uniform stress given in Ref.[13]; $f_{A'seg}$, $f_{B'seg}$ and $f_{C'seg}$ are the non-dimensional stress intensity factors due to the compression yield stress $-\sigma_s$ in the plastic zones, which can be obtained by using Eqs.(10-12).

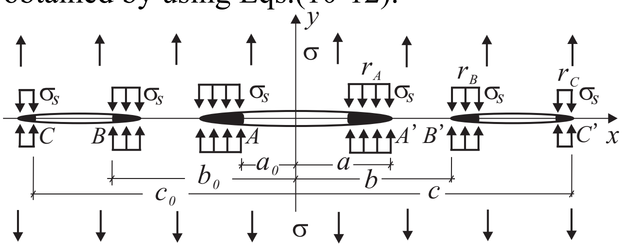


Fig.7. Strip-yield model for three collinear cracks, with separated plastic zones.

Figure 8a shows the plastic zone sizes r_A as a function of the applied stress σ/σ_s obtained from solving Eqs.(17) for three equal-length collinear cracks. The results are normalized by the plastic zone size of a single center Dugdale crack of the same length. The plastic zone sizes for crack tip *B* and *C* were given in Ref.[8].

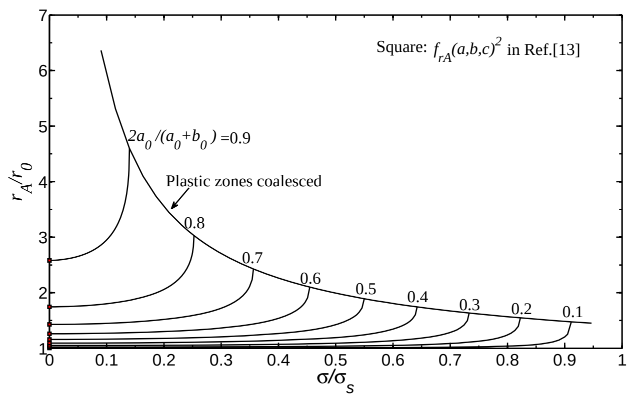


Fig.8. Variation of the normalized plastic zone sizes r_A for three equal-length collinear cracks with the applied stress σ/σ_s, $r_0=a_0[\sec(0.5\pi\sigma/\sigma_s)-1]$.

Having determined the plastic zone sizes, crack tip opening displacements can be obtained by the WFM and superposition of two crack opening displacement solutions. One is due to remote uniform stress σ, Ref.[11], the other is due to segments uniform compressive yield stress $-\sigma_s$ in the plastic zone, which can be obtained by using Eqs.(14-15).

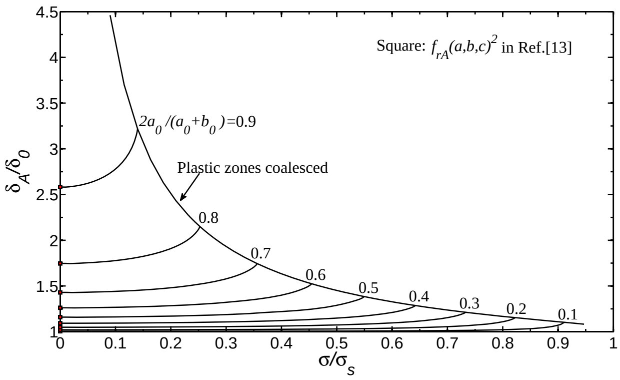


Fig.9. Variation of the normalized crack tip opening displacement for three equal-length collinear cracks with the applied stress σ/σ_s, $\delta_0=8a_0\cdot\sigma_s/(\pi E)\cdot\ln[\sec(0.5\pi\sigma/\sigma_s)]$.

Figure 9 shows the crack tip opening displacements δ_A for three equal-length collinear cracks as a function of the applied stress σ/σ_s. The results are normalized by the crack tip opening displacement δ_0 of a single center crack. The crack tip opening displacements for crack tip *B* and *C* were given in Ref.[8].

4 Conclusions

An analytical approach, the weight function method for dealing with the MSD problems is presented in this paper. The following conclusions can be made:

For special collinear crack configurations which can be treated as a single crack problem, e.g. three collinear cracks with strip yield plastic zones critical coalescence in a finite sheet and two equal-length collinear cracks in an infinite sheet, the strip yield model solutions can be easily solved by the weight functions for a single crack. The results are in perfect agreement with existing solutions in the literature.

Weight function formulas for more general collinear cracks have been derived, which are markedly different from those for the single crack case. With the derived weight functions, the key fracture mechanics parameters, stress intensity factors and crack opening displacement for the three collinear cracks under arbitrary load conditions are easily computed by a simple quadrature. Furthermore, the strip yield plastic zone sizes and crack tip opening displacements are presented.

The present WFM for MSD is characterized by its versatility, simplicity, reliability and cost-effectiveness, thus provides a powerful and efficient tool for residual strength and fatigue life predictions for MSD-contained aircraft structures.

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