ANALYTICAL NATURAL FREQUENCIES OF TAPERED WINGS

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Abstract

The bending frequencies of a wing are calculated based on the model of a beam clamped at the root and free at the tip; since the mass and the moment of inertia (per unit span) vary along the span, a non-uniform beam is considered. For a swept-back wing with straight leading- and trailing-edges, the chord is a linear function of the span; the same linear function of the span applies to thickness, in the case of constant thickness-to-chord ratio. Thus, the bending modes of a non-uniform beam are considered, with mass and moment of inertia respectively quadratic and quartic functions of the span. There is no exact solution expressible in finite terms using elementary functions, and thus power series expansions are used. The boundary conditions, that the wing is clamped at the root and free at the tip, lead to the natural bending frequencies. The fundamental bending frequency is calculated for a delta wing, and compared with a rectangular wing, with the same span, mean chord and thickness, mass density and Young modulus. It is shown that the fundamental frequency is higher by a factor 11.32 for the delta wing., i.e., it is stiffer because it has a higher proportion of the mass near the root; it is also shown that the case of the tapered swept-back wing is intermediate between the delta and the rectangular wing.

1 Introduction

The representation of a wing as a beam of constant cross-section is adequate for a rectangular wing, with airfoil section and material properties constant along the span. Retaining the latter case of an homogeneous wing, but with non-uniform chord and/or thickness, the mass and moment of inertia of the section vary along the span. Thus the model must be extended to a beam of non-uniform cross-section, e.g., for a swept-back wing. This is particularly true for a delta wing, for which the mass and moment of inertia vanish at the tip.

The study of wing bending in general, and its natural vibration frequency in particular has been of interest since many years and it is a standard topic of elasticity [1–3], vibrations [4–6] and aeroelasticity [7–9]. For example, the analytical study of vibration modes of cantilever beams has already been well documented by Volterra and Zachmanoglou [10]. A detailed analytical study of natural vibration frequency of bending bars subject to different boundary conditions has been made by Chen [11], including numerical results for a truncated conical bar. More recently, Balakrishnan and Iliff [12] developed an aeroelastic analytical model for the bending-torsion dynamics of a slender high aspect-ratio wing in inviscid subsonic airflow. It comprised of a cantilever beam model as structural model, and a potential field model as linear aerodynamic model, and it studied the aeroelastic modes and flutter instability in two-dimensions. The study of physical phenomena involving any combination of solid mechanics, dynamics and fluid mechanics are becoming more and more recurrent in aerospace, namely in the study of airplanes.

The knowledge of exact analytical solutions of dynamic response of rather simplistic wing structures offers valuable data. It can be used as benchmark to validate high-fidelity computational fluid-structure interaction models, that upon validation, can then be used to tackle complex wing configurations. The work presented here intends to provide such reference data for sweptback wings.

This paper is divided into five main sections. Section 2 lays out a few basic geometric and physical function definitions for straight leading- and trailing-edges wings. The equation of transverse vibrations is applied to such wings and general expressions of amplitude and frequency of oscillation are derived in Sec. 3. The solutions for tapered wings are determined in Sec. 4, while the particular cases of rectangular and delta wings are included in Sec. 5 and Sec. 6, respectively. Finally, the paper ends with some remarks about the findings of the work presented.

2 Spanwise distribution of mass and moment of inertia

The starting point for the representation of a wing with arbitrary planform and airfoil section as a non-uniform beam, is to specify the mass and moment of inertia (per unit span) as a function of the spanwise coordinate y. In the case of Fig. 1 of a sweptback (or sweptforward) wing, with straight leading- and trailing-edges, the chord at spanwise section y is given by

$$c(y) = c_r + (c_t - c_r)\frac{y}{L}, \quad 0 \le y \le L, \tag{1}$$

where L is the semi-span and c_r, c_t are the chord at the root and tip. The latter are related to the mean chord, $\bar{c} \equiv (c_r + c_t)/2$, and taper ratio, $\lambda \equiv c_t/c_r$, by

$$c_r = \frac{2\bar{c}}{1+\lambda}, \quad c_t = \frac{2\bar{c}\lambda}{1+\lambda}, \tag{2}$$

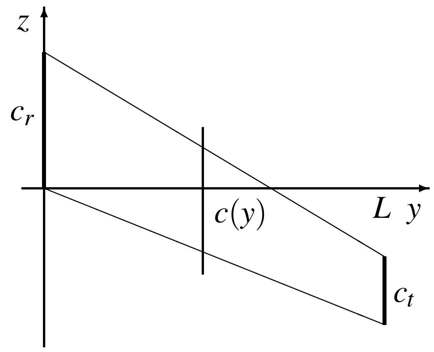

Fig. 1 Sweptback wing with straight leading- and trailing-edges.

and thus the chord (1) is given by

$$c(y) = \frac{2\bar{c}}{1+\lambda}\left[1 + (\lambda - 1)\frac{y}{L}\right], \tag{3}$$

as the function of the spanwise coordinate. If the wing sections have a constant thickness-to-chord ratio, then (3) also applies to the thickness distribution along the span,

$$e(y) = \frac{2\bar{e}}{1+\lambda}\left[1 + (\lambda - 1)\frac{y}{L}\right], \tag{4}$$

where $\bar{e}$ is the mean thickness; in such case, the thickness-to-chord ratio is $e(y)/c(y) = \bar{e}/\bar{c} = const$.

Assuming that the section is homogeneous with mass density μ, the mass per unit span,

$$m(y) = \mu c(y) e(y), \tag{5}$$

is given by a quadratic function (6a) using the expressions (3),(4),

$$m(y) = m_0\left[1 + (\lambda - 1)\frac{y}{L}\right]^2, \tag{6a}$$

$$m_0 \equiv m(0) = \frac{4\mu\bar{c}\bar{e}}{(1+\lambda)^2}, \tag{6b}$$

where (6b) would be the constant value for a rectangular wing. Likewise, the moment of inertia per unit span relative to the z-axis, for a rectangle with height equal to the mean thickness of the airfoil $e(y)$ and length equal to the chord $c(y)$, as illustrated in Fig. 2, is given by

$$I(y) = \frac{1}{12}\mu c(y)\left[e(y)\right]^3. \tag{7}$$

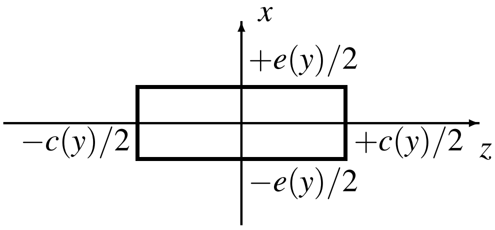

Fig. 2 Moment of inertia per unit span for an arbitrary section at spanwise position y.

Substituting (3) and (4) in (7) leads to a quartic function (8a),

$$I(y) = I_0 \left[1 + (\lambda - 1)\frac{y}{L}\right]^4 , \tag{8a}$$

$$I_0 \equiv I(0) = \frac{4}{3}\frac{\mu \bar{c} \bar{e}^3}{(1+\lambda)^4} , \tag{8b}$$

where (8b) would be the constant value for a rectangular wing. The radius of gyration,

$$r(y) = \sqrt{\frac{I(y)}{m(y)}} = r_0 \left[1 + (\lambda - 1)\frac{y}{L}\right] , \tag{9a}$$

$$r_0 \equiv r(0) = \sqrt{\frac{I_0}{m_0}} = \frac{\bar{e}}{\sqrt{3}(1+\lambda)} , \tag{9b}$$

varies along the span as the chord (3) and thickness (4).

3 Transverse vibrations of beam with non-uniform cross-section

The transverse or vertical displacement $X(y,t)$ of an elastic beam, with Young modulus $E(y)$, mass $m(y)$ and moment of inertia $I(y)$ per unit span, satisfies the equation of bending waves [13],

$$m(y)\frac{\partial^2 X(y,t)}{\partial t^2} = \frac{\partial^2}{\partial y^2}\left[E(y)I(y)\frac{\partial^2 X(y,t)}{\partial y^2}\right] . \tag{10}$$

For a homogeneous wing, the Young modulus is constant, and for a swept wing with straight leading- and trailing-edges and constant thickness-to-chord ratio, the substitution of the mass (6a) and moment of inertia (8a) in (10) lead to the linear partial differential equation with non-uniform coefficients,

$$\left[1 + (\lambda - 1)\frac{y}{L}\right]^2 \frac{\partial^2 X(y,t)}{\partial t^2} = \frac{EI_0}{m_0}\frac{\partial^2}{\partial y^2}\left\{\left[1 + (\lambda - 1)\frac{y}{L}\right]^4 \frac{\partial^2 X(y,t)}{\partial y^2}\right\} . \tag{11}$$

Since the coefficients do not depend on time, there are sinusoidal oscillations with frequency ω expressed as

$$X(y,t) = F(y) cos(\omega t) , \tag{12a}$$

whose amplitude satisfies a linear ordinary differential equation with variable coefficients,

$$\frac{d^2}{dy^2}\left\{\left[1 + (\lambda - 1)\frac{y}{L}\right]^4 \frac{d^2 F(y)}{dy^2}\right\} + \frac{(\omega/r_0)^2}{E}\left[1 + (\lambda - 1)\frac{y}{L}\right]^2 F(y) = 0 , \tag{12b}$$

where the reference radius of gyration (9b) was introduced. The change of independent variable, with $\lambda \neq 1$,

$$z \equiv 1 + (\lambda - 1)\frac{y}{L} , \tag{13a}$$

$$F(y) \equiv G(z) , \tag{13b}$$

which implies that $\frac{d}{dy} = \frac{\lambda-1}{L}\frac{d}{dz}$ and $\frac{d^2}{dy^2} = \left(\frac{\lambda-1}{L}\right)^2 \frac{d^2}{dz^2}$, transforms the coefficients in (12a) to powers in (14),

$$\frac{d^2}{dz^2}\left[z^4 \frac{d^2 G}{dz^2}\right] + \Omega^2 z^2 G = 0 , \tag{14}$$

where the only parameter is the dimensionless frequency,

$$\Omega \equiv \frac{\omega/r_0}{\sqrt{E}}\left(\frac{L}{\lambda - 1}\right)^2 = \frac{\omega L^2}{\bar{e}}\sqrt{\frac{3}{E}}\frac{\lambda + 1}{(\lambda - 1)^2} , \tag{15}$$

which involves the frequency of oscillation ω, material properties through the Young modulus E, and wing geometry through the semi-span L, mean thickness $\bar{e}$ and taper ratio λ. Note that the mass density μ and mean chord $\bar{c}$ have dropped out because they do not appear in the radius of gyration (9b).

Upon the expansion of (14), the resulting linear fourth-order ordinary differential equation with power coefficients (16),

$$z^4 G'''' + 8z^3 G''' + 12z^2 G'' + \Omega^2 z^2 G = 0 , \tag{16}$$

has no exact solution which can be expressed in finite terms using only elementary functions. Since the only singularities are $z = 0, \infty$, and $z = 0$

is a regular singularity, there are solutions as Frobenius-Fuchs series [14]:

$$G(z)=\sum_{n=0}^{\infty}a_n(\sigma)z^{n+\sigma}\,,0\leq\lambda<z\leq 1<\infty\,, \tag{17}$$

which has infinite radius of convergence, $0<z<\infty$, and thus covers the region of interest $\lambda\leq z\leq 1$ in (13a) corresponding to $0\leq y\leq L$. Thus the solution (17) will be needed at most only in the unit interval $0\leq z\leq 1$, in the case of the delta wing with zero taper ratio ($\lambda=0$) and tip chord ($c_t=0$). Substituting the Frobenius-Fuchs series (17) in the differential equation (16), and equating to zero the coefficients of the powers of z, leads to the recurrence formula:

$$(n+\sigma)(n+\sigma-1)$$
$$[12+(n+\sigma-2)(n+\sigma+5)]a_n(\sigma)+\Omega^2a_{n-2}(\sigma)=0\,. \tag{18}$$

Note that (18) implies that the coefficients decay like $a_n\sim o(n^{-4})$, ensuring the uniform and absolute convergence of the series (17) for finite $z<\infty$ [15]. Setting $n=0,1$ in (18), leads to

$$n=0:\quad(\sigma-1)\sigma(\sigma+1)(\sigma+2)a_0(\sigma)=0\,, \tag{19a}$$
$$n=1:\quad\sigma(\sigma+1)(\sigma+2)(\sigma+3)a_1(\sigma)=0\,, \tag{19b}$$

for the even and odd modes, respectively involving a_{2n} and a_{2n+1}. If $a_0(\sigma)=0$, then by (18) all $a_{2n}(\sigma)=0$, and similarly if $a_1(\sigma)=0$, then by (18) all $a_{2n+1}(\sigma)=0$, and a trivial solution $G(z)=0$ would result from (17). Thus at least one of $a_0(\sigma)\neq 0\neq a_1(\sigma)$ cannot vanish, implying from (19) that

$$a_0(\sigma)\neq 0:\quad\sigma_m=1,0,-1,-2\,, \tag{20a}$$
$$a_1(\sigma)\neq 0:\quad\sigma_m=0,-1,-2,-3\,. \tag{20b}$$

For each of the four values of the index m corresponds a particular solution of (17) as a power series:

$$m=1,2,3,4:\quad G_m(z)=\sum_{n=0}^{\infty}a_n(\sigma_m)z^{n+\sigma_m}\,. \tag{21a}$$

From the difference of the indices in (20a) and (20b), logarithmic solutions may occur [16–18].

Since the four particular solutions (21a) are linearly independent, the general solution is given by their linear combination:

$$a_0(\sigma_m)=1:\quad G(z)=\sum_{m=1}^{4}C_mG_m(z)\,, \tag{21b}$$

where the arbitrary constants C_m are determined by boundary conditions and incorporate the coefficients $a_0(\sigma_m)$, which can be put equal to unity.

4 Clamped-free boundaries and natural frequencies

The general solutions (21b) must satisfy boundary conditions, which will specify the natural frequencies. The four boundary conditions are determined by setting the beam:

(i) clamped at the root $y=0,z=1$, i.e., zero displacement (22a) and slope (22b):

$$I:\quad X(0,t)=0\quad\Rightarrow F(0)=0\quad\Rightarrow G(1)=0\,, \tag{22a}$$

$$II:\quad\frac{\partial X(0,t)}{\partial y}=0\quad\Rightarrow F'(0)=0\quad\Rightarrow G'(1)=0; \tag{22b}$$

where the definitions (12a) and (13) have been used.

(ii) free at the tip $y=L,z=\lambda$, i.e., zero bending moment (23a) and transverse force (23b):

$$III:\quad M(y\to L)=\lim_{y\to L}EI(y)\frac{\partial^2X(y,t)}{\partial y^2}=0$$
$$\Rightarrow\lim_{z\to\lambda}h(z)G''(z)=0\,, \tag{23a}$$

$$IV:\quad V(y\to L)=\frac{\partial M}{\partial y}=\lim_{y\to L}\frac{\partial}{\partial y}\left[EI(y)\frac{\partial^2X}{\partial y^2}\right]=0$$
$$\Rightarrow\lim_{z\to\lambda}\frac{\lambda-1}{L}\frac{d}{dz}\left[h(z)G''(z)\right]=0\,, \tag{23b}$$

where

$$h(z)=EI_0z^4\left(\frac{dz}{dy}\right)^2=\frac{4\mu\bar{c}\bar{e}^3}{3}E\left(\frac{z}{1+\lambda}\right)^4\left(\frac{\lambda-1}{L}\right)^2. \tag{23c}$$

In addition to the definitions mentioned above, (8a) and (8b) have also been used.

The boundary conditions at the wing root (22) readily apply to (21b):

$$I: \quad G(1) = \sum_{m=1}^{4} C_m G_m(1) = 0\,, \tag{24a}$$

$$II: \quad G'(1) = \sum_{m=1}^{4} C_m G'_m(1) = 0\,. \tag{24b}$$

Concerning the boundary conditions at the wing tip $z \to \lambda$ (23), the case of the delta wing $\lambda = 0$ will be excluded (it will be addressed subsequently in section 6), so that,

$$III: \quad \lim_{z\to\lambda} h(z)G''(z) = h(\lambda)G''(\lambda) = h(\lambda)\sum_{m=1}^{4} C_m G''_m(\lambda) = 0\,, \tag{25a}$$

$$IV: \quad \lim_{z\to\lambda} \frac{\lambda-1}{L}\frac{d}{dz}\left[h(z)G''(z)\right] = \frac{\lambda-1}{L}h(\lambda)\sum_{m=1}^{4} C_m G'''_m(\lambda) = 0; \tag{25b}$$

in the derivation of

$$\lim_{z\to\lambda}\frac{d}{dz}\left[h(z)G''(z)\right] = h(\lambda)\sum_{m=1}^{4} C_m G'''_m(\lambda) + h'(\lambda)\sum_{m=1}^{4} C_m G''_m(\lambda) = 0\,, \tag{25c}$$

the last term vanishes by (25a), and thus (25c) reduces to (25b). The four boundary conditions (24) and (25) form a linear homogeneous system of equations in $(C_1,C_2,C_3,C_4) \neq (0,0,0,0)$, which cannot be all zero. Hence the determinant of coefficients must vanish:

$$\begin{vmatrix} G_1(1) & G_2(1) & G_3(1) & G_4(1) \\ G'_1(1) & G'_2(1) & G'_3(1) & G'_4(1) \\ G''_1(\lambda) & G''_2(\lambda) & G''_3(\lambda) & G''_4(\lambda) \\ G'''_1(\lambda) & G'''_2(\lambda) & G'''_3(\lambda) & G'''_4(\lambda) \end{vmatrix} = H(\Omega,\lambda) = 0\,. \tag{26}$$

For fixed variable $z = 1$ or $z = \lambda$, the particular solution (21a) depend though the coefficients (18) only on the dimensionless frequency defined by (15). Thus, for each taper ratio λ, the roots of the determinant (26) specify the natural frequencies $\Omega_n(\lambda)$, of which the real root with smaller modulus is the fundamental frequency $\Omega_1(\lambda)$.

Before proceeding to calculate the natural frequencies, the case of the delta wing $\lambda = 0$, excluded from (25), is considered. The following analysis is similar for even and odd modes, and the former will be considered next for a delta wing. The two boundary conditions at the wing root (22) are unchanged (24) for the delta wing, but at the wing tip, the condition of zero bending moment (23a) leads to

$$III: \quad \lim_{\lambda\to 0} h(\lambda)G''_m(\lambda) = \lim_{\lambda\to 0} h(\lambda)\frac{d^2}{d\lambda^2}\sum_{n=0}^{\infty} a_n(\sigma_m)\lambda^{n+\sigma_m} \sim \lim_{\lambda\to 0}\lambda^2\frac{d^2}{d\lambda^2}\lambda^{\sigma_m} = \lim_{\lambda\to 0}\sigma_m(\sigma_m-1)\lambda^{\sigma_m} = 0\,, \tag{27a}$$

where the leading term of (21a) was considered by setting $n = 0$, $a_0(\sigma_m) = 1$, and $h(\lambda) \sim \lambda^4$ for $z = \lambda$ in (23c). Using (20a), this expression tends to zero for $m = 1,2$ or $\sigma_m = 1,0$, but to infinity for $m = 3,4$ or $\sigma_m = -1,-2$, so $G''_3(0) = \infty$ and $G''_4(0) = \infty$ must be excluded by setting $C_3 = 0$ and $C_4 = 0$; the condition of zero transverse force (23b) leads to

$$IV: \quad \lim_{\lambda\to 0}\frac{\lambda-1}{L}\frac{d}{d\lambda}\left[h(\lambda)G''(\lambda)\right] = -\frac{1}{L}\lim_{\lambda\to 0}\frac{d}{d\lambda}\left[h(\lambda)\frac{d^2}{d\lambda^2}\sum_{n=0}^{\infty} a_n(\sigma_m)\lambda^{n+\sigma_m}\right] \sim \lim_{\lambda\to 0}\frac{d}{d\lambda}\left[\lambda^4\left(\frac{d^2}{d\lambda^2}\lambda^{\sigma_m}\right)\right] = \lim_{\lambda\to 0}\sigma_m(\sigma_m-1)(\sigma_m+2)\lambda^{\sigma_m+1} = 0\,, \tag{27b}$$

where the leading term of (21a) was considered as in (27a). Using (20a), this expression tends to zero for $\sigma_m = 1,0,-2$ corresponding to $m = 1,2,4$, and is finite for $\sigma_m = -1$ corresponding to $m = 3$, so the solution G_3 is excluded by setting $C_3 = 0$. Thus, in the case of a delta wing, (24) simplifies to

$$\lambda = 0: C_3 = 0 = C_4, \quad \begin{bmatrix} G_1(1) & G_2(1) \\ G'_1(1) & G'_2(1) \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = 0\,, \tag{28}$$

because there are at most two non-zero constants of integration $(C_1,C_2) \neq (0,0)$. They cannot both vanish, so the determinant in (28) must vanish,

$$\lambda = 0: \quad G_1(1)G_2'(1) - G_2(1)G_1'(1) = H(\Omega, 0) = 0, \tag{29}$$

and this is the condition that determines the natural frequencies $\Omega_n(0)$ in the case of the delta wing.

5 Comparison with the corresponding rectangular wing planform

The particular case, distinct from the preceding, which is the simplest, is the rectangular wing, with the same mean chord and thickness,

$$\lambda = 1: \quad \bar{c} = c(y) = const, \tag{30a}$$

$$\lambda = 1: \quad \bar{e} = e(y) = const, \tag{30b}$$

for which the mass and moment of inertia per unit span are constant,

$$\lambda = 1: \quad \bar{m} = \mu\bar{c}\bar{e}, \tag{31a}$$

$$\lambda = 1: \quad \bar{I} = \frac{1}{12}\mu\bar{c}\bar{e}^3. \tag{31b}$$

In this case, the equation of transverse vibrations (10) has constant coefficients,

$$\lambda = 1: \quad \bar{m}\frac{\partial^2 X}{\partial t^2} = E\bar{I}\frac{\partial^4 X}{\partial y^4}, \tag{32a}$$

and in the case of constant frequency ω (12a), it leads to

$$\lambda = 1: \quad \frac{d^4 F}{dy^4} + \frac{\omega^2 \bar{m}}{E\bar{I}}F = 0. \tag{32b}$$

The change of independent variable

$$\rho \equiv \frac{y}{L}, \tag{33a}$$

$$F(y) \equiv H(\rho), \tag{33b}$$

which implies that $\frac{d}{dy} = \frac{1}{L}\frac{d}{d\rho}$ and $\frac{d^2}{dy^2} = \frac{1}{L^2}\frac{d^2}{d\rho^2}$, leads to

$$H'''' + \bar{\Omega}^2 H = 0, \tag{34a}$$

where the dimensionless frequency,

$$\bar{\Omega} \equiv \omega L^2\sqrt{\frac{\bar{m}}{E\bar{I}}} = \frac{\omega L^2}{\bar{e}}\sqrt{\frac{12}{E}}, \tag{34b}$$

replaces (15), which is not valid for $\lambda = 1$; note that the change of variable (13a) also does not apply for $\lambda = 1$, and was replaced by (33a).

The case of the rectangular wing is the simplest because (34a) has elementary solutions:

$$\alpha \equiv \sqrt{\bar{\Omega}}: \quad H(\rho) = B_1\cos(\alpha\rho) + B_2\sin(\alpha\rho) + B_3\cosh(\alpha\rho) + B_4\sinh(\alpha\rho). \tag{35}$$

The boundary conditions of clamping at the root $y = 0, \rho = 0$ (22) can be expressed as

$$I: \quad X(0,t) = 0 \Rightarrow F(0) = 0 \Rightarrow H(0) = B_1 + B_3 = 0, \tag{36a}$$

$$II: \quad \frac{\partial X(0,t)}{\partial y} = 0 \Rightarrow F'(0) = 0 \Rightarrow H'(0) = \alpha(B_2 + B_4) = 0, \tag{36b}$$

which leaves two out of four constants of integration independent:

$$H(\rho) = B_1[\cos(\alpha\rho) - \cosh(\alpha\rho)] + B_2[\sin(\alpha\rho) - \sinh(\alpha\rho)]. \tag{37}$$

The boundary conditions at the tip $y = L, \rho = 1$ (23),

$$III: \frac{\partial^2 X(L,t)}{\partial y^2} = 0 \Rightarrow F''(1) = 0 \Rightarrow H''(1) = -B_1(\cos\alpha + \cosh\alpha) - B_2(\sin\alpha + \sinh\alpha) = 0, \tag{38a}$$

$$IV: \frac{\partial^3 X(L,t)}{\partial y^3} = 0 \Rightarrow F'''(1) = 0 \Rightarrow H'''(1) = B_1(\sin\alpha - \sinh\alpha) - B_2(\cos\alpha + \cosh\alpha) = 0, \tag{38b}$$

form a linear homogeneous system, which has a non-trivial solution if the determinant of coefficients vanishes:

$$(B_1, B_2) \neq (0,0): \begin{vmatrix} \cos\alpha + \cosh\alpha & \sin\alpha + \sinh\alpha \\ \sinh\alpha - \sin\alpha & \cos\alpha + \cosh\alpha \end{vmatrix} = 2(1 + \cos\alpha\cosh\alpha) = 0. \tag{39a}$$

Thus the natural frequencies for the rectangular wing are the roots of

$$\lambda = 1: \quad \operatorname{sech}(\sqrt{\bar{\Omega}}) = -\cos(\sqrt{\bar{\Omega}}), \tag{39b}$$

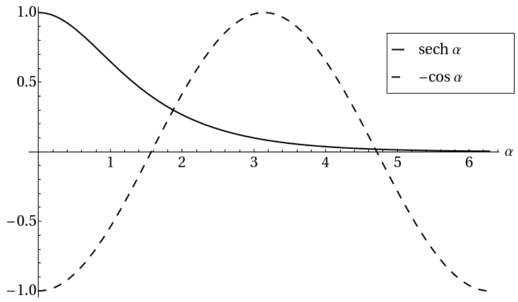

Fig. 3 Natural dimensionless frequencies $\bar{\Omega}_n = \alpha_n^2$ of rectangular wing determined by the intersections of $\operatorname{sech}\alpha$ and $-\cos\alpha$.

and are illustrated in Fig. 3. The dimensionless frequency of the fundamental mode $\alpha_1 = 1.875104$ is calculated in Tab. 1, leading to $\bar{\Omega}_1 = \alpha_1^2 = 3.51602$. Higher-modes were computed using numerically and are summarized in Tab. 2.

6 Fundamental natural frequency for delta wing

The calculation of the fundamental frequency of bending oscillations is less simple for a non-rectangular wing because it involves the four non-elementary functions (21a), of which only two are needed in the case (29) of a delta wing.

The first fundamental $G_1(z)$ in (21a) corresponds to the index $m = 1$. For the coefficients of even order (20a), this corresponds to $\sigma_1 = 1$, thus the recurrence formula (18) yields

$$\sigma_1 = 1: a_n(1) = -\frac{\Omega^2 a_{n-2}(1)}{n(n+1)(n+2)(n+3)}. \quad (40a)$$

This determines the coefficients of even order starting with $a_0(1) = 1$:

$$\begin{aligned} a_{2n}(1) &= \frac{(-1)^n \Omega^{2n}}{2n(2n+1)(2n+2)(2n+3).(2n-2)(2n-1)(2n)(2n+1)\ldots} \\ &= \frac{2.3.(-1)^n \Omega^{2n}}{(2n)!!(2n+1)!!(2n+2)!!(2n+3)!!}, \end{aligned} \quad (40b)$$

where the double factorial notation has been introduced: $n!! = n(n-2)(n-4)\ldots$. The coefficients of even order (40b) lead to the solution

$$\begin{aligned} G_1(z) &= \sum_{n=0}^{\infty} a_{2n}(1) z^{2n+1} \\ &= a_0(1)z + a_2(1)z^3 + a_4(1)z^5 + \ldots \\ &= z - \frac{\Omega^2 z^3}{120} + \frac{\Omega^4 z^5}{100800} + O(\Omega^6 z^7), \end{aligned} \quad (41)$$

whose first three terms have been written explicitly. For the coefficients of odd order (20b), substituting the first index $\sigma_1 = 0$ in the recurrence formula (18) leads to

$$\sigma_1 = 0: \quad a_n(0) = -\frac{\Omega^2 a_{n-2}(0)}{(n-1)n(n+1)(n+2)}. \quad (42a)$$

This determines the coefficients of odd order starting with $a_1(0) = 1$:

$$\begin{aligned} a_{2n+1}(0) &= \frac{(-1)^n \Omega^{2n}}{(2n)(2n+1)(2n+2)(2n+3).(2n-2)(2n-1)(2n)(2n+1)\ldots} \\ &= \frac{2.3.(-1)^n \Omega^{2n}}{(2n)!!(2n+1)!!(2n+2)!!(2n+3)!!}, \end{aligned} \quad (42b)$$

The coefficients of odd order (42b) lead to the solution

$$\begin{aligned} G_1(z) &= \sum_{n=0}^{\infty} a_{2n+1}(0) z^{2n+1} \\ &= a_1(0)z + a_3(0)z^3 + a_5(0)z^5 + \ldots \\ &= z - \frac{\Omega^2 z^3}{120} + \frac{\Omega^4 z^5}{100800} + O(\Omega^6 z^7), \end{aligned} \quad (43)$$

which reverts to the same solution as (41), since the indexes σ_1 in (20a) and (20b) differ by unity and the recurrence formula just replaces $2n$ by $2n+1$.

The second fundamental $G_2(z)$ in (21a) corresponds to the index $m = 2$. For the coefficients of even order (20a), this corresponds to $\sigma_2 = 0$, thus the recurrence formula (18) yields

$$\sigma_2 = 0: \quad a_n(0) = -\frac{\Omega^2 a_{n-2}(0)}{(n-1)n(n+1)(n+2)}. \quad (44a)$$

This determines the coefficients of even order starting with $a_0(0) = 1$:

$$\begin{aligned} a_{2n}(0) &= \frac{(-1)^n \Omega^{2n}}{(2n-1)2n(2n+1)(2n+2).(2n-3)(2n-2)(2n-1)2n\ldots} \\ &= \frac{2(-1)^n \Omega^{2n}}{(2n-1)!!(2n)!!(2n+1)!!(2n+2)!!}. \end{aligned} \quad (44b)$$

The coefficients of even order (44b) lead to the solution

$$
\begin{aligned}
G_2(z) &= \sum_{n=0}^{\infty} a_{2n}(0) z^{2n} \\
&= a_0(0) + a_2(0) z^2 + a_4(0) z^4 + \dots \\
&= 1 - \frac{\Omega^2 z^2}{24} + \frac{\Omega^4 z^4}{8640} + O(\Omega^6 z^6) \,, \qquad (45)
\end{aligned}
$$

whose first three terms have been written explicitly. For the coefficients of odd order (20b), the substitution of the second index $\sigma_2 = -1$ in the recurrence formula (18) and the expansion of the particular solution lead to a solution similar to (45) using the second root in (20b), for the same reasons explained for the first fundamental.

The study of even (20a) and odd (20b) modes is similar, and the former are considered next. For a delta wing (29), the functions (41) and (45) are needed, together with their derivative. Differentiating (41) with respect to z and substituting the coefficients using (40b) results in

$$
\begin{aligned}
G_1'(z) &= \sum_{n=0}^{\infty} (2n+1) a_{2n}(1) z^{2n} \\
&= a_0(1) + 3a_2(1) z^2 + 5a_4(1) z^4 + \dots \\
&= 1 - \frac{\Omega^2 z^2}{40} + \frac{\Omega^4 z^4}{20160} + O(\Omega^6 z^6) \,, \qquad (46a)
\end{aligned}
$$

and similarly, differentiating (45) with respect to z and substituting the coefficients using (44b) yields

$$
\begin{aligned}
G_2'(z) &= \sum_{n=0}^{\infty} (2n) a_{2n}(0) z^{2n-1} \\
&= 2a_2(0) z + 4a_4(0) z^3 + \dots \\
&= \frac{\Omega^2 z}{12} + \frac{\Omega^4 z^3}{2160} + O(\Omega^6 z^5) \,. \qquad (46b)
\end{aligned}
$$

The condition (29) that specifies the natural frequencies of the delta wing can then be used.

The simplification of condition (29) for even modes is detailed in Appendix A and its numerical evaluation is included in Appendix B. The higher order approximations are calculated in Table 3 and lead to the fundamental frequency for the delta wing:

$$
\begin{aligned}
0 \le \lambda \le 1: \quad & 3.51602 = \bar{\Omega}_1 \\
& = \Omega_1(1) \le \Omega_1(\lambda) \le \Omega_1(0) = 19.91343 \,, \qquad (47)
\end{aligned}
$$

which is higher than for the rectangular wing, because the delta wing is stiffer, i.e., it has a larger fraction of the mass near the root. The result (47) can be converted from dimensionless Ω to dimensional ω frequency using (15) and (34b):

$$
\begin{aligned}
\lambda = 1: \; \bar{\omega}_1 &= \frac{\bar{\Omega}_1 \bar{e}}{L^2} \sqrt{\frac{E}{12}} \\
&= \frac{3.51602 \times 0.1}{6^2} \sqrt{\frac{70 \times 10^9}{12}} = 745.9 \, rad/s \,, \qquad (48a)
\end{aligned}
$$

$$
\begin{aligned}
\lambda = 0: \; \omega_1 &= \frac{\Omega_1(0) \bar{e}}{L^2} \sqrt{\frac{E}{3}} \\
&= \frac{19.91343 \times 0.1}{6^2} \sqrt{\frac{70 \times 10^9}{3}} = 8449.5 \, rad/s \,, \qquad (48b)
\end{aligned}
$$

$$
0 < \lambda < 1: \; \omega_1 = \frac{\Omega_1(\lambda) \bar{e}}{L^2} \sqrt{\frac{E}{3} \frac{(\lambda - 1)^2}{\lambda + 1}} \,, \qquad (48c)
$$

where the values (48a) and (48b) were calculated for an aluminum wing $E = 70$ GPa, with semi-span $L = 6$ m and mean thickness $\bar{e} = 0.1$ m. The fundamental frequency varies most between the delta and rectangular wing,

$$
\frac{\Omega_1}{\bar{\Omega}_1} = \frac{\Omega_1(0)}{\Omega_1(1)} = 5.66 \,, \qquad (49a)
$$

$$
\frac{\omega_1}{\bar{\omega}_1} = 2 \frac{\Omega_1}{\bar{\Omega}_1} = 11.32 \,, \qquad (49b)
$$

and for a swept wing with intermediate taper ratio,

$$
\lambda = 1/2: \quad \omega = \frac{\Omega \bar{e}}{L^2} \sqrt{\frac{E}{108}} \,, \qquad (50)
$$

lies in the range (50).

7 Discussion and conclusions

The natural bending frequency of a sweptback wing was derived using the governing differential equation for the unsteady deflection of a beam. The frequencies were obtained by casting the problem in the form of an eigenvalue problem, which translated into a root finding problem, $H(\Omega, \lambda) = 0$, once the proper expressions were derived for the sweptback wing.

As expected, for wings with the same span and mean chord (and thus area) and material, the fundamental natural bending frequency is higher for a delta planform, when compared to a rectangular planform, by a factor of 11.32. A tapered planform exhibits a frequency that lies within these two cases.

The results shown were restricted to wings with straight leading- and trailing-edges but a generalization is possible at the expense of a more complex derivation.

Copyright Statement

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A Calculation of eigenvalues for the delta wing

This appendix presents a method of calculating not only the fundamental but also the higher-order modes of a delta wing, by finding the roots of (29), where the factors are given by (41), (45), (46a) and (46b),

$$\left[\sum_{n=0}^{\infty} a_{2n}(1)\right]\left[\sum_{m=0}^{\infty}(2m)a_{2m}(0)\right] = \left[\sum_{n=0}^{\infty} a_{2n}(0)\right]\left[\sum_{m=0}^{\infty}(2m+1)a_{2m}(1)\right], \quad (51)$$

where $a_0(0) = 1 = a_0(1)$. Since the series are uniformly convergent, the rule of multiplication [15, 19],

$$\left(\sum_{n=0}^{\infty} b_n\right)\left(\sum_{m=0}^{\infty} c_m\right) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} b_m c_{n-m}, \quad (52)$$

applies to both sides of (51) resulting

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\Big[a_{2m}(1)(2n-2m)a_{2n-2m}(0) - a_{2m}(0)(2n-2m+1)a_{2n-2m}(1)\Big] = 0. \quad (53)$$

Substituting (40b) and (44b) in (53) yields a series of powers of Ω^2,

$$P_\infty(\Omega^2) \equiv 2.2.3.\sum_{n=0}^{\infty}(-1)^n d_n \Omega^{2n} = 0, \quad (54)$$

with coefficients

$$d_n \equiv \sum_{m=0}^{n}\left\{\frac{1}{(2m)!!(2m+1)!!(2m+2)!!}\frac{1}{(2n-2m)!!(2n-2m+1)!!(2n-2m+2)!!}\left[\frac{2n-2m}{(2m+3)!!(2n-2m-1)!!} - \frac{2n-2m+1}{(2m-1)!!(2n-2m+3)!!}\right]\right\}, \quad (55)$$

whose roots $\pm\Omega_n$ are the natural frequencies.

The successive approximations to the eigenvalues can be obtained by considering the series (54) truncated after $N+1$ terms, which is a polynomial of degree N in Ω^2:

$$P_N(\Omega^2) \equiv 2.2.3.\sum_{n=0}^{N}(-1)^n d_n \Omega^{2n} = d_N \Pi_{m=1}^{N}\left[\Omega^2 - (\Omega_m^{(N)})^2\right] = 0, \quad (56)$$

whose roots $\pm\Omega_1^{(N)}, \ldots, \pm\Omega_N^{(N)}$ are approximations to the first $2N$ eigenvalues. By increasing the degree of the polynomial, $N = 1, 2, \ldots$, more eigenvalues are found, and better approximations are obtained, e.g., the successive approximations to the fundamental frequency are $\Omega_1^{(1)}, \Omega_1^{(2)}, \Omega_1^{(3)}, \ldots$, which tend to the exact value:

$$\Omega_1 \equiv \lim_{N\to\infty} \Omega_1^{(N)}. \quad (57)$$

This process of successive approximations is illustrated in Tab. 3, for the fundamental frequency and next five harmonics. For a given N, the estimate is more accurate for the fundamental frequency $\Omega_1^{(N)}$ than for the higher harmonics.

B Calculation of natural frequencies

$\alpha_1 \equiv \sqrt{\bar{\Omega}_1}$	$f(\alpha) = \operatorname{sech}\alpha + \cos\alpha$
2	-1.5×10^{-1}
1.8	$+9.5\times10^{-2}$
1.9	-3.1×10^{-2}
1.87	$+6.3\times10^{-3}$
1.88	-6.1×10^{-3}
1.875	$+1.3\times10^{-4}$
1.8751	$+5.0\times10^{-6}$
1.87511	-7.4×10^{-6}
1.875105	-1.2×10^{-6}
1.875104	$+8.5\times10^{-8}$
$\alpha_1 = 1.875104,\quad \bar{\Omega}_1 = 3.51602$	

Table 1 Calculation of the fundamental dimensionless frequency $\bar{\Omega}_1$ of a rectangular wing.

α_1	α_2	α_3	α_4	α_5	α_6
1.87510	4.69409	7.85476	10.99554	14.13717	17.27876

$\bar{\Omega}_1$	$\bar{\Omega}_2$	$\bar{\Omega}_3$	$\bar{\Omega}_4$	$\bar{\Omega}_5$	$\bar{\Omega}_6$
3.51602	22.03449	61.69721	120.90192	199.85953	298.55553

Table 2 Fundamental dimensionless frequency and higher harmonics $\bar{\Omega}_n$ of a rectangular wing, where $\bar{\Omega}_n = \alpha_n^2 : f(\alpha_n) = 0$.

N	$\pm\Omega_1^{(N)}$	$\pm\Omega_2^{(N)}$	$\pm\Omega_3^{(N)}$	$\pm\Omega_4^{(N)}$	$\pm\Omega_5^{(N)}$	$\pm\Omega_6^{(N)}$
1	4.47214	-	-	-	-	-
2	-	-	-	-	-	-
3	13.06453	-	-	-	-	-
4	-	-	-	-	-	-
5	18.43387	-	-	-	-	-
6	20.34592	23.35716	-	-	-	-
7	19.89385	27.35107	32.73381	-	-	-
8	19.91421	26.40442	-	-	-	-
9	19.91341	26.45232	43.54070	-	-	-
10	19.91343	26.45003	-	-	-	-
11	19.91343	26.45011	47.27125	56.12092	61.31242	-
⋮						
20	19.91343	26.45011	47.40761	54.01666	83.06464	92.04460
⋮						
100	19.91343	26.45011	47.40761	54.01666	83.06464	92.04460

Table 3 Calculation of the fundamental dimensionless frequency and higher harmonics Ω_n of a delta wing, to successively higher orders in $\Omega^{(2N)}$.