THE SYNTHESIS OF WORST–CASE DISTURBANCES FOR CRITICAL FLIGHT CONDITIONS GENERATION

Alexander S. Filatyev, Olga V. Yanova
Central Aerohydrodynamic Institute (TsAGI), RUSSIA
e-mail: filatyev@yandex.ru

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Abstract

The method for determination of worst-case profiles of random disturbances distributed along the trajectory (at a given event probability) is described. The analytical synthesis of the worst-case disturbances profiles is obtained. The method allows to effectively build attainability domains, to analyze specific effects of various disturbances on the criterion under consideration, to significantly reduce the necessary calculations as compared to the Monte-Carlo technique.

1 Introduction

For the maximum efficiency, reliability and safety of aircraft flights in atmosphere it is necessary to define the influence of random disturbances. For real problems the number of accountable random factors can be very large. Only the dispersion of atmospheric thermodynamic parameters defined by canonical decompositions contains some tens of random parameters.

The Monte-Carlo method of statistic simulation [1] is a traditional engineering approach for the investigation of aircraft trajectories dispersion under effect of random disturbances. However, investigation of complex dynamic systems by a given method to obtain an estimate of probability $P(z > z^*)$ of some parameter z exceeding its permissible boundary z^* with required accuracy ε can require an unacceptably large number of realizations $N > \frac{1-P}{P\varepsilon^2}$ [2].

The so-called "guaranteed" approach or the "sandwich" method, used in some cases as the last resort, is reduced to the application of combinations of limit values of random parameters. Although such approach allows to sharply decrease the amount of computation, the obtained estimates of parameter scatter are greatly exaggerated. Methods based on the minimax approach [2]-[4] allow the estimations of output parameters of the problem to move closer to the reality. In this case the game problem is considered where the player, specifying values of unknown random factors, is nature. Application of these methods is particularly effective if it is possible to synthesize an optimal control of nature.

The method developed in this paper is ideologically close to the minimax approach. On one hand, assumptions used do not go beyond frames accepted in practical applications. On the other hand, they allow to obtain an analytical synthesis of "optimal" distribution of the random disturbances, leading to the maximal (worst) deviation of the functional from the nominal value. In contrast to a "sandwich" method, a probability of the combination of random factors is given.

The following assumptions are made:

- random disturbances are small, so it is possible to use the Bliss formula [5], linking the functional variation with disturbances of the right parts of the equations of motion and the solution of the conjugate system;
- distributions of random factors and the probability of an aggregate of random events are given.

At the same time

- the number of random factors can be arbitrary;

- the amount of computation and complexity of the method are minimal - the trajectory simulation is carried out only once in nominal terms (without disturbances);
- any simplification of the equations of motion is not required;
- the solution of the conjugate system of equations does not require an iterative procedure, because of its linearity for a stated phase trajectory; so the transversality conditions can be easily transferred to the left trajectory end, reducing the boundary problem to the Cauchy one.

Note that in the case of constructing the nominal trajectory with the help of the maximum principle, the "ready" solution of the conjugate system can be used for purposes of this study.

The developed technique has been realized within the program complex ASTER of the through optimization of branching aircraft trajectories [6], [7].

An example of the critical disturbance profiles is given for two widespread problems: the launch of a space vehicle (SV) and the dispersion of the fall points of the separated part (SP) of the first stage of this SV under the influence of random atmospheric disturbances including wind. A comparison of the results obtained with estimates of statistical modeling by the Monte Carlo is given. The possibility of significant (several orders of magnitude) reduction in the amount of computation, needed to evaluate the effect of random disturbances on the functional and the trajectory, is demonstrated.

2 The Problem Statement

The aircraft motion is described by a normal system of ordinary differential equations in the noninertial start frame of reference [8], [9]:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, \mathbf{u}, \boldsymbol{\varepsilon}, t), \tag{1}$$

where $\mathbf{x} = \{\mathbf{r}, \mathbf{v}, m\}^{\mathrm{T}} \in \mathbf{X}$ is the state vector, $\mathbf{r}$ is the radius vector, $\mathbf{v}$ is the velocity vector, m is the mass, $\mathbf{f}$ is the right part vector, $(\ )^{\mathrm{T}}$ is the transposition, $\mathbf{u} \in \mathbf{U}$ is the control vector, $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{sys} + \Delta\boldsymbol{\varepsilon}$ is the disturbance vector-function with systematic $\boldsymbol{\varepsilon}_{sys}$ and random $\Delta\boldsymbol{\varepsilon}$ components, $t \in [t_i, t_f]$ is the time.

Let's name the aircraft control and trajectory without random disturbances ($\Delta\boldsymbol{\varepsilon} = 0$) as nominal ones.

The effect of random disturbances is specified by the vector $\boldsymbol{\xi}$ of random factors:

$$\Delta\boldsymbol{\varepsilon} = \Delta\boldsymbol{\varepsilon}(\boldsymbol{\xi}).$$

It is required to find the worst combination of random factors $\boldsymbol{\xi}_{wrst}$, where the maximal (the worst) change $\delta\Phi$ of a functional $\Phi \equiv F(\mathbf{x})\big|_{t_f}$ is realized:

$$\boldsymbol{\xi}_{wrst} = \arg\max_{\boldsymbol{\xi} \in \mathbf{I}_\sigma} \delta\Phi(\boldsymbol{\xi}), \tag{2}$$

where $\mathbf{I}_\sigma$ is the set defined by the given event probability P_k.

A functional variation $\delta\Phi$, caused by a random disturbances influence, could be presented with the Bliss formula [5]:

$$\delta\Phi = \boldsymbol{\Psi}^{\mathrm{T}} \delta\mathbf{x}\Big|_i + \int_{t_i}^{t_f} \boldsymbol{\Psi}^{\mathrm{T}} \delta_{\xi}\mathbf{f} dt. \tag{3}$$

Here $\boldsymbol{\Psi} = \{\mathbf{P}, \mathbf{S}, P_m\}^{\mathrm{T}}$ is the conjugate vector with components, corresponding to $\mathbf{r}$, $\mathbf{v}$, and m respectively, $\delta_{\xi}\mathbf{f}$ is the $\mathbf{f}$ variation, caused by the $\boldsymbol{\xi}$. The conjugate vector $\boldsymbol{\Psi}$ is obtained from the solution of the Cauchy problem:

$$\begin{gathered} \dot{\boldsymbol{\Psi}} = -\frac{\partial \mathbf{f}}{\partial \mathbf{x}}^{\mathrm{T}} \boldsymbol{\Psi}, \\ \boldsymbol{\Psi}^{\mathrm{T}}(t_f) = \nabla\Phi(\mathbf{x}_f) - \frac{\dot{\Phi}(\mathbf{x}(t_f))}{\dot{G}(\mathbf{x}(t_f))} \nabla G(\mathbf{x}_f), \end{gathered} \tag{4}$$

where $G(\mathbf{x}(t_f)) = 0$ is the condition of implicit determination of the right end of the phase trajectory.

The conjugate variables $\boldsymbol{\Psi}$, defined according (3), (4), are the influence functions on the functional Φ of variations of phase variables in each trajectory point.

The vectors $\mathbf{x}, \mathbf{u}$ in (3), (4) correspond to the nominal trajectory without random disturbances ($\Delta\boldsymbol{\varepsilon} = 0$).

3 The Disturbances Model

There are considered disturbances of: an atmosphere density ρ, an atmosphere pressure p, and a horizontal wind with the velocity vector $\mathbf{W}$:

$$\begin{aligned}\boldsymbol{\varepsilon}^{\mathrm{T}} &= \{\rho, p, \mathbf{W}^{\mathrm{T}}\},\\ \boldsymbol{\varepsilon}^{\mathrm{T}}_{sys} &= \{\rho_{sys}, p_{sys}, \mathbf{W}^{\mathrm{T}}_{sys}\},\\ \Delta\boldsymbol{\varepsilon}^{\mathrm{T}} &= \{\Delta\rho, \Delta p, \Delta\mathbf{W}^{\mathrm{T}}\},\\ \Delta\mathbf{W} &= \Delta W_\lambda \mathbf{e}_\lambda + \Delta W_\varphi \mathbf{e}_\varphi,\end{aligned} \tag{5}$$

where ΔW_λ is a longitude projection (in the east direction), ΔW_φ is a lateral projection (to the north), $\mathbf{e}_\lambda$ and $\mathbf{e}_\varphi$ are unit vectors in corresponding directions.

Random components are defined in the form of a canonical decomposition:

$$\begin{aligned}\Delta\bar{\rho} &= \frac{\Delta\rho}{\rho_{ном}} = \kappa_1(n_t)\sum_{j=1}^{k} b_j^{\rho}(\mathbf{x})\xi_j,\\ \Delta\bar{p} &= \frac{\Delta p}{p_{ном}} = \kappa_1(n_t)\sum_{j=1}^{k} b_j^{p}(\mathbf{x})\xi_j,\\ \Delta W_\lambda &= \kappa_2(n_t)\sum_{j=k+1}^{k+m} b_j^{W}(\mathbf{x})\xi_j,\\ \Delta W_\varphi &= \kappa_2(n_t)\sum_{j=k+1}^{k+m} b_j^{W}(\mathbf{x})\xi_{j+m},\end{aligned} \tag{6}$$

where $\rho_{ном}$, $p_{ном}$ are standard atmosphere density and pressure, n_t is a date of flight, $k_1(n_t)$, $k_2(n_t)$, $\mathbf{b}^{\rho}(\mathbf{x})$, $\mathbf{b}^{p}(\mathbf{x})$, $\mathbf{b}^{W}(\mathbf{x})$ are known vector functions, ξ is a vector of independent random numbers distributed under the central normal law with the unit dispersion with the dimension $n_\xi = k + 2m$.

4 The Synthesis of Worst-Case Disturbances

From (3) in view of (5), (6) a functional variation $\delta\Phi$ could be written down as

$$\begin{aligned}&\delta\Phi = \boldsymbol{\Phi}_\xi^{\mathrm{T}}\boldsymbol{\xi},\\ &\Phi_{\xi j} = \kappa_1(n_t)\int_{t_i}^{t_f}\mathbf{S}^{\mathrm{T}}\left(b_j^{\bar{\rho}}(\mathbf{x})\frac{\partial\mathbf{f}}{\partial\bar{\rho}} + b_j^{\bar{p}}(\mathbf{x})\frac{\partial\mathbf{f}}{\partial\bar{p}}\right)dt,\\ &\qquad j = 1,\ldots,k,\\ &\Phi_{\xi j} = \kappa_2(n_t)\int_{t_i}^{t_f}\mathbf{S}^{\mathrm{T}}\frac{\partial\mathbf{f}}{\partial\mathbf{W}}\mathbf{e}_\lambda b_j^{W}(\mathbf{x})dt,\\ &\qquad j = k+1,\ldots,k+m,\\ &\Phi_{\xi\, j+m} = \kappa_2(n_t)\int_{t_i}^{t_f}\mathbf{S}^{\mathrm{T}}\frac{\partial\mathbf{f}}{\partial\mathbf{W}}\mathbf{e}_\varphi b_j^{W}(\mathbf{x})dt,\\ &\qquad j = k+1,\ldots,k+m.\end{aligned} \tag{7}$$

Let the set $\mathbf{I}_\sigma$ of random factors $\boldsymbol{\xi}$ is a hypersphere with the radius κ_σ, limited to a surface with equal probability:

$$\mathbf{I}_\sigma = \left\{\boldsymbol{\xi} : \|\boldsymbol{\xi}\| \overset{\Delta}{=} \sqrt{\sum_{j=1}^{n_\xi}\xi_j^2} \le \kappa_\sigma\right\}. \tag{8}$$

In this case $\boldsymbol{\xi}_{wrst}$ is defined from the solution of the system

$$\begin{cases}\Phi_{\xi j} + 2\lambda_\xi\xi_j = 0,\ j = 1,\ldots,n_\xi,\\ \sum_{j=1}^{n_\xi}\xi_j^2 - \kappa_\sigma^2 = 0,\end{cases} \tag{9}$$

where λ_ξ is the Lagrange coefficient.

From (9) the synthesis of a vector $\boldsymbol{\xi}_{wrst}$ is obtained:

$$\xi_{j\,wrst} = -\frac{\Phi_{\xi j}}{2\lambda_\xi},\ j = n_\xi,\ \lambda_\xi = -\frac{1}{2\kappa_\sigma}\sqrt{\sum_{j=1}^{n_\xi}\Phi_{\xi j}^2}\,. \tag{10}$$

The hypersphere radius κ_σ is defined by the given event probability P_k of a performance of a (8) condition. The P_k is equal to the product of probabilities P_{kj} of hitting independent components of the vector $\boldsymbol{\xi}_{wrst}$ into $[-\kappa_\sigma, \kappa_\sigma]$:

$$P_k = \prod_{j=1}^{n_\xi} P_{kj}\left(|\xi_j| \le \kappa_\sigma\right). \tag{11}$$

It is follows from (11):

$$P_{kj}\left(|\xi_j| \le \kappa_\sigma\right) = \sqrt[n_\xi]{P_k} = \frac{1}{\sqrt{2\pi}}\int_{-\kappa_\sigma}^{\kappa_\sigma} e^{-\frac{1}{2}\tau^2}d\tau,\ j = 1,\ldots,n_\xi,$$

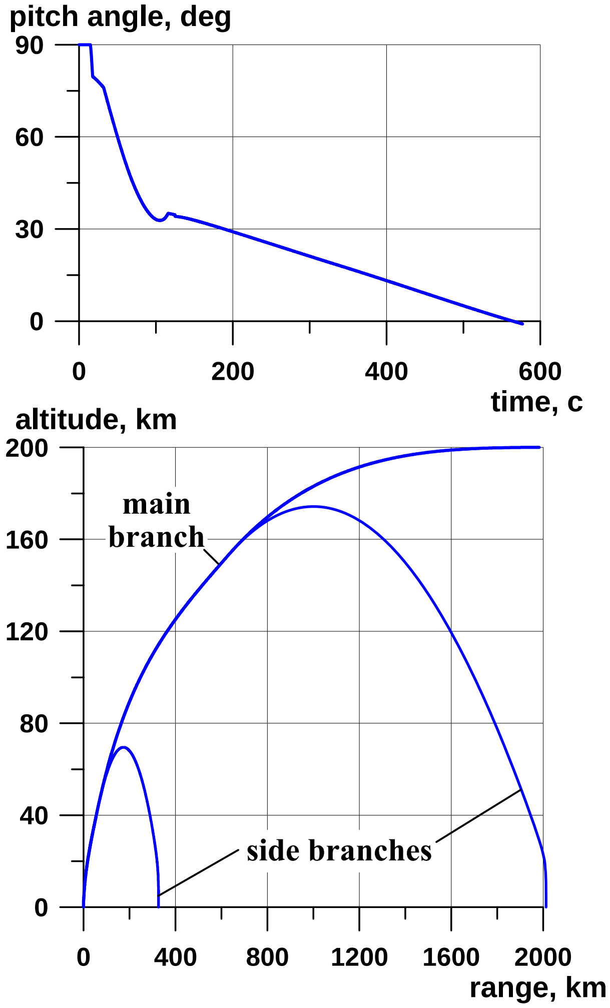

Fig.1 The optimal time-history of the pitch angle and optimal trajectory of the space vehicle considered.

whence with the help of function tables of the normal distribution the hypersphere radius κ_σ is defined for given P_k and n_ξ.

5 Critical Atmospheric Disturbances for the Space Vehicles Launch

In accordance with the developed technique, the influence of random disturbances is evaluated on the active trajectory phase of a three-stage SV with vertical start in the disturbed atmosphere. Initial masses of stages are as 1: 0.3516: 0.1078. Initial thrust-to-weight ratios of stages are 1,374, 1.000 and 0,826 respectively on the nominal trajectory. Mass ratios of separated parts to the initial SV mass are 0.0431, 0.0202 and 0.0052 respectively. SV is injected to the circular satellite orbit with the altitude $h_{orb} = 200\,\text{км}$ and the inclination

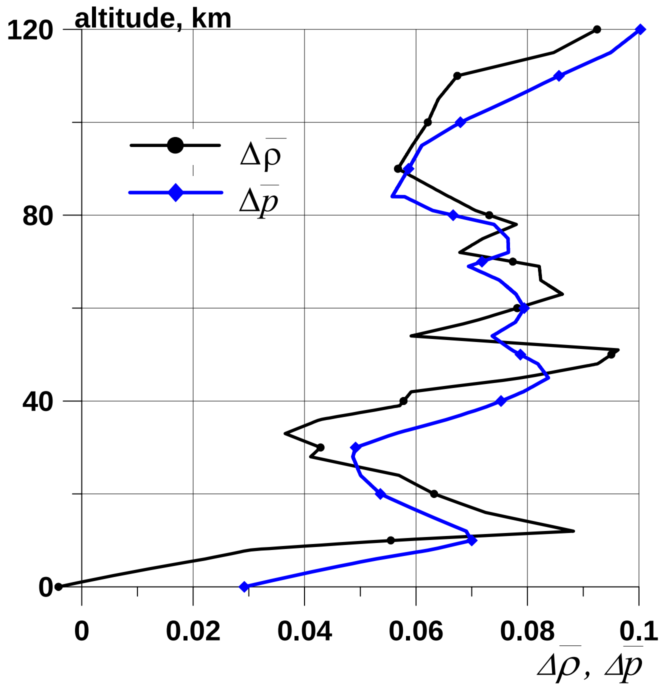

Fig. 2 The worst-case profiles of random disturbances of an atmospheric density ρ and pressure p for the probability P_k=0 .9973.

$i_{orb} = 51.6^\circ$ from the Baikonur launch site in January.

Figure 1 shows the nominal ($\Delta\varepsilon = 0$) branching injection trajectory and optimal time-history of the pitch angle of SV. The main trajectory branch corresponds to the insertion trajectory, side branches correspond to passive SP trajectories.

The vector ξ (6) has dimension of $n_\xi = 39$ ($k = 15,\ m = 12$). The level of events probability is

$$P_k = 0.9973. \tag{12}$$

The vector of random factors ξ_{wrst} that determines the maximal (in absolute value) decrease of the SV injected mass

$$\overline{\Delta m_f} = \frac{|\Delta m_f|}{m_f|_{\Delta\varepsilon=0}}$$

is calculated.

The corresponding worst disturbances profiles of atmospheric density and pressure are shown in Fig. 2. To obtain these profiles in accordance with the developed method is sufficient to calculate the SV nominal trajectory.

For comparison, a statistical simulation of N=10^4 SV injected trajectories is performed under disturbances of atmospheric density and

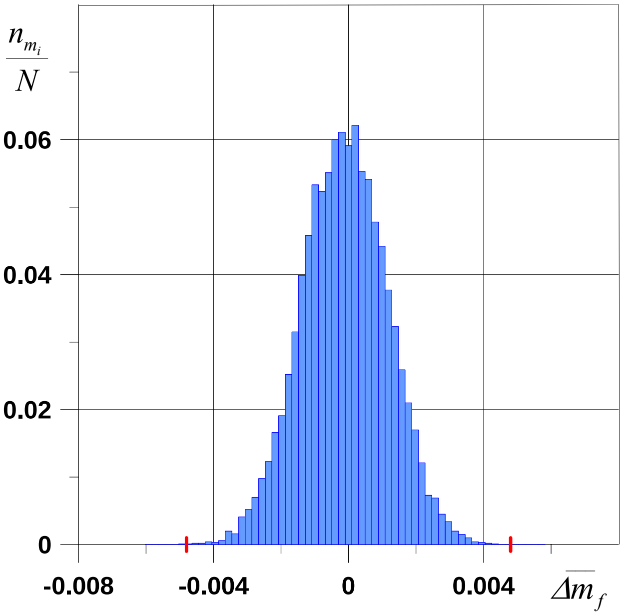

Fig. 3 Distribution bar chart for variations of injected mass $\Delta \overline{m}_f$ due to disturbances of atmospheric density and pressure. | corresponds to the injected mass estimation in accordance with (7), (10).

pressure (5), (6). The histogram obtained of the probability distribution of injected mass variations is shown in Fig. 3. The estimation of the maximal change of injected mass (with a given events probability (12)), obtained in accordance with (7), (10), is shown. As seen from the comparison, estimates obtained using the two methods practically coincide, while the amounts of computation related as 1: 10000.

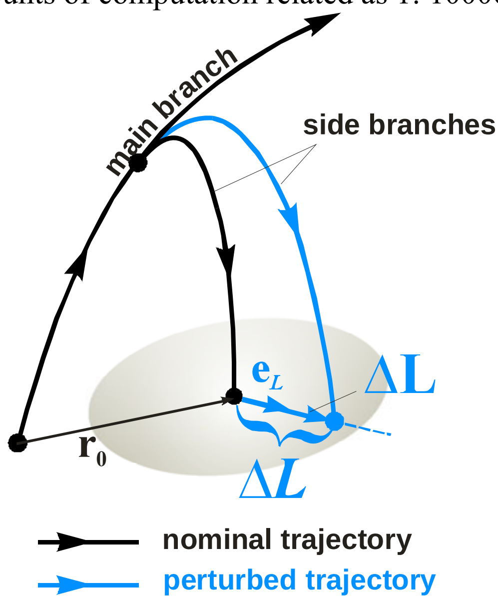

Fig. 4.

6 Critical Atmospheric Disturbances for the analysis of a dispersion of Separated Parts Fall Points

To calculate a boundary of a dispersion ellipse of the first-stage SP fall points it is required to find such combination of random factors, which results in the maximal deviation of the fall point in various directions $\mathbf{e}_L$ (see Fig. 4):

$$\Delta L_{max} = \max_{\xi} \Delta L .$$

The 3D critical profile of a random wind $\Delta W(\mathbf{x}, \xi_{wrst})$ for the probability (12) of a combination of all random events for the January atmosphere conditions is shown in Fig. 5. Such wind profile provides the maximum longitudinal deviation of the first-stage SP fall point. This profile represents the hodograph of the horizontal wind velocity vector depending on the flight altitude h.

Rotating the unit vector $\mathbf{e}_L$ around the nominal aiming point $\mathbf{r}_0$ in a local horizontal plane, the vector $\mathbf{\Delta L} = \Delta L_{max}\mathbf{e}_L$ circumscribes the boundary of the dispersion area of SP fall points D (see Fig. 6). Note that the calculation of this boundary in accordance with the developed method is carried out without integration of disturbed trajectories. It is still enough to have one nominal trajectory.

The dispersion ellipse boundary, calculated in accordance with the developed method, and fall points, obtained as a result of the statistic simulation of 10^4 falling trajectories by the Monte-Carlo engineering method, are compared in Fig. 6. The results are in a good agreement. At the same time the developed method is 10,000 times more efficient.

7 Conclusions

The analytical synthesis of the "optimal control" for random disturbances, which causes the worst trajectory deviations from rated conditions, is obtained. It allows to reduce the necessary calculations for an estimation of the random disturbances effect in ~10^5–10^7 times in comparison with the widespread Monte-Carlo method.

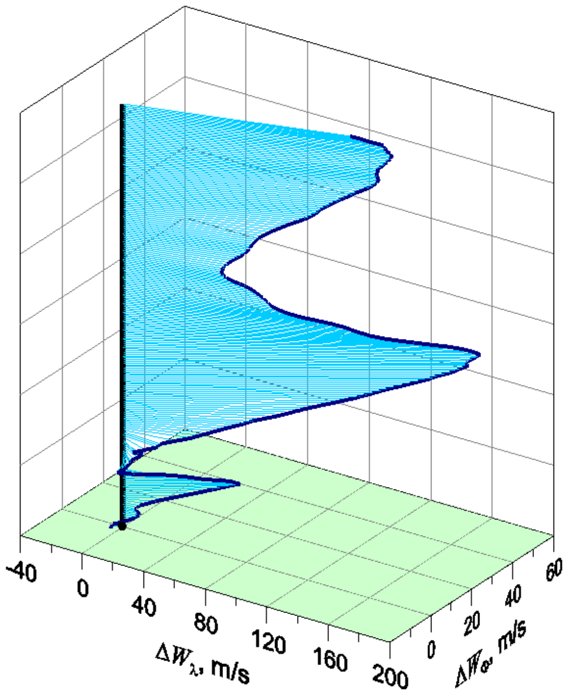

Fig. 5 The 3D profile of the critical random disturbances, providing the maximal fall point longitudinal deviation for SV launch in January.

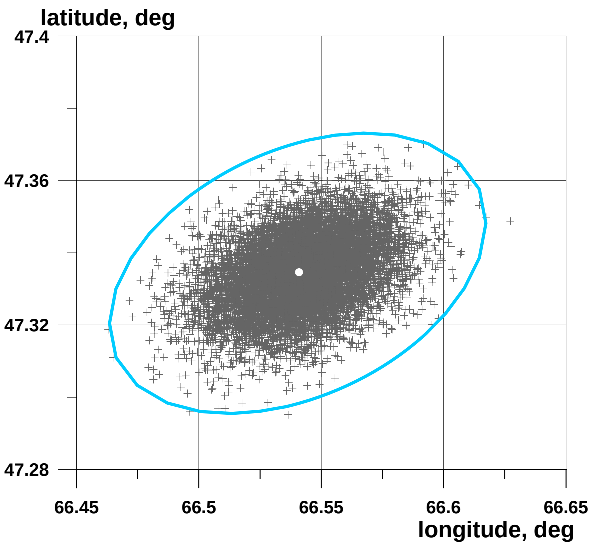

Fig. 6 The comparison of the boundary of the dispersion ellipse, calculated by the developed method (—), with the ellipse evaluation by integration of 10^4 trajectories in the frame of Monte-Carlo method (+).

8 Acknowledgments

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References

[1] Ермаков С.М., Михайлов Г.А. Статистическое моделирование М., Наука, 1982.

[2] Kuzmin V.P., Yaroshevsky V.A. Estimation of phase co-ordinates large deviation of dynamic system caused by random perturbations. Moscow, Nauka, Publishing Company Fizmatlit, 1995.

[3] Кейн В.М. Оптимизация систем управления по минимаксному критерию. – М.Ж Наука, 1985.

[4] Malyshev V.V., Kibzun F.I., Karp K.A. Minimax approach for complex technical system stochastic modeling. AMSE Press. – 1988. – V. 1, No 3.

[5] Bliss G.A. Mathematics for Exterior Ballistics. N. Y., 1944.

[6] Filatyev, A.S., Yanova, O.V., and Golikov, A.A. ASTER – Indirect Optimization of Branched Injection Trajectories of Aerospace Vehicles, 1st ESA Workshop on Astrodynamics Tools and Techniques, 17-18 July 2001, ESTEC, Noordwijk, The Netherlands.

[7] Filatyev, A.S. and Yanova, O.V. ASTER Program Package for the Thorough Trajectory Optimization, AIAA 2001-4391, 41st AIAA GN&C Conference, 2001, Montreal, Canada.

[8] Filatyev, A.S. Optimization of spacecraft ascent using aerodynamic forces. 43rd Congress of the International Astronautical Federation, August 28 - September 5, 1992/ Washington, DC, IAF - 92-0022.

[9] Filatyev, A.S. “Paradoxes” of Optimal Solutions in Problems of Space Vehicle Injection and Reentry. Acta Astronautica, Vol.47, No.1, pp.11-18, 2000.

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