FREQUENCY DOMAIN FLUTTER SOLUTION TECHNIQUE USING COMPLEX MU-ANALYSIS

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Abstract

Appling perturbation to dynamic pressure in the nominal aeroelastic equation of motion, a mu-omega method is presented for flutter solution. Utilizing the structured singular value mu as stability margin indicator, this method simply employs frequency domain aerodynamics to predict the nominal flutter margin through frequency domain mu-analysis. The error of flutter dynamic pressure is selected as convergence criterion during iterations. Discontinuity of the real mu with respect to purely real perturbation is found in a two dimensional wing model with steady aerodynamics when using mu-omega method. To prevent this problem, complex perturbation to dynamic pressure is considered mathematically and the complex mu-analysis is employed in this study. Numerical examples demonstrated that the flutter results obtained by the mu-omega method correlate well with those of the p-k method.

1 General Introduction

Flutter is a typical dynamic instability phenomenon in fluid structural interaction systems such as atmosphere flight vehicles with aerodynamic surfaces and the mathematical model can always reduce to a nonlinear eigenvalue problem. A historical review of several solutions of flutter eigenvalue has been made by Heeg in [1], wherein the classical k method and p-k method were discussed [2,3]. For k method always assumes structural damping g which lacks physical meaning, thus the p-k method are more widely used, because it is directly related to decay rate which has more physical meanings. However, as no p-domain aerodynamic is available (i.e. the unsteady aerodynamics is usually obtained in frequency domain by panel methods); the decay rate computed by the p-k method is only valid near the critical flutter speed. Recent years a g-method is proposed by Chen [4], in which the unsteady aerodynamics is treated as analytic function in p-domain, and the frequency domain aerodynamics is employed with a damping perturbation approach.

In the last decade, Lind and Brenner introduced structured singular value (SSV) μ into the aeroelastic research field and developed a μ method [5]. In the μ method, aeroelastic system is parameterized at a nominal dynamic pressure with perturbation to dynamic pressure, which makes it possible to solve nominal flutter problem by μ analysis.

However, rational function approximation (RFA) technique [6,7] of the frequency domain aerodynamics is needed in the μ method, which sounds like a p method that employs a p-domain aerodynamics. Borglund et al. [8-11] demonstrated that the RFA technique is not necessary in robust flutter analysis and proposed a μ-k method to make direct use of frequency domain aerodynamics, but the perturbation to dynamic pressure is excluded to facilitate the robust flutter analysis.

As a combined work of [5] and [7], the authors proposed a μ method to make a nominal flutter solution in frequency domain in the previous studies, and the error of reduced frequency at the flutter speed is selected as the convergence criterion. The results coincide well with the p-k method for the test cases [12].

The μ-ω method simply solves the flutter margin through frequency domain μ analysis of the system frequency response matrix, thus it utilizes frequency domain aerodynamics directly, and no assumption (i.e. small decaying rate) needs to be made other than the classical methods mentioned above. Using μ as flutter margin indicator, one increase the initial guess airspeed or dynamic pressure carefully and the μ-ω method predicts the flutter speed until the error of flutter reduced frequency converges to meet an acceptable accuracy level.

In this work, the discontinuity of μ analysis with respect to purely real perturbation to dynamics pressure is concerned and a complex perturbation to dynamic pressure is introduced mathematically, the complex μ analysis is employed which can guarantee the continuity of μ and predict accurate flutter results.

2 Formulation of the μ-ω Method

2.1 Equation of Aeroelastic Motion

The equation of aeroelastic motion in frequency domain reads

$$[-\omega^2\boldsymbol{M} + j\omega\boldsymbol{B} + \boldsymbol{K} - q_\infty \boldsymbol{Q}(k)]\{\overline{\boldsymbol{\eta}}\} = 0 \qquad (1)$$

where $\boldsymbol{M}, \boldsymbol{B}, \boldsymbol{K} \in \mathbf{R}^{n\times n}$ stand for the generalized mass, damping and stiffness matrices, respectively, $\boldsymbol{Q}(k)$ stands for the generalized aerodynamic influence coefficient matrix which is a tabular function of reduced frequency k. $\overline{\boldsymbol{\eta}} \in \mathbf{C}^n$ is generalized displacement and q_∞ is dynamic pressure.

Lind and Brenner proposed a real perturbation to the dynamic pressure, which is expressed as

$$q_\infty = q_0(1+\delta_q) \qquad (2)$$

where $\delta_q \in \mathrm{R}$ and $\| \delta_q \|_\infty \le 1$.

Substitute Eq. (2) into Eq. (1) and transfer the perturbed term to the right hand side of Eq. (1), one can get

$$[-\omega^2\boldsymbol{M} + j\omega\boldsymbol{B} + \boldsymbol{K} - q_0 \boldsymbol{Q}(k)]\{\overline{\boldsymbol{\eta}}\} = \delta_q q_0[\boldsymbol{Q}(k)]\{\overline{\boldsymbol{\eta}}\} \qquad (3)$$

2.2 Development of the μ Framework

As note in [12], the μ framework for Eq. (3) can be formulated by the following procedure.

First introduce two signals $\{\boldsymbol{w}\}$ and $\{\boldsymbol{z}\}$,

$$\{\boldsymbol{w}\} = \delta_q \boldsymbol{I}_n \{\boldsymbol{z}\} \qquad (4)$$

$$\{\boldsymbol{z}\} = q_0[\boldsymbol{Q}(k)]\{\overline{\boldsymbol{\eta}}\} \qquad (5)$$

and define the flutter matrix $\boldsymbol{F}_0$ at q_0 as

$$\boldsymbol{F}_0 = -\omega^2\boldsymbol{M} + j\omega\boldsymbol{B} + \boldsymbol{K} - q_0\boldsymbol{Q}(k) \qquad (6)$$

Then Eq. (3) can be rewritten as follows.

$$[\boldsymbol{F}_0]\{\overline{\boldsymbol{\eta}}\} = \{\boldsymbol{w}\} \qquad (7)$$

Solving Eq. (7) for $\overline{\boldsymbol{\eta}}$, we get

$$\{\overline{\boldsymbol{\eta}}\} = [\boldsymbol{F}_0]^{-1}\{\boldsymbol{w}\} \qquad (8)$$

Now we Substitute Eq. (8) into Eq. (5), and the formulation is given by

$$\{\boldsymbol{z}\} = [\boldsymbol{P}]\{\boldsymbol{w}\} \qquad (9)$$

$$\text{where } \boldsymbol{P} = q_0\boldsymbol{Q}(k)\boldsymbol{F}_0^{-1} \qquad (10)$$

The relationships between signals $\{\boldsymbol{w}\}$ and $\{\boldsymbol{z}\}$, i.e. Eq. (4) and Eq. (9), essentially form a typical μ framework as depicted in Fig. 1.

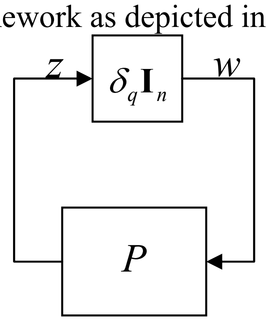


Fig. 1. Diagram of the μ Framework of Nominal Aeroelastic System.

Now we can use the frequency domain μ-analysis to determine the critical flutter point with the frequency response matrix $\boldsymbol{P}(j\omega)$. Before that, let's first recall the definition of μ.

2.3 Definition of μ and the μ-ω Method

For a plant model $\boldsymbol{P}$ formulated in the framework depicted in Fig. 1 with respect to a structured uncertainty set

$$\mathbf{\Delta} = \{\Delta : \Delta = \delta_q I_n, \delta_q \in R, \| \delta_q \|_\infty \le 1\} \quad (11)$$

structured singular value μ is defined as

$$\mu_{\mathbf{\Delta}}(P) = 1/\min\{\bar{\sigma}(\Delta) : \Delta \in \mathbf{\Delta}, \det(I - P\Delta) = 0\} \quad (12)$$

If there is no $\Delta \in \mathbf{\Delta}$ that makes $I - P\Delta$ singular, then $\mu_{\mathbf{\Delta}}(P) = 0$.

The peak value of μ, β_u is used to predict the flutter margin.

$$\beta_u = \sup_{\omega} \mu_{\mathbf{\Delta}}(P(j\omega)) \quad (13)$$

$$|\delta_q| \le 1/\beta_u \quad (14)$$

$$q_f = q_0(1 + 1/\beta_u) \quad (15)$$

where Eq. (14) stands for the maximum perturbation in Eq. (2) to ensure the robust stability of uncertain aeroelastic model, Eq. (3).

In the μ-ω method, one increases the initial guess airspeed or dynamic pressure q_0 carefully until the predicted flutter dynamic pressure q_f converges to an acceptable accuracy level.

3 Analysis and Main Results

3.1 Analytical μ Analysis of 2D Wing System

For a two dimensional (2D) wing aeroelastic system, when steady aerodynamics is considered, the system equation of motion and the generalized matrices can be formulated as follows [13].

$$\ddot{h}_0 + x_\alpha \ddot{\alpha} + c_{h_0} \dot{h}_0 + \omega_0^2 h_0 = -2q_0\alpha \quad (16)$$

$$x_\alpha \ddot{h}_0 + r_\alpha^2 \ddot{\alpha} + c_\alpha \dot{\alpha} + r_\alpha^2 \alpha = (1+2a)q_0\alpha$$

$$\boldsymbol{M} = \begin{bmatrix} 1 & x_\alpha \\ x_\alpha & r_\alpha^2 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} c_{h_0} & 0 \\ 0 & c_\alpha \end{bmatrix},$$

$$\boldsymbol{K} = \begin{bmatrix} \omega_0^2 & 0 \\ 0 & r_\alpha^2 \end{bmatrix}, \boldsymbol{\eta} = \begin{Bmatrix} h_0 \\ \alpha \end{Bmatrix},$$

$$\boldsymbol{Q} = \begin{bmatrix} 0 & -2 \\ 0 & 1+2a \end{bmatrix}$$

where $q_0 = Q_0/\mu_0$ is non-dimensional dynamic pressure.

Let $\Delta = \boldsymbol{I}\lambda$ in Eq. (3) and solve for λ

$$\det(\boldsymbol{I} - \boldsymbol{P}\Delta) = 0 \quad (17)$$

one can obtain

$$\lambda = -1 - \frac{c_1 + jd_1}{a_1 + jb_1} = -1 - \frac{(a_1c_1 + b_1d_1) + j(a_1d_1 - b_1c_1)}{a_1^2 + b_1^2}$$

where

$$a_1 = -q_0(1 + 2a + 2x_\alpha)\omega^2 + q_0(1+2a)\omega_0^2$$

$$b_1 = q_0(1+2a)c_{h_0}\omega$$

$$c_1 = (x_\alpha^2 - r_\alpha^2)\omega^4 + [r_\alpha^2(1+\omega_0^2) + c_{h_0}c_\alpha]\omega^2 \cdot$$

$$d_1 = (c_{h_0}r_\alpha^2 + c_\alpha)\omega^3 - (c_{h_0}r_\alpha^2 + \omega_0^2 c_\alpha)\omega$$

According to the definition of Eq. (12), if there exists a real λ, $\mu = 1/|\lambda|$, else $\mu = 0$.

Now the existence of real λ is equal to

$$a_1d_1 - b_1c_1 = -q_0\omega(a_0\omega^4 + b_0\omega^2 + c_0) = 0 \quad (18)$$

where

$$a_0 = (1+2a)c_{h_0}x_\alpha^2 + 2(c_\alpha + c_{h_0}r_\alpha^2)x_\alpha + (1+2a)c_\alpha$$
$$b_0 = c_\alpha(1+2a)c_{h_0}^2 - 2x_\alpha r_\alpha^2 c_{h_0} - 2c_\alpha(1+2a+x_\alpha)\omega_0^2$$
$$c_0 = c_\alpha(1+2a)\omega_0^4$$

Assuming $a > -0.5$ (the elastic axis locates aft 1/4 chord stream wisely), it is concluded that Eq. (18) has three non-negative real roots at most, i.e. $0, \omega_1, \omega_2$ when structural damping is omitted. This means the real μ has at most three non-zero value along the positive frequency axis, i.e. the real μ is discontinuous.

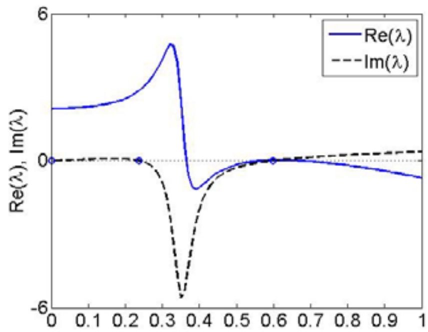
3.2 Numerical μ Analysis of 2D Wing System

With the parameters employed in [13] (see Table 1), real part and image part of λ is shown in Fig. 2 with respect to a given dynamic pressure $Q_0 = 4.0$, and numerical value of real and complex μ is shown in Fig. 3. Figure 2 depicts that the image part of λ crosses the horizontal axis three times. It is illustrate that real μ only has non-zero values at three discrete frequency points in Fig. 3, while the complex μ is a continuous function of frequency.

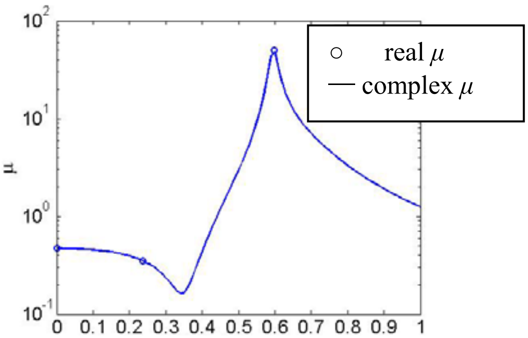
The predicted flutter results by analytical method are shown in Table 2 compared with the results by real and complex μ.

Table 1 Non-dimensional parameters for 2D wing

parameters	value
μ_0	20.0
a	-0.1
x_α	0.25
r_α^2	0.5
ω_0^2	0.2
c_{h0}	0.1
c_α	0.1



Nondimensional frequency ω/ω_α
Fig. 2. Diagrams of μ v.s. ω/ω_α



Nondimensional frequency ω/ω_α
Fig. 3. Diagrams of μ v.s. ω/ω_α

Table 2 Flutter results predicted for $Q_0 = 4.0$

Method	β_μ	Q_F	ω_F / ω_α
Analytic	-	4.0802	0.5982
Real μ	49.9057	4.0802	0.5982
Complex μ	47.9566	4.0834	0.6000

3.3 The Complex μ-ω Method and Numerical Application

It is implied that when the computed dynamic pressure is close to the flutter point, complex μ can predict a flutter result with certain accuracy. Based on this fact, the complex μ-ω method can be described as:

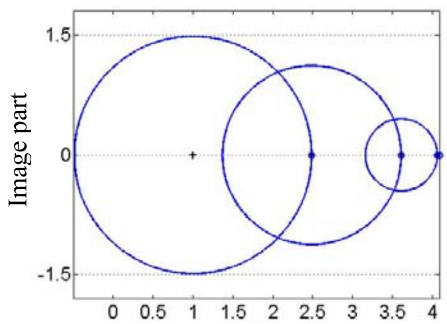
A nominal flutter solution by increasing the initial guess airspeed or dynamic pressure carefully every time, utilize a frequency domain μ analysis to search a minimum W_q that destabilizes the aeroelastic system with respect to purely complex dynamic pressure perturbation.

In this study, we just use the predicted flutter dynamic pressure in each step as initial guess of dynamic pressure in the next step. With a convergence level of

$$|Q_F / Q_0 - 1| \times 100\% < 1\% \qquad (19)$$

the flutter results predicted with complex μ is shown in Table 3. It is demonstrated that the algorithm converges well and the final results coincide well with the analytical results shown in Table 2.

An interesting graph of the predicted flutter dynamic pressure is illustrated in Fig. 4, wherein the center point of each disc stands for initial guess dynamic pressure Q_0 in the first column of Table 3, and the radius of each disc stands for the predicted flutter margin. This graph shows a good convergence property of the complex μ-ω method with rapidly decreasing radius within 4 iterations.



Real part of Q_0
Fig. 4. Predicted Flutter Dynamic Pressure on Complex Surface

Table 3 Flutter results predicted with complex μ

Q_0	Predicted Flutter Margin	Q_F	ω_F / ω_α
1.0000	1.4887	2.4887	0.9900
2.4887	1.1214	3.6101	0.8400
3.6101	0.4528	4.0629	0.5800

4.0629	0.0234	4.0863	0.6000

4 Conclusions

The discontinuity problem of real μ analysis in flutter solution is noted in this report and an alternative way that using complex μ analysis to get a continuous μ is explored. It is found that, when using real μ analysis in flutter solution, the discontinuity problem of real μ that results from purely real perturbation to dynamic pressure should be carefully examined. The complex perturbation to dynamic pressure is mathematically introduced to make a continuous μ. The algorithm developed for flutter solution with complex μ analysis can converge to the flutter result with satisfied accuracy, which is already applied to several examples successfully in another work [14].

Acknowledgments

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