FULLY ISOTROPIC LAMINATES WITH UP TO TWENTY SEVEN PLIES

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Abstract

The present study provides a number of fully isotropic stacking sequences for relatively thin composite laminates made of (0°, +60°, -60°) layers with up to 27 plies. Five independent fully isotropic solutions with one anti-symmetric case are found for 18-ply laminates. For 27-ply laminates, 340 independent fully isotropic solutions with thirty six anti-symmetric cases are found.

1 Introduction

Most fiber-reinforced composites are usually constructed with multiple orthotropic layers in which the reinforcing fibers are aligned in pre-determined directions. In most field applications with single material type, the thickness of each layer is usually an integer-multiple of the unit ply thickness. Even though the elastic constants of a laminate can be tailored by combining and adjusting the fiber orientation in each layer, it is challenging to find fully isotropic stacking sequences for relatively thin laminates.

A particular class of anti-symmetric angle-ply laminates with $B_{ij}=D_{16}=D_{26}=0$ was presented by Caprino and Visconti [1]. Their analysis utilizes the fact that $B_{ij}=0$ for any symmetric laminate, and $A_{16}=A_{26}=0$ for any balanced laminate, and $D_{16}=D_{26}=B_{11}=B_{22}=B_{12}=B_{66}=0$ for any anti-symmetric laminate. Their study shows that the simplest laminate is the anti-symmetric angle-ply laminate with 8 plies, $[(\pm\theta)_S/(\mp\theta)_S]$, in which all coupling terms, A_{16}, A_{26}, B_{ij}, D_{16}, and D_{26} are rigorously zero.

Gunnink [2] presented a simple comment on the result of Caprino and Visconti [1], which states "if laminate B is anti-symmetric with respect to laminate A and if laminate A as well as B is balanced, then A_{16}, A_{26}, B_{ij}, D_{16}, and D_{26} of the entire laminate AB become rigorously zero."

Jong-Won Lee et al. [3] presented a simple analytical method to obtain fully orthotropic laminates (FOL's) with anti-symmetric stacking sequences. In their study, Pascal's number triangle and Pythagorean theorem are found to be very closely related to the analytical stacking sequences sufficient to guarantee FOL's.

Vannucci & Verchery [4] proposed some new kinds of fully isotropic laminates (FIL's) obtained by the so-called polar method. They found five independent solutions for the 18 ply-laminates, one independent solution for the 24-ply laminates, 219 solutions for the 27-ply laminates, and 29 solutions for the 30-ply laminates.

Valot and Vannucci [5] presented some numerical solutions for FOL's made of 6 to 18 plies stacked anti-symmetrically. One of their solution is identical to $[(\pm\theta)_S/(\mp\theta)_S]$ previously presented by Caprino and Visconti [1].

Jong-Won Lee et al. [6] proposed some sufficient conditions for analytically obtaining FOL's. They studied a number of specially chosen anti-symmetric stacking sequences in which all coupling stiffness components including B_{16} and B_{26} become rigorously zero. They also presented a systematic way to insert fully orthotropic cores with arbitrary thicknesses and arbitrary elastic constants between plies to provide a stepping stone toward further generalization.

In the present study, a relatively simple analytical method is presented for obtaining FIL's made of (0°, +60°, -60°) layers with up to 27 plies.

2 Mathematical Modeling

Let's consider the laminates made of (0°, +60°, -60°) layers with m plies for each fiber orientation, where m is a natural number no greater than 9. In order to obtain FOL's initially, some plies must be shuffled properly so that D_{16}=D_{26}=0, B_{11}=B_{22}=B_{12}=0, and B_{16}=B_{26}=0. It is also self-evident that the contribution of each fiber orientation must be kept identical to make the entire laminate fully isotropic. In other words, the entire laminate stiffness components must be invariant with respect to the in-plane tensor transformation.

It has been well known that the stiffness matrix for FOL's is given by

$$\begin{Bmatrix} N_1 \\ N_2 \\ N_6 \\ M_1 \\ M_2 \\ M_6 \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 & 0 & 0 \\ A_{12} & A_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{11} & D_{12} & 0 \\ 0 & 0 & 0 & D_{12} & D_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \\ \kappa_1 \\ \kappa_2 \\ \kappa_6 \end{Bmatrix} \qquad (1)$$

where ε_i and κ_i (i=1, 2, 6) are the strains and the curvatures of the laminate mid-plane, respectively. The stiffness components, A_{ij}, B_{ij}, and D_{ij} are calculated from

$$A_{ij} = \int Q_{ij} dz \qquad (2\text{-}1)$$

$$B_{ij} = \int Q_{ij} z dz \qquad (2\text{-}2)$$

$$D_{ij} = \int Q_{ij} z^2 dz \qquad (2\text{-}3)$$

where

$$\begin{aligned} Q_{11} &= U_1 + U_2 \cos 2\theta + U_3 \cos 4\theta \\ Q_{22} &= U_1 - U_2 \cos 2\theta + U_3 \cos 4\theta \\ Q_{12} &= U_4 - U_3 \cos 4\theta \\ Q_{16} &= (U_2 \sin 2\theta)/2 + U_3 \sin 4\theta, \\ Q_{26} &= (U_2 \sin 2\theta)/2 - U_3 \sin 4\theta, \\ Q_{66} &= U_5 - U_3 \cos 4\theta \end{aligned} \qquad (3)$$

and

$$\begin{aligned} U_1 &= (3Q_{xx} + 3Q_{yy} + 2Q_{xy} + 4Q_{ss})/8, \\ U_2 &= (Q_{xx} - Q_{yy})/2, \\ U_3 &= (Q_{xx} + Q_{yy} - 2Q_{xy} - 4Q_{ss})/8, \\ U_4 &= (Q_{xx} + Q_{yy} + 6Q_{xy} - 4Q_{ss})/8, \\ U_5 &= (Q_{xx} + Q_{yy} - 2Q_{xy} + 4Q_{ss})/8 \end{aligned} \qquad (4)$$

The coordinates for the laminate and the ply are defined in Figure 1.

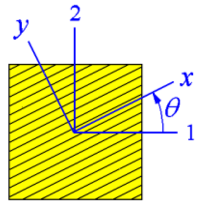

Fig. 1. Laminate and ply coordinates

For FIL's, the following relationships must be satisfied.

$$A_{11} = A_{22} \qquad (5\text{-}1)$$

$$A_{16} = A_{26} = 0 \qquad (5\text{-}2)$$

$$A_{66} = (A_{11} - A_{12})/2 \qquad (5\text{-}3)$$

$$B_{11} = B_{22} = B_{12} = 0 \qquad (5\text{-}4)$$

$$B_{16} = B_{26} = 0 \qquad (5\text{-}5)$$

$$D_{16} = D_{26} = 0 \qquad (5\text{-}6)$$

$$D_{ij} = A_{ij} H^2/12 \qquad (5\text{-}7)$$

where H is the entire laminate thickness.

As an example, let's consider the (0°, +60°, -60°)$_{m=3}$ stacking sequence shown in Table 1 in which the algebraic manipulation becomes much simpler with [(k-9/2)^3-(k-9/2-1)^3]/3-1/12 instead of [(k-9/2)^3-(k-9/2-1)^3]/3.

Table 1. (0°, +60°, -60°)$_{m=3}$ Laminate

k	1	2	3	4	5	6	7	8	9
[(k-9/2)^2-(k-9/2-1)^2]/2	-4	-3	-2	-1	0	1	2	3	4
[(k-9/2)^3-(k-9/2-1)^3]/3-1/12	16	9	4	1	0	1	4	9	16

In order to make D_{16}=D_{26}=0 and D_{ij} invariant with respect to the in-plane tensor transformation, the 9 boxes on the third row in Table 1 must be divided into three groups so that each group occupies three boxes and the numbers in each group sum to 20. Since (16+4+0) is the only combination satisfying this condition, it is obvious that no FIL solution exists for (0°, +60°, -60°)$_{m=3}$ laminates.

For (0°, +60°, -60°)$_{m=4}$ laminates shown in Table2, it is more convenient to handle [(k-6)^2-(k-6-1)^2] and [(k-12/2)^3-(k-12/2-1)^3-1]/6 instead of [(k-6)^2-(k-6-1)^2]/2 and [(k-12/2)^3-(k-12/2-1)^3]/3 respectively.

Table 2. (0°, +60°, -60°)$_{m=4}$ Laminate

k	1	2	3	4	5	6	7	8	9	10	11	12
[(k-6)^2-(k-6-1)^2]	-11	-9	-7	-5	-3	-1	1	3	5	7	9	11
[(k-6)^3-(k-6-1)^3]/6- 1/6	15	10	6	3	1	0	0	1	3	6	10	15

Since the twelve boxes on the third row in Table 2 sum to 70 which is not a multiple of 3, no FIL solution exists for (0°, +60°, -60°)$_{m=4}$

laminates. Likewise, it can be shown that FIL solutions do not exist for m=5, 7, and 8.

Therefore, the FIL solutions with up to 27 plies are possible only for $(0°, +60°, -60°)_{m=6}$ and $(0°, +60°, -60°)_{m=9}$ stacking sequences.

3 FIL's with $(0°, +60°, -60°)_{m=6}$ (18 plies)

When a laminate is constructed with (0°, +60°, -60°) layers with m plies for each fiber orientation, the total number of plies becomes $3m$. Since this type of laminates is always balanced, i. e. the number of +60° plies is equal to that of -60° plies, (5-1), (5-2) and (5-3) are satisfied trivially.

In order to satisfy (5-6) and (5-7), the summation of all numbers occupying the third row in Table3 must be a multiple of 3 so that the numbers on the third row can be split into three groups which sum to identical values.

Table 3. $(0°, +60°, -60°)_{m=6}$ with symmetric 0°'s

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$[(k-9)^2-(k-9-1)^2]$	-17	-15	-13	-11	-9	-7	-5	-3	-1	1	3	5	7	9	11	13	15	17
$[(k-9)^3-(k-9-1)^3]/6-1/6$	36	28	21	15	10	6	3	1	0	0	1	3	6	10	15	21	28	36

The LHS summation of the numbers on the second row not occupied by 0° plies must cancel out the RHS counterpart to satisfy (5-4). The LHS summation of the numbers on the second row occupied by +60° plies must cancel out the RHS counterpart occupied by +60° ones to satisfy (5-5). The same must be true for $-\theta$ plies.

Among numerous $(0°, +60°, -60°)_{m=6}$ laminates, we can select a special case in which 0° plies are arranged symmetrically so that D_{16}=D_{26}=0 and D_{ij}'s are invariant with respect to the in-plane tensor transformation of elastic constants. On the third row of Table 3, the only symmetric combination of six boxes which sum to 80 is (36+3+1+1+3+36).

The remaining 12 boxes on the third row of Table 3 must be divided into 2 groups, one with the appropriate six boxes for +60° plies and the other with the remaining six boxes for -60° plies, so that each group must sum to 80. At the same time, the second row boxes for +60° and -60° plies must be summed to 0, separately. The only possible combination for the second row satisfying these two conditions is [(+1)×(-15 -13)+(+1)×(1+7+9+11)] together with [(-1)×(-11-9-7-1)-(-1)×(13-15)] as shown in Table 4. This sequence can be written as $[0/+/+/-/-/-/00/-/+/0/0/+/+/+/-/-/0]_{total}$ and the resulting stiffness components are fully isotropic.

Table 4. $(0°, +60°, -60°)_{m=6}$ FIL #1 (Anti-Sym.)

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$[(k-9)^2-(k-9-1)^2]$	-17	-15	-13	-11	-9	-7	-5	-3	-1	1	3	5	7	9	11	13	15	17
$[(k-9)^3-(k-9-1)^3]/6-1/6$	36	28	21	15	10	6	3	1	0	0	1	3	6	10	15	21	28	36
Stack. Seq.	0	+	+	-	-	-	0	0	-	+	0	0	+	+	+	-	-	0

Now the 36 in the right hand side of the third row in Table 4 can be split into 21 and 15, together with merging two 3's into the 6 in the left hand side of the same row as shown in Table 5. The resulting distribution of 0° plies is (36, 6, 1, 1, 15, 21). Simple shuffling with the remaining 12 boxes in the third row yields an asymmetric FIL of which the sequence is given by $[0/+/-/+/-/0/-/0/+/-/0/+/+/-/0/0/+/-]_{total}$.

Table 5. $(0°, +60°, -60°)_{m=6}$ FIL #2 (Asym.)

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$[(k-9)^2-(k-9-1)^2]$	-17	-15	-13	-11	-9	-7	-5	-3	-1	1	3	5	7	9	11	13	15	17
$[(k-9)^3-(k-9-1)^3]/6-1/6$	36	28	21	15	10	6	3	1	0	0	1	3	6	10	15	21	28	36
Stack. Seq.	0	+	-	+	-	0	-	0	+	-	0	+	+	-	0	0	+	-

The 1, 15, and 21 in the right hand side of the third row in Table 5 can be replaced by the 3, 6 and 28 in the same side of the same row as shown in Table 6. Simple shuffling with the remaining 12 boxes in the third row yields $[0/+/-/+/-/0/-/0/+/-/+/0/0/-/+/ +/0/-]_{total}$ which is another asymmetric FIL.

Table 6. $(0°, +60°, -60°)_{m=6}$ FIL #3 (Asym.)

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$[(k-9)^2-(k-9-1)^2]$	-17	-15	-13	-11	-9	-7	-5	-3	-1	1	3	5	7	9	11	13	15	17
$[(k-9)^3-(k-9-1)^3]/6-1/6$	36	28	21	15	10	6	3	1	0	0	1	3	6	10	15	21	28	36
Stack. Seq.	0	+	-	+	-	0	-	0	+	-	+	0	0	-	+	+	0	-

Since 1+3+6=0+0+10, the 1 in the LHS with the 3, 6, and 28 in the right hand side of the third row in Table 6 can be replaced by two 0's and the 10 and 28 in the right hand side of the same row as shown in Tables 7 and 8. Simple shuffling with the remaining 12 boxes in the same row gives two additional FIL's, i.e., $[0/+/-/-/+/0/+/-/0/0/-/+/-/0/+/+/0/-]_{total}$ and [0/+/+/-/-

/0/-/-/0/0/+/+/+/0/-/+/0/-]$_{total}$ as shown in Tables 7 and 8, respectively.

Table 7. (0°, +60°, -60°)$_{m=6}$ FIL #4 (Asym.)

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$[(k-9)^2-(k-9-1)^2]$	-17	-15	-13	-11	-9	-7	-5	-3	-1	1	3	5	7	9	11	13	15	17
$[(k-9)^3-(k-9-1)^3]/6-1/6$	36	28	21	15	10	6	3	1	0	0	1	3	6	10	15	21	28	36
Stack. Seq.	0	+	-	-	+	0	+	-	0	0	-	+	-	0	+	+	0	-

Table 8. (0°, +60°, -60°)$_{m=6}$ FIL #5 (Asym.)

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$[(k-9)^2-(k-9-1)^2]$	-17	-15	-13	-11	-9	-7	-5	-3	-1	1	3	5	7	9	11	13	15	17
$[(k-9)^3-(k-9-1)^3]/6-1/6$	36	28	21	15	10	6	3	1	0	0	1	3	6	10	15	21	28	36
Stack. Seq.	0	+	+	-	-	0	-	-	0	0	+	+	+	0	-	+	0	-

The five independent FIL's of (0°, +60°, -60°)$_{m=6}$ shown in Tables 4~8 are identical to the result of Vannucci and Verchery [4].

4 (0°, +60°, -60°)$_{m=9}$ with Symmetric 0°'s

The present study has found that (0°, +60°, -60°)$_{m=9}$ stacking sequence has 340 independent FIL solutions among which thirty six cases are anti-symmetric. A simple and straightforward summation scheme for obtaining those thirty six anti-symmetric solutions is presented herein.

The method to obtain Table 1 for m=3 can be extended for m=9 as shown in Table 9.

Table 9. (0°, +60°, -60°)$_{m=9}$ with sym. 0°'s

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
α	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13
β	169	144	121	100	81	64	49	36	25	16	9	4	1	0	1	4	9	16	25	36	49	64	81	100	121	144	169
0	0					0		0				0		0		0				0		0					0
		0		0					0			0		0		0			0					0		0	
		0				0	0			0				0				0			0	0				0	
			0	0				0		0				0				0		0				0	0		
$\alpha=[(k-9/2)^2-(k-9/2-1)^2]/2,\ \beta=[(k-9/2)^3-(k-9/2-1)^3]/3-1/12$																											

A 0° ply is assumed to occupy the laminate mid-plane to reduce the computational effort significantly. The remaining eight 0° plies must be arranged symmetrically with respect to the laminate mid-plane to obtain anti-symmetric FIL's. Table 9 shows the four symmetric arrangements of 0° plies in which four boxes in the LHS as well as four boxes the RHS of the third row sum to 273 identically.

4.1 Type01 FIL's with (0°, +60°, -60°)$_{m=9}$

The stacking sequence on the fourth row in Table 9 has ten FIL solutions among which nine solutions are anti-symmetric. The remaining 18 boxes on the second row must be divided into two groups so that each one sums to 31+31=62 for the nine anti-symmetric solutions, Type01-1A to 9A, as shown in Table 10.

Table 10. Type01 FIL with (0°, +60°, -60°)$_{m=9}$

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
α	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13
β	169	144	121	100	81	64	49	36	25	16	9	4	1	0	1	4	9	16	25	36	49	64	81	100	121	144	169
1A	0	+	-	+	+	0	-	0	-	-	-	0	-	0	+	0	+	+	+	0	+	0	-	-	+	-	0
2A	0	+	+	-	-	0	+	0	-	-	-	0	+	0	-	0	+	+	+	0	-	0	+	+	-	-	0
3A	0	+	+	-	-	0	-	0	+	-	+	0	-	0	+	0	-	+	-	0	+	0	+	+	-	-	0
4A	0	+	-	+	-	0	-	0	+	+	-	0	-	0	+	0	+	-	-	0	+	0	+	-	+	-	0
5A	0	+	-	-	+	0	+	0	-	-	+	0	-	0	+	0	-	+	+	0	-	0	-	+	+	-	0
6A	0	-	+	+	+	0	-	0	-	-	-	0	+	0	-	0	+	+	+	0	+	0	-	-	-	+	0
7A	0	-	+	+	-	0	+	0	-	-	+	0	-	0	+	0	-	+	+	0	-	0	+	-	-	+	0
8A	0	-	+	-	+	0	+	0	-	+	-	0	-	0	+	0	+	-	+	0	-	0	-	+	-	+	0
9A	0	-	-	+	+	0	+	0	+	-	-	0	-	0	+	0	+	+	-	0	-	0	-	-	+	+	0
10	0	+	-	-	+	0	+	0	-	+	-	0	-	0	+	0	-	-	+	0	+	0	+	+	-	-	0

An asymmetric solution, Type01-10, is obtained by dividing the same 18 boxes on the second row into two groups so that each one sums to 32+30=62

4.2 Type02 FIL's with (0°, +60°, -60°)$_{m=9}$

The stacking sequence on the fifth row in Table 9 has nine anti-symmetric and three asymmetric FIL solutions. The remaining 18 boxes on the second row must be divided into two groups so that each one sums to 31+31=62 for the nine anti-symmetric solutions, Type02-1A to 9A, and an asymmetric solution, Type02-10, as shown in Table 11.

Table 11. Type02 FIL with (0°, +60°, -60°)$_{m=9}$

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
α	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13
β	169	144	121	100	81	64	49	36	25	16	9	4	1	0	1	4	9	16	25	36	49	64	81	100	121	144	169
1A	+	0	+	0	-	-	+	-	0	-	-	0	-	0	+	0	+	+	0	+	-	+	+	0	-	0	-
2A	+	0	+	0	-	-	-	+	0	-	-	0	+	0	-	0	+	+	0	-	+	+	+	0	-	0	-
3A	+	0	+	0	-	-	-	-	0	+	+	0	-	0	+	0	-	-	0	+	+	+	+	0	-	0	-
4A	+	0	-	0	+	+	-	-	0	-	-	0	+	0	-	0	+	+	0	+	+	-	-	0	+	0	-
5A	+	0	-	0	+	-	-	+	0	-	+	0	-	0	+	0	-	+	0	-	+	+	-	0	+	0	-
6A	+	0	-	0	-	+	+	-	0	-	+	0	-	0	+	0	-	+	0	+	-	-	+	0	+	0	-
7A	+	0	-	0	-	+	-	+	0	+	-	0	-	0	+	0	+	-	0	-	+	-	+	0	+	0	-
8A	-	0	+	0	+	+	-	-	0	-	+	0	-	0	+	0	-	+	0	+	+	-	-	0	-	0	+
9A	-	0	+	0	+	-	+	-	0	+	-	0	-	0	+	0	+	-	0	+	-	+	-	0	-	0	+
10	+	0	+	0	-	-	-	-	0	+	+	0	-	0	-	0	+	+	0	+	+	-	-	0	+	0	-
11	-	0	+	0	-	+	+	+	0	-	-	0	-	0	+	0	+	-	0	-	-	+	+	0	+	0	-
12	-	0	+	0	+	-	+	+	0	-	-	0	-	0	-	0	+	+	0	-	+	+	-	0	+	0	-

Two additional asymmetric solutions, Type02-11 and Type02-12, are obtained by dividing the same 18 boxes on the second row into two groups so that each one sums to 32+30=62 and 33+29=62, respectively, as shown in Table 11.

4.3 Type03 FIL's with (0°, +60°, -60°)$_{m=9}$

The stacking sequence on the sixth row in Table 9 has nine anti-symmetric and four asymmetric FIL solutions. The remaining 18 boxes on the second row must be divided into two groups so that each group sums to 30+30=60 for the nine anti-symmetric solutions, Type03-1A to 9A as shown in Table 12.

Table 12. Type03 FIL with (0°, +60°, -60°)$_{m=9}$

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
α	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13
β	169	144	121	100	81	64	49	36	25	16	9	4	1	0	1	4	9	16	25	36	49	64	81	100	121	144	169
1A	+	0	+	-	-	0	0	+	-	0	-	-	-	0	+	+	+	0	+	-	0	0	+	+	-	0	-
2A		0	+	+	+	0	0	-	-	0	-	-	-	0	+	+	+	0	+	+	0	0	-	-	-	0	+
3A	+	0	+	-	-	0	0	-	+	0	-	-	+	0	-	+	+	0	-	+	0	0	+	+	-	0	-
4A	+	0	-	+	-	0	0	+	-	0	-	-	+	0	-	+	+	0	+	-	0	0	+	-	+	0	-
5A	+	0	-	+	-	0	0	-	+	0	-	+	-	0	+	-	+	0	-	+	0	0	+	-	+	0	-
6A	+	0	-	-	+	0	0	+	-	0	-	+	-	0	+	-	+	0	+	-	0	0	-	+	+	0	-
7A	+	0	-	-	+	0	0	-	+	0	+	-	-	0	+	+	-	0	-	+	0	0	-	+	+	0	-
8A	-	0	+	+	-	0	0	+	-	0	+	-	-	0	+	+	-	0	+	-	0	0	+	-	-	0	+
9A	-	0	-	+	+	0	0	+	+	0	-	-	-	0	+	+	+	0	-	-	0	0	-	-	+	0	+
10	-	0	+	+	+	0	0	-	-	0	-	-	+	0	+	-	-	0	+	+	0	0	+	+	-	0	-
11	+	0	+	-	-	0	0	+	-	0	-	+	-	0	+	-	+	0	+	-	0	0	+	+	-	0	-
12	+	0	+	-	-	0	0	-	+	0	+	-	-	0	+	+	-	0	-	+	0	0	+	+	-	0	-
13	-	0	+	+	-	0	0	+	+	0	-	-	-	0	+	-	-	0	+	+	0	0	+	-	+	0	-

An asymmetric solution, Type03-10, is obtained by dividing the same 18 boxes on the second row into two groups so that each group sums to 31+29=60. Three asymmetric solutions, Type03-11 to Type03-13, are additionally obtained by dividing the same 18 boxes on the second row into two groups so that each group sums to 33+28=60 as shown in Table12.

4.4 Type04 FIL's with (0°, +60°, -60°)$_{m=9}$

The stacking sequence on the seventh row in Table 9 has nine anti-symmetric and two asymmetric FIL solutions as shown in Table 13.

Table 13. Type04 FIL with (0°, +60°, -60°)$_{m=9}$

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
α	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13
β	169	144	121	100	81	64	49	36	25	16	9	4	1	0	1	4	9	16	25	36	49	64	81	100	121	144	169
1A	+	+	0	0	-	-	-	0	+	0	-	-	-	0	+	+	+	0	-	0	+	+	+	0	0	-	-
2A	+	-	0	0	+	+	-	0	-	0	-	-	-	0	+	+	+	0	+	0	+	-	-	0	0	+	-
3A	+	+	0	0	-	-	-	0	-	0	+	+	-	0	+	-	-	0	+	0	+	+	+	0	0	-	-
4A	+	-	0	0	+	-	+	0	-	0	-	-	+	0	-	+	+	0	+	0	-	+	-	0	0	+	-
5A	+	-	0	0	+	-	-	0	+	0	+	-	-	0	+	+	-	0	-	0	+	+	-	0	0	+	-
6A	+	-	0	0	-	+	+	0	-	0	-	+	-	0	+	-	+	0	+	0	-	-	+	0	0	+	-
7A	-	+	0	0	+	+	-	0	-	0	-	-	+	0	-	+	+	0	+	0	+	-	-	0	0	-	+
8A	-	+	0	0	+	-	+	0	-	0	-	+	-	0	+	-	+	0	+	0	-	+	-	0	0	-	+
9A	-	+	0	0	-	+	+	0	-	0	+	-	-	0	+	+	-	0	+	0	-	-	+	0	0	-	+
10	-	+	0	0	+	-	+	0	-	0	+	-	-	0	-	+	+	0	+	0	-	-	+	0	0	+	-
11	+	-	0	0	+	+	-	0	-	0	-	+	-	0	-	-	+	0	+	0	+	+	+	0	0	-	-

The remaining boxes on the second row must be divided into two groups so that each group sums to 30+30=60 for the nine anti-symmetric solutions, Type04-1A to 9A as shown in Table 13.

In addition, two asymmetric solutions, Type04-10 and 11 are obtained by dividing the same 18 boxes on the second row into two groups so that each group sums to 31+29=60 and 32+28, respectively.

5 (0°, +60°, -60°)$_{m=9}$ with Asymmetric 0°'s

As before, a 0° ply is placed on the laminate mid-plane and four boxes in the LHS and four boxes in the RHS on the third row of Table 9 are assumed to be occupied by 0° plies. However, the symmetric arrangement of the remaining eight 0° plies need not be kept for obtaining asymmetric FIL's classified as Type05 to Type51. And yet the numbers on the third row in Table 3 occupied by 0° plies must sum to 546.

5.1 Type05~06 FIL's with (0°, +60°, -60°)$_{m=9}$

For these types, four boxes in the LHS and four boxes in the RHS are occupied asymmetrically by 0° plies, and each group sum to 273 as shown in Table 14 and 15.

Table 14. Type05 FIL with (0°, +60°, -60°)$_{m=9}$

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
α	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13
β	169	144	121	100	81	64	49	36	25	16	9	4	1	0	1	4	9	16	25	36	49	64	81	100	121	144	169
1	0	+	+	-	-	0	+	0	-	-	-	0	+	0	+	0	-	-	0	+	+	+	+	0	-	0	-
2	0	+	+	-	-	0	+	0	-	-	-	0	+	0	-	0	+	+	0	+	+	-	-	0	+	0	-
3	0	+	-	+	-	0	-	0	+	+	-	0	-	0	+	0	+	-	0	-	+	-	+	0	+	0	-
4	0	-	+	-	+	0	+	0	-	+	-	0	-	0	+	0	+	-	0	+	-	+	-	0	-	0	+
5	0	+	+	+	-	0	-	0	-	-	-	0	-	0	+	0	+	+	0	+	-	+	-	0	+	0	-

Table 15. Type06 FIL with (0°, +60°, -60°)$_{m=9}$

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
α	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13
β	169	144	121	100	81	64	49	36	25	16	9	4	1	0	1	4	9	16	25	36	49	64	81	100	121	144	169
1	+	-	0	0	+	-	-	0	+	0	+	-	-	0	+	+	-	0	-	+	0	0	-	+	+	0	-
2	-	+	0	0	+	+	-	0	-	0	-	-	+	0	+	-	+	0	+	-	0	0	-	+	+	0	-
3	-	+	0	0	-	+	+	0	-	0	+	-	-	0	+	+	-	0	+	-	0	0	+	-	-	0	-
4	-	+	0	0	-	+	+	0	+	0	-	-	-	0	+	-	-	0	+	+	0	0	+	-	+	0	-
5	+	-	0	0	+	-	+	0	-	0	-	+	-	0	+	-	-	0	+	+	0	0	+	+	-	0	-

5.2 Type07~10 FIL's with (0°, +60°, -60°)$_{m=9}$

For these types, four boxes in the LHS and four boxes in the RHS are occupied by 0° plies on the third row of Table 9 and sum to 263+283 or 243+303. The entire boxes occupied by 0° plies sum to 546 as do the previous types. Four types exist for the combination of numbers as listed in Table 16.

Table 16. Type07~10 FIL with (0°, +60°, -60°)$_{m=9}$

# of Type	0° plies		No. of FIL Solutions
	LHS + Center + RHS	Summation	
07	$11^2+9^2+6^2+5^2+0^2+3^2+7^2+9^2+12^2$	263+0+283=546	2
08	$13^2+7^2+6^2+3^2+0^2+1^2+7^2+8^2+13^2$	263+0+283=546	1
09	$11^2+9^2+5^2+4^2+0^2+2^2+3^2+11^2+13^2$	243+0+303=546	3
10	$11^2+8^2+7^2+3^2+0^2+2^2+3^2+11^2+13^2$	243+0+303=546	6

5.3 Type11~14 FIL's with (0°, +60°, -60°)$_{m=9}$

For these types, four boxes in the LHS and four boxes in the RHS are occupied by 0° plies on the third row of Table 9 and sum to 264+282 or 284+262. And yet the entire boxes occupied by 0° plies sum to 546. Four types exist for the combination of numbers as listed in Table 17.

Table 17. Type11~14 FIL with (0°, +60°, -60°)$_{m=9}$

# of Type	0° plies		No. of FIL Solutions
	LHS + Center + RHS	Summation	
11	$12^2+10^2+4^2+2^2+0^2+2^2+3^2+10^2+13^2$	264+0+282=546	9
12	$12^2+10^2+4^2+2^2+0^2+1^2+4^2+11^2+12^2$	264+0+282=546	6
13	$12^2+10^2+6^2+2^2+0^2+4^2+5^2+10^2+11^2$	284+0+262=546	10
14	$13^2+9^2+5^2+3^2+0^2+4^2+5^2+10^2+11^2$	284+0+262=546	4

5.4 Type15~26 FIL's with (0°, +60°, -60°)$_{m=9}$

For these types, four boxes in the LHS and four boxes in the RHS are occupied by 0° plies on the third row of Table 9 and sum to 255+291, 265+281, 275+271, 285+261, or 295+251. The entire boxes occupied by 0° plies sum to 546 as do previous cases. Twelve types of FIL solutions are listed in Table 18.

Table 18. Type15~26 FIL with (0°, +60°, -60°)$_{m=9}$

# of Type	0° plies		No. of FIL Solutions
	LHS + Center + RHS	Summation	
15	$11^2+10^2+5^2+3^2+0^2+1^2+5^2+11^2+12^2$	255+0+291=546	2
16	$11^2+9^2+7^2+2^2+0^2+1^2+5^2+11^2+12^2$	255+0+291=546	3
17	$10^2+9^2+7^2+5^2+0^2+4^2+5^2+9^2+13^2$	255+0+291=546	5
18	$10^2+9^2+7^2+5^2+0^2+3^2+7^2+8^2+13^2$	255+0+291=546	5
19	$12^2+9^2+6^2+2^2+0^2+1^2+6^2+10^2+12^2$	265+0+281=546	2
20	$12^2+9^2+7^2+1^2+0^2+1^2+7^2+10^2+11^2$	275+0+271=546	4
21	$13^2+9^2+4^2+3^2+0^2+1^2+7^2+10^2+11^2$	275+0+271=546	4
22	$12^2+10^2+5^2+4^2+0^2+4^2+8^2+9^2+10^2$	285+0+261=546	6
23	$13^2+8^2+6^2+4^2+0^2+4^2+8^2+9^2+10^2$	285+0+261=546	4
24	$12^2+11^2+4^2+2^2+0^2+2^2+7^2+8^2+12^2$	285+0+261=546	5
25	$12^2+11^2+4^2+2^2+0^2+2^2+6^2+10^2+11^2$	285+0+261=546	5
26	$13^2+11^2+2^2+1^2+0^{+}+1^2+5^2+9^2+12^2$	295+0+251=546	1

5.5 Type27~41 FIL's with (0°, +60°, -60°)$_{m=9}$

For these types, four boxes in the LHS and four boxes in the RHS are occupied by 0° plies on the third row of Table 9 and sum to 246+300, 276+270, 286+260, or 316+230. The entire boxes occupied by 0° plies sum to 546 as do previous cases. Fifteen types exist for the combination of numbers as listed in Table 19.

Table 19. Type27~41 FIL with (0°, +60°, -60°)$_{m=9}$

# of Type	0° plies		No. of FIL Solutions
	LHS + Center + RHS	Summation	
27	$10^2+9^2+8^2+1^2+0^2+1^2+3^2+11^2+13^2$	246+0+300=546	2
28	$11^2+10^2+4^2+3^2+0^2+1^2+3^2+11^2+13^2$	246+0+300=546	5
29	$13^2+6^2+5^2+4^2+0^2+1^2+3^2+11^2+13^2$	246+0+300=546	4
30	$10^2+9^2+7^2+4^2+0^2+1^2+7^2+9^2+13^2$	246+0+300=546	5
31	$11^2+8^2+6^2+5^2+0^2+1^2+7^2+9^2+13^2$	246+0+300=546	3
32	$13^2+9^2+5^2+1^2+0^2+1^2+5^2+10^2+12^2$	276+0+270=546	4
33	$13^2+9^2+5^2+1^2+0^2+2^2+4^2+9^2+13^2$	276+0+270=546	5
34	$13^2+9^2+5^2+1^2+0^2+1^2+6^2+8^2+13^2$	276+0+270=546	4
35	$11^2+9^2+7^2+5^2+0^2+6^2+7^2+8^2+11^2$	276+0+270=546	4
36	$11^2+9^2+7^2+5^2+0^2+5^2+8^2+9^2+10^2$	276+0+270=546	2
37	$11^2+10^2+8^2+1^2+0^2+3^2+7^2+9^2+11^2$	286+0+260=546	1
38	$13^2+8^2+7^2+2^2+0^{+}+3^2+7^2+9^2+11^2$	286+0+260=546	5
39	$11^2+10^2+8^2+1^2+0^2+4^2+6^2+8^2+12^2$	286+0+260=546	7
40	$13^2+8^2+7^2+2^2+0^2+4^2+6^2+8^2+12^2$	286+0+260=546	13
41	$13^2+11^2+5^2+1^2+0^{+}+6^2+7^2+8^2+9^2$	316+0+230=546	3

5.6 Type42~51 FIL's with (0°, +60°, -60°)$_{m=9}$

For these types, four boxes in the LHS and four boxes in the RHS are occupied by 0° plies on the third row of Table 9 and sum to 267+279, 277+269, 287+259, 297+249, or 307+239. The entire boxes occupied by 0° plies sum to 546 as do previous cases. Ten types exist for the combination of numbers as listed in Table 20.

Table 20. Type42~51 FIL with (0°, +60°, -60°)$_{m=9}$

# of Type	0° plies		No. of FIL Solutions
	LHS + Center + RHS	Summation	
42	$11^2+9^2+8^2+1^2+0^2+2^2+5^2+9^2+13^2$	267+0+279=546	4
43	$13^2+8^2+5^2+3^2+0^2+2^2+5^2+9^2+13^2$	267+0+279=546	5
44	$13^2+9^2+4^2+1^2+0^2+1^2+3^2+10^2+13^2$	267+0+279=546	5
45	$11^2+9^2+7^2+4^2+0^2+3^2+7^2+10^2+11^2$	267+0+279=546	0
46	$11^2+9^2+7^2+4^2+0^2+5^2+6^2+7^2+13^2$	267+0+279=546	5
47	$12^2+9^2+6^2+4^2+0^2+5^2+6^2+8^2+12^2$	277+0+269=546	2
48	$13^2+9^2+6^2+1^2+0^2+3^2+5^2+9^2+12^2$	287+0+259=546	1
49	$13^2+9^2+6^2+1^2+0^2+4^2+5^2+7^2+13^2$	287+0+259=546	2
50	$12^2+10^2+7^2+2^2+0^2+6^2+7^2+8^2+10^2$	297+0+249=546	6
51	$13^2+11^2+4^2+1^2+0^2+3^2+7^2+9^2+10^2$	307+0+239=546	2

6 Type52~71 FIL's with (0°, +60°, -60°)$_{m=9}$

For these types, three boxes in the LHS and five boxes in the RHS are occupied by 0° plies on the third row of Table 9 and sum to 302+244, 282+264, 293+253, 314+232, 315+231, 326+220, 317+229, 308+238, 299+247, 339+207, or 329+217. The entire boxes occupied by 0° plies sum to 546 as do previous cases. Twenty types for the combination of numbers are listed in Table 21.

Table 21. Type52~71 FIL with (0°, +60°, -60°)$_{m=9}$

# of Type	0° plies		No. of FIL Solutions
	LHS + Center + RHS	Summation	
52	$11^2+10^2+9^2+0^2+1^2+3^2+7^2+8^2+11^2$	302+0+244=546	3
53	$11^2+10^2+9^2+0^2+1^2+4^2+5^2+9^2+11^2$	302+0+244=546	6
54	$13^2+8^2+7^2+0^2+1^2+2^2+3^2+9^2+13^2$	282+0+264=546	4
55	$12^2+10^2+7^2+0^2+1^2+3^2+5^2+7^2+13^2$	293+0+253=546	4
56	$13^2+12^2+1^2+0^2+1^2+2^2+3^2+7^2+13^2$	314+0+232=546	5
57	$13^2+9^2+8^2+0^2+1^2+5^2+6^2+7^2+11^2$	314+0+232=546	3
58	$12^2+11^2+7^2+0^2+1^2+5^2+6^2+7^2+11^2$	314+0+232=546	4
59	$13^2+11^2+5^2+0^2+2^2+3^2+5^2+7^2+12^2$	315+0+231=546	5
60	$13^2+11^2+5^2+0^2+1^2+3^2+6^2+8^2+11^2$	315+0+231=546	4
61	$13^2+11^2+5^2+0^2+2^2+3^2+4^2+9^2+11^2$	315+0+231=546	6
62	$13^2+11^2+6^2+0^2+2^2+4^2+6^2+8^2+10^2$	326+0+220=546	8
63	$13^2+11^2+6^2+0^2+1^2+5^2+7^2+8^2+9^2$	326+0+220=546	3
63	$13^2+12^2+2^2+0^2+1^2+2^2+4^2+8^2+12^2$	317+0+229=546	6
65	$12^2+10^2+8^2+0^2+1^2+4^2+6^2+8^2+11^2$	308+0+238=546	6
66	$12^2+10^2+8^2+0^2+2^2+4^2+5^2+7^2+12^2$	308+0+238=546	7
67	$13^2+9^2+7^2+0^2+1^2+3^2+4^2+10^2+11^2$	299+0+247=546	3
68	$13^2+9^2+7^2+0^2+2^2+3^2+4^2+7^2+13^2$	299+0+247=546	5
69	$13^2+9^2+7^2+0^2+1^2+4^2+5^2+6^2+13^2$	299+0+247=546	5
70	$13^2+11^2+7^2+0^2+4^2+5^2+6^2+7^2+9^2$	339+0+207=546	5
71	$13^2+12^2+4^2+0^2+1^2+4^2+6^2+8^2+10^2$	329+0+217=546	6

7 Concluding Remarks

The five independent FIL's of (0°, +60°, -60°)$_{m=6}$ obtained by Vannucci and Verchery [4] are confirmed by the present study.

In contrast to the 219 independent FIL's of (0°, +60°, -60°)$_{m=9}$ stacking sequences obtained by Vannucci and Verchery [4], 340 FIL solutions are obtained by the present study. The authors are still checking whether the present result is the complete set of the fully isotropic solutions of (0°, +60°, -60°)$_{m=9}$ stacking sequences.

The present method will be extended for obtaining FIL's made of (0°, +45°, -45°, 90°) layers up to 24 plies and (0°, +72°, -72°, +144°, -144°) layers up to 30 plies.

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References

[1] Caprino G, and Crivelli Visconti I. A Note on Specially Orthotropic Laminates. *Journal of Composite Materials*, Vol. 16, pp 395~399. Sept. 1982.

[2] Gunnink J W. Comments on "A Note on Specially Orthotropic Laminates". *Journal of Composite Materials*, Vol. 17, pp 508~510, Nov. 1983.

[3] Lee J, Kim J and Choi Y. Analysis of Specially Anti-Symmetric Stacking Sequences with Pascal′s Triangle and Pythagorean Theorem. *2002 Spring Conference, The Korean Society for Aeronautical and Space Science*, 2002.

[4] Vannucci P, Verchery G. A New Method for Generating Fully Isotropic Laminates. *Composite Structures*, Vol. 58, pp 75~82, 2002.

[5] Valot E, Vannucci P. “Some Exact Solutions for Fully Orthotropic Laminates. *Composite Structures*, Vol. 69, pp 157~166, 2005.

[6] Lee J, Kim J, and Kim H. Analytical Laminate Stacking Sequences for Perfect Orthotropy. 6th Asian-Pacific Conference on Aerospace Technology and Science (APCATS 2009), , Huangshan, China, Nov. 2009.

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