THERMO-MECHANICAL COUPLING IN MULTILAYERED PLATES AND SHELLS

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Abstract

This paper proposes a fully-coupled thermo-mechanical analysis of multilayered plates and shells. In the proposed refined plate/shell models, the temperature is considered as a primary variable of the problem as the displacement and it is directly obtained from the governing equations. Such models are very promising for multilayered structures because they permit both equivalent single layer and layer wise approaches and they have the order of expansion in the thickness direction as a free parameter (from linear to fourth order). Three different problems can be analyzed: evaluation of temperature field effects in the free vibration analysis of multilayered plates and shells; evaluation of temperature field effects in the stress analysis of multilayered structures subjected to mechanical loads; thermal stress analysis of multilayered structures with imposed sovra-temperature.

1 Introduction

Several thermal environments are applied to aerospace structures, typical examples are high temperatures, high gradients and cyclic temperature changes [1]. The well-known mechanical loads and the described thermal variations are the most important causes of failure mechanisms in aerospace structures [2], [3]. For these reasons, the effects of both high-temperature and mechanical loads must be included in the structural models for the analysis of multilayered plates and shells [4].

The thermo-mechanical models proposed in this work are defined as fully-coupled because the temperature field is considered a primary variable of the problem as the displacement [5], [6], in this way the effects of the thermal field can be evaluated in the static and dynamic analysis of multilayered plates and shells [7], and such models can also be applied to the thermal stress analysis of aerospace structures [8] without the necessity of a priori defining the temperature profile (by assuming it linear in the thickness direction or by solving the Fourier heat conduction equation).

In the open literature, a small amount of work has been devoted to the coupled thermo-mechanical analysis of structures (both thermoelastic and thermoplastic analysis), and only few of them give numerical results. Some interesting works about this topic are Yang et al. [9], Altay and Dokmeci [10], [11], Cannarozzi and Ubertini [12]. A first tentative to evaluate the thermo-mechanical coupling in plates and shells has been made by the authors in [7]: the case of an isotropic plate with an applied mechanical load has been considered, as just suggested in Nowinski book [1] the coupling is about 0.5% (both static and dynamic cases). An exhaustive discussion about the variational statements which permit the coupling between different physical fields (mechanical, thermal, electric and magnetic) has been given by the authors in [13], then these variational statements have been refined and applied to the thermo-mechanical cases in [5] and [6]: details about the formulation and further comments about the results (not given in the present work for sake of brevity) can be found in these works.

The static and dynamic analysis have been accomplished using several higher-order two-dimensional theories, obtained in the framework of Carrera's Unified Formulation (CUF) [14]. In the case of multilayered structures, these models can be equivalent single layer or layer wise, and the order of expansion in the thickness direction is taken as a free parameter (N=1 to N=4). In the proposed multilayered structures, the use of layer wise kinematics results mandatory in order to recover the typical zigzag form of displacements and temperature through the thickness direction. The governing equations are obtained by extending the Principle of Virtual Displacements (PVD) to the thermo-mechanical coupling by simply adding the internal thermal work: consistent constitutive equations must be considered in this case. The governing equations are solved in closed form using Navier's solution.

2 Geometrical Relations

The aerospace structures considered in this work are the well-known plate and shell geometries. They are defined as two-dimensional structures because one dimension (in general the thickness) is negligible with respect to the other two in the in-plane directions. Such structures are considered as multilayered made, see Fig. 1.

The geometrical relations for shells, in the case of thermo-mechanical problems, link the mechanical strains with the displacement vector (first two lines in Eq. 1) and the spatial gradient of temperature with the scalar temperature (second two lines in Eq. 1). The relations split in in-plane (p) and out-of-plane (n) components are:

$$\begin{aligned}
\boldsymbol{\epsilon}_{pG}^k &= [\epsilon_{\alpha\alpha}, \epsilon_{\beta\beta}, \gamma_{\alpha\beta}]^{kT} = (\boldsymbol{D}_p^k + \boldsymbol{A}_p^k)\ \boldsymbol{u}^k\ ,\\
\boldsymbol{\epsilon}_{nG}^k &= [\gamma_{\alpha z}, \gamma_{\beta z}, \epsilon_{zz}]^{kT} = (\boldsymbol{D}_{np}^k + \boldsymbol{D}_{nz}^k - \boldsymbol{A}_n^k)\ \boldsymbol{u}^k\ ,\\
\boldsymbol{\vartheta}_{pG}^k &= [\vartheta_\alpha, \vartheta_\beta]^{kT} = -\boldsymbol{D}_{tp}^k\ \theta^k\ ,\\
\boldsymbol{\vartheta}_{nG}^k &= [\vartheta_z]^k = -\boldsymbol{D}_{tn}^k\ \theta^k\ ,
\end{aligned} \qquad (1)$$

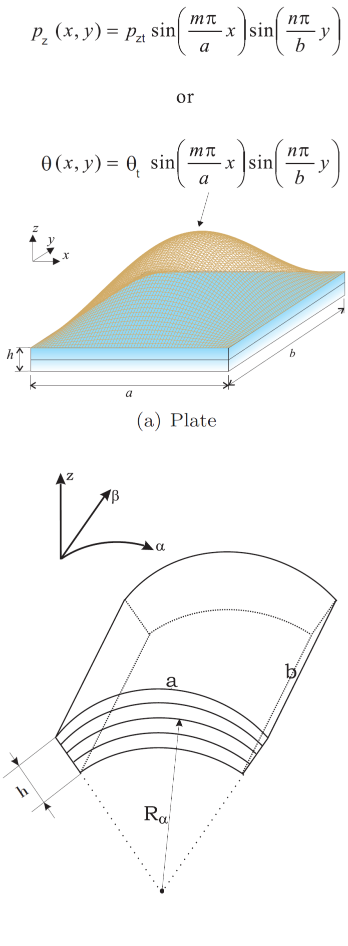
the displacement components are **u**=(u,v,w) for each layer k, and the sovra-temperature θ is a scalar for each layer k. The meaning of the involved matrices is:

$$\boldsymbol{A}_p^k = \begin{bmatrix} 0 & 0 & \frac{1}{H_\alpha^k R_\alpha^k} \\ 0 & 0 & \frac{1}{H_\beta^k R_\beta^k} \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{A}_n^k = \begin{bmatrix} \frac{1}{H_\alpha^k R_\alpha^k} & 0 & 0 \\ 0 & \frac{1}{H_\beta^k R_\beta^k} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (2)$$

$$\boldsymbol{D}_p^k = \begin{bmatrix} \frac{\partial_{\alpha_k}}{H_\alpha^k} & 0 & 0 \\ 0 & \frac{\partial_{\beta_k}}{H_\beta^k} & 0 \\ \frac{\partial_{\beta_k}}{H_\beta^k} & \frac{\partial_{\alpha_k}}{H_\alpha^k} & 0 \end{bmatrix}, \quad \boldsymbol{D}_{np}^k = \begin{bmatrix} 0 & 0 & \frac{\partial_{\alpha_k}}{H_\alpha^k} \\ 0 & 0 & \frac{\partial_{\beta_k}}{H_\beta^k} \\ 0 & 0 & 0 \end{bmatrix}, \qquad (3)$$

$$\boldsymbol{D}_{nz}^k = \begin{bmatrix} \partial_{z_k} & 0 & 0 \\ 0 & \partial_{z_k} & 0 \\ 0 & 0 & \partial_{z_k} \end{bmatrix}, \quad \boldsymbol{D}_{tp}^k = \begin{bmatrix} \frac{\partial_{\alpha_k}}{H_\alpha^k} \\ \frac{\partial_{\beta_k}}{H_\beta^k} \end{bmatrix}, \quad \boldsymbol{D}_{tn}^k = \begin{bmatrix} \partial_{z_k} \end{bmatrix},$$

matrices **A** containes the information for the shell geometry (radii of curvature Rα and Rβ, and parametric coefficients Hα and Hβ). Matrices **D** containes the differential operators.



(a) Plate

(b) Cylidrical shell

Figure 1. Geometry and notations for multilayered plates (a) and shells (b).

Geometrical relations for shells in Eq. 1 degenerate in those for plates when the radii of curvature are infinite: so matrices in Eq. 2 are zero and parametric coefficients Hα and Hβ in matrices of Eq. 3 equal one. Details about shell and plate geometries can be found in [15].

3 Carrera's Unified Formulation

Carrera's Unified Formulation (CUF) [14] permits to obtain, in a unified manner, a large variety of plate/shell theories. According to CUF, the governing equations are written in terms of a few fundamental nuclei which do not formally depend on the order of expansion N used in the thickness direction and on the description of variables (equivalent single layer (ESL) or layer wise (LW)). The application of a two-dimensional method for plates/shells permits the unknown variables to be expressed as a set of thickness functions that only depend on the thickness coordinate z and the correspondent variable which depends on the in-plane coordinates α and β. The generic variable **f**(α,β,z), for instance a displacement, and its variation δ**f**(α,β,z) are written therefore according to the following general expansion:

$$\boldsymbol{f}(\alpha,\beta,z) = F_\tau(z)\boldsymbol{f}_\tau(\alpha,\beta)\,, \quad \delta\boldsymbol{f}(\alpha,\beta,z) = F_s(z)\delta\boldsymbol{f}_s(\alpha,\beta)\,, \tag{4}$$

with τ,s=1,…,N, and bold letters denote arrays. (α,β) are the in-plane curvilinear coordinates and z the thickness one. In the case of plate geometry the curvilinear coordinates (α,β,z) are replaced with the rectilinear ones (x, y, z). The summing convention, with repeated indexes τ and s, is assumed. The order of expansion N goes from first to fourth-order, and depending on the used thickness functions, a theory can be: ESL, when the variable is assumed for the whole multilayer and a Taylor expansion is employed as the thickness functions F(z); LW, when the variable is considered independent in each layer and a combination of Legendre polynomials are used as the thickness functions F(z). In the thermo-mechanical models, proposed in this work, displacements can be modelled in both ESL or LW form, temperature is always considered in LW form. A two-dimensional thermo-mechanical model is defined therefore as ESL or LW, depending on the choice made for the displacement vector.

3.1 Equivalent Single Layer Approach

The displacement **u**=(u,v,w) is described according to equivalent single layer (ESL) description if the unknowns are the same for the whole multilayered plate/shell [14]. The z expansion is obtained via Taylor polynomials, that is:

$$\begin{aligned} u &= F_0\, u_0 + F_1\, u_1 + \ldots + F_N\, u_N = F_\tau\, u_\tau\,, \\ v &= F_0\, v_0 + F_1\, v_1 + \ldots + F_N\, v_N = F_\tau\, v_\tau\,, \\ w &= F_0\, w_0 + F_1\, w_1 + \ldots + F_N\, w_N = F_\tau\, w_\tau \end{aligned} \tag{5}$$

with τ=0,1,…,N; N is the order of expansion that ranges from 1 (linear) to 4:

$$F_0 = z^0 = 1,\ \ F_1 = z^1 = z,\ \ \ldots,\ \ F_N = z^N\,. \tag{6}$$

Eq. 5 can be written in a vectorial form:

$$\boldsymbol{u}(\alpha,\beta,z) = F_\tau(z)\boldsymbol{u}_\tau(\alpha,\beta)\,, \quad \delta\boldsymbol{u}(\alpha,\beta,z) = F_s(z)\delta\boldsymbol{u}_s(\alpha,\beta)\,, \tag{7}$$

with τ,s=0,1,…,N.

Simpler theories, such those which discard the ε_{zz} effect, can be obtained from refined ESL models: it is sufficient to impose that the transverse displacement w is constant in z. First order Shear Deformation Theory (FSDT) is obtained from an ESL model with linear expansion in the thickness direction z, by imposing a constant transverse displacement w in z. Classical Lamination Theory (CLT) is obtained from FSDT via an opportune penalty technique which imposes an infinite transverse shear rigidity. All the ESL theories, with constant or linear transverse displacement w, which means zero or constant transverse normal strain ε_{zz}, show Poisson's locking phenomena: it can be overcame via plane stress conditions in constitutive equations [16].

3.2 Layer Wise Approach

When each layer of a multilayered plate/shell is described as independent plates/shells [14], a layer wise (LW) approach is accounted for. The displacement **u**=(u,v,w) is described for each layer k, in this way the zigzag form of displacement, in multilayered transverse-anisotropy structures, is easily obtained. The z expansion for displacement components is made for each layer k:

$$\begin{aligned} u^k &= F_0\, u_0^k + F_1\, u_1^k + \ldots + F_N\, u_N^k = F_\tau\, u_\tau^k\,, \\ v^k &= F_0\, v_0^k + F_1\, v_1^k + \ldots + F_N\, v_N^k = F_\tau\, v_\tau^k\,, \\ w^k &= F_0\, w_0^k + F_1\, w_1^k + \ldots + F_N\, w_N^k = F_\tau\, w_\tau^k \end{aligned} \tag{8}$$

with τ,s=0,1,…,N, N is the order of expansion that ranges from 1 (linear) to 4. k=1,….,N_l, where N_l indicates the number of layers. Eq. 8, written in a vectorial form, is:

$$\boldsymbol{u}^k(\alpha,\beta,z) = F_\tau(z)\boldsymbol{u}^k_\tau(\alpha,\beta) \,, \quad \delta\boldsymbol{u}^k(\alpha,\beta,z) = F_s(z)\delta\boldsymbol{u}^k_s(\alpha,\beta) \,, \tag{9}$$

with τ,s=t,b,r and k=1,….,N_l. t and b indicate the top and bottom of each layer k, respectively; r indicates the higher order of expansion in the thickness direction: r=2,…,N. In this case the thickness functions F_τ and F_s are a combination of Legendre polynomials (for details see [14]) in order to easily obtain the compatibility conditions for displacements at each layer interface. In LW models, even if a linear expansion in *z* is considered for the transverse displacement *w*, Poisson's locking phenomena does not appear: the transverse normal strain ε_{zz} is piece-wise constant in the thickness direction [15].

In the case of thermo-mechanical problems, the primary variables are the displacement vector **u**=(u,v,w) and the scalar sovra-temperature θ (temperature T_1 referred to the external room temperature T_0, $\theta = T_1 - T_0$). By considering the higher spatial gradient of the temperature field, the variable θ in each layer k is always modeled as LW:

$$\theta^k(\alpha,\beta,z) = F_\tau(z)\theta^k_\tau(\alpha,\beta) \,, \quad \delta\theta^k(\alpha,\beta,z) = F_s(z)\delta\theta^k_s(\alpha,\beta) \,, \tag{10}$$

with τ,s=t,b,r and k=1,….,N_l. The thickness functions are a combination of Legendre polynomials as in Eq. 9. The sovra-temperature θ can be considered as an external load [8] or as a primary variable [5], [6] . A two-dimensional model for thermo-mechanical problems is defined as ESL or LW depending on the choice made for the displacement vector: the temperature is always considered in LW form.

4 Constitutive Equations

Constitutive equations, for the thermo-mechanical problem, are obtained in according to that reported in [5], [6] and [13]. The coupling between the mechanical and thermal fields can be determined by using thermodynamical principles and Maxwell's relations [9]-[13]. For this aim, it is necessary to define a *Gibbs free-energy function G* and a *thermomechanical enthalpy density H*:

$$G(\epsilon_{ij},\theta) = \sigma_{ij}\epsilon_{ij} - \eta\theta \,, \quad H(\epsilon_{ij},\theta,\vartheta_i) = G(\epsilon_{ij},\theta) - F(\vartheta_i) \,, \tag{10}$$

where σ_{ij} and ε_{ij} are the stress and strain components. η is the variation of entropy per unit of volume, and θ the sovra-temperature considered with respect to the reference temperature T_0. The function F is the dissipation function, it depends by the spatial temperature gradient:

$$F(\vartheta_i) = \frac{1}{2}\kappa_{ij}\vartheta_i\vartheta_j - \tau_0\dot{h}_i \,, \tag{11}$$

where κ_{ij} is the symmetric, positive semidefinite conductivity tensor. In the second term, τ_0 is a thermal relaxation parameter which multiplies the temporal derivative of the heat flux h_i. The thermal relaxation parameter is omitted in the present work.

The thermomechanical enthalpy density H can be expanded in order to obtain a quadratic form for a linear interaction:

$$H(\epsilon_{ij},\theta,\vartheta_i) = \frac{1}{2}Q_{ijkl}\epsilon_{ij}\epsilon_{kl} - \lambda_{ij}\epsilon_{ij}\theta - \frac{1}{2}\chi\theta^2 - \frac{1}{2}\kappa_{ij}\vartheta_i\vartheta_j \,, \tag{12}$$

where Q_{ijkl} is the elastic coefficients tensor considered for an orthotropic material in the problem reference system. λ_{ij} are the thermo-mechanical coupling coefficients, $\chi=\rho/C_vT_0$ where ρ is the material density, C_v is the specific heat per unit mass and T_0 is the reference temperature.

The constitutive equations are obtained by considering the following relations:

$$\sigma_{ij} = \frac{\partial H}{\partial \epsilon_{ij}} \,, \quad \eta = -\frac{\partial H}{\partial \theta} \,, \quad h_i = -\frac{\partial H}{\partial \vartheta_i} \,. \tag{13}$$

By considering Eqs. 12 and 13, the constitutive equations for the thermo-mechanical problem are obtained:

$$\sigma_{ij} = Q_{ijkl}\epsilon_{kl} - \lambda_{ij}\theta \,, \quad \eta = \lambda_{ij}\epsilon_{ij} + \chi\theta \,, \quad h_i = \kappa_{ij}\vartheta_j \,. \tag{14}$$

The split stress and strain components vectors are:

$$\boldsymbol{\sigma}^k_{pC} = \begin{Bmatrix} \sigma_{\alpha\alpha} \\ \sigma_{\beta\beta} \\ \sigma_{\alpha\beta} \end{Bmatrix}^k \,, \quad \boldsymbol{\sigma}^k_{nC} = \begin{Bmatrix} \sigma_{\alpha z} \\ \sigma_{\beta z} \\ \sigma_{zz} \end{Bmatrix}^k \,,$$

$$\boldsymbol{\epsilon}_{pG}^k = \left\{ \begin{array}{c} \epsilon_{\alpha\alpha} \\ \epsilon_{\beta\beta} \\ \gamma_{\alpha\beta} \end{array} \right\}^k , \quad \boldsymbol{\epsilon}_{nG}^k = \left\{ \begin{array}{c} \gamma_{\alpha z} \\ \gamma_{\beta z} \\ \epsilon_{zz} \end{array} \right\}^k . \quad (15)$$

The vectors (3x1) of heat flux and spatial gradient of the temperature, split in in-plane and out-of-plane components, are:

$$\boldsymbol{h}_{pC}^k = \left\{ \begin{array}{c} h_\alpha \\ h_\beta \end{array} \right\}^k , \quad \boldsymbol{h}_{nC}^k = \left\{ h_z \right\}^k , \quad (16)$$

$$\boldsymbol{\vartheta}_{pG}^k = \left\{ \begin{array}{c} \vartheta_\alpha \\ \vartheta_\beta \end{array} \right\}^k , \quad \boldsymbol{\vartheta}_{nG}^k = \left\{ \vartheta_z \right\}^k .$$

By considering Eqs. 14, their split equations in vectorial form are:

$$\begin{aligned} \boldsymbol{\sigma}_{pC}^k &= \boldsymbol{Q}_{pp}^k \boldsymbol{\epsilon}_{pG}^k + \boldsymbol{Q}_{pn}^k \boldsymbol{\epsilon}_{nG}^k - \boldsymbol{\lambda}_p^k \theta^k , \\ \boldsymbol{\sigma}_{nC}^k &= \boldsymbol{Q}_{np}^k \boldsymbol{\epsilon}_{pG}^k + \boldsymbol{Q}_{nn}^k \boldsymbol{\epsilon}_{nG}^k - \boldsymbol{\lambda}_n^k \theta^k , \\ \eta_C^k &= \boldsymbol{\lambda}_p^{kT} \boldsymbol{\epsilon}_{pG}^k + \boldsymbol{\lambda}_n^{kT} \boldsymbol{\epsilon}_{nG}^k + \chi^k \theta^k , \\ \boldsymbol{h}_p^k &= \boldsymbol{\kappa}_{pp}^k \boldsymbol{\vartheta}_{pG}^k + \boldsymbol{\kappa}_{pn}^k \boldsymbol{\vartheta}_{nG}^k , \\ \boldsymbol{h}_n^k &= \boldsymbol{\kappa}_{np}^k \boldsymbol{\vartheta}_{pG}^k + \boldsymbol{\kappa}_{nn}^k \boldsymbol{\vartheta}_{nG}^k . \end{aligned} \quad (17)$$

The explicit forms of the split matricies in Eqs. 17 are:

Elastic coefficients matrices:

$$\boldsymbol{Q}_{pp}^k = \left[\begin{array}{ccc} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{array} \right]^k , \quad \boldsymbol{Q}_{pn}^k = \left[\begin{array}{ccc} 0 & 0 & Q_{13} \\ 0 & 0 & Q_{23} \\ 0 & 0 & Q_{36} \end{array} \right]^k ,$$

$$\boldsymbol{Q}_{np}^k = \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ Q_{13} & Q_{23} & Q_{36} \end{array} \right]^k , \quad \boldsymbol{Q}_{nn}^k = \left[\begin{array}{ccc} Q_{55} & Q_{45} & 0 \\ Q_{45} & Q_{44} & 0 \\ 0 & 0 & Q_{33} \end{array} \right]^k . \quad (18)$$

Thermo-mechanical coupling coefficients:

$$\boldsymbol{\lambda}_p^k = \left[\begin{array}{c} \lambda_1 \\ \lambda_2 \\ \lambda_6 \end{array} \right]^k , \quad \boldsymbol{\lambda}_n^k = \left[\begin{array}{c} 0 \\ 0 \\ \lambda_3 \end{array} \right]^k . \quad (19)$$

Conductivity coefficients:

$$\boldsymbol{\kappa}_{pp}^k = \left[\begin{array}{cc} \kappa_{11} & \kappa_{12} \\ \kappa_{12} & \kappa_{22} \end{array} \right]^k , \quad \boldsymbol{\kappa}_{pn}^k = \left[\begin{array}{c} 0 \\ 0 \end{array} \right]^k ,$$

$$\boldsymbol{\kappa}_{np}^k = \left[\begin{array}{cc} 0 & 0 \end{array} \right]^k , \quad \boldsymbol{\kappa}_{nn}^k = \left[\kappa_{33} \right]^k . \quad (20)$$

In order to use the relations given in Eqs. 17 in the proposed variational statements, that will be presented in the next section, it is convenient to split them in in-plane components (subscript p) and out-of-plane components (subscript n). Other two new subscripts are introduced: the subscript C for those variables, in the variational statements, which need the substitution of constitutive equations; the subscript G for those variables, in constitutive equations, which need the substitution of geometrical relations. Constitutive equations discussed for shell geometry, can be also used for plate configurations simply replacing the curvilinear coordinates (α,β,z) with the rectilinear ones (x,y,z).

5 Considered Variational Statements

In this section two different extensions of the Principle of Virtual Displacements (PVD) are given: the first is for the partially coupled thermo-mechanical analysis, the second one is for the fully coupled thermo-mechanical problem. In the case of a partially coupled analysis the PVD is the same of the mechanical case, but the stresses are considered as an algebraic addition of the pure mechanical and pure thermal parts [8]. For the fully coupled analysis, the virtual internal thermal work is added to the virtual internal mechanical one [5], [6].

5.1 PVD for Partially Coupled Thermo-Mechanical Case

In the case of the thermal stress analysis of plates and shells, a possible extension of the PVD considers the temperature as an external load without any coupling between the mechanical and thermal fields [8]. In the variational statement obtained in Eq. 22 the stresses are seen as an algebraic addition of mechanical (d) and thermal (t) contributions:

$$\begin{aligned} \boldsymbol{\sigma}_{pC}^k &= \boldsymbol{\sigma}_{pd}^k - \boldsymbol{\sigma}_{pt}^k = \boldsymbol{Q}_{pp}^k \boldsymbol{\epsilon}_{pG}^k + \boldsymbol{Q}_{pn}^k \boldsymbol{\epsilon}_{nG}^k - \boldsymbol{\lambda}_p^k \theta^k , \\ \boldsymbol{\sigma}_{nC}^k &= \boldsymbol{\sigma}_{nd}^k - \boldsymbol{\sigma}_{nt}^k = \boldsymbol{Q}_{np}^k \boldsymbol{\epsilon}_{pG}^k + \boldsymbol{Q}_{nn}^k \boldsymbol{\epsilon}_{nG}^k - \boldsymbol{\lambda}_n^k \theta^k , \end{aligned} \quad (21)$$

where the arrays λ_p and λ_n permit the partial coupling between the mechanical field and the temperature.

By considering a laminate of N_l layers, and the integral on the volume V_k of each layer k as an integral on the in plane domain Ω_k plus the integral in the thickness-direction domain A_k, it is possible to write:

$$\sum_{k=1}^{N_l}\int_{\Omega_k}\int_{A_k}\left\{\delta\boldsymbol{\epsilon}_{pG}^{k\ T}(\boldsymbol{\sigma}_{pd}^k-\boldsymbol{\sigma}_{pt}^k)+\delta\boldsymbol{\epsilon}_{nG}^{k\ T}(\boldsymbol{\sigma}_{nd}^k-\boldsymbol{\sigma}_{nt}^k)\right\}$$

$$d\Omega_k dz=\sum_{k=1}^{N_l}\delta L_e^k-\sum_{k=1}^{N_l}\delta L_{in}^k\ , \quad (22)$$

where δL_e and δL_{in} are the external and inertial virtual works at the k-layer level, respectively. The governing equations are:

$$\delta\boldsymbol{u}_s^k:\ \boldsymbol{K}_{uu}^{k\tau s}\,\boldsymbol{u}_\tau^k\ =\ -\boldsymbol{M}_{uu}^{k\tau s}\,\ddot{\boldsymbol{u}}_\tau^k\ -\ \boldsymbol{K}_{u\theta}^{k\tau s}\,\theta_\tau^k\ +\ \boldsymbol{p}_{us}^k\ , \quad (23)$$

with related boundary conditions on the layer edge Γ_k:

$$\boldsymbol{\Pi}_{uu}^{k\tau s}\,\boldsymbol{u}_\tau^k\ -\ \boldsymbol{\Pi}_{u\theta}^{k\tau s}\,\theta_\tau^k\ =\ \boldsymbol{\Pi}_{uu}^{k\tau s}\,\bar{\boldsymbol{u}}_\tau^k\ -\ \boldsymbol{\Pi}_{u\theta}^{k\tau s}\,\bar{\theta}_\tau^k\ , \quad (24)$$

the three terms at the right of the equal in Eq. 23 are the inertial load, the thermal load and the mechanical load, respectively. From Eqs. 23 and 24, simply discarding the thermal contribution, it is possible to obtain the governing equations and the boundary conditions for the pure mechanical case: the variational statement in this case is obtained from Eq. 22 simply discarding the thermal contribution for the stresses. In order to define the thermal load, the temperature profile must be a priori given: by linearly assuming it in the thickness direction (θ_a) or by calculating it with solving the Fourier heat conduction equation (θ_c).

5.2 PVD for Fully Coupled Thermo-Mechanical Case

In case of fully coupling between the thermal and mechanical fields, the variational statement is the PVD with the introduction of the virtual internal thermal work. This variational statement is:

$$\sum_{k=1}^{N_l}\int_{\Omega_k}\int_{A_k}\left\{\delta\boldsymbol{\epsilon}_{pG}^{k\ T}\boldsymbol{\sigma}_{pC}^k+\delta\boldsymbol{\epsilon}_{nG}^{k\ T}\boldsymbol{\sigma}_{nC}^k-\delta\theta^k\eta_C^k-\delta\boldsymbol{\vartheta}_{pG}^{k\ T}\boldsymbol{h}_{pC}^k-\right.$$

$$\left.\delta\boldsymbol{\vartheta}_{nG}^{k\ T}\boldsymbol{h}_{nC}^k\right\}d\Omega_k dz\ =\sum_{k=1}^{N_l}\delta L_e^k-\sum_{k=1}^{N_l}\delta L_{in}^k\ , \quad (25)$$

where δL_e and δL_{in} are the external and inertial virtual works at the k-layer level, respectively.

The governing equations have the following form:

$$\delta\boldsymbol{u}_s^k:\quad \boldsymbol{K}_{uu}^{k\tau s}\ \boldsymbol{u}_\tau^k+\boldsymbol{K}_{u\theta}^{k\tau s}\ \theta_\tau^k=\boldsymbol{p}_{us}^k-\boldsymbol{M}_{uu}^{k\tau s}\ \ddot{\boldsymbol{u}}_\tau^k$$

$$\delta\theta_s^k:\quad \boldsymbol{K}_{\theta u}^{k\tau s}\ \boldsymbol{u}_\tau^k+\boldsymbol{K}_{\theta\theta}^{k\tau s}\ \theta_\tau^k=\boldsymbol{p}_{\theta s}^k\ . \quad (26)$$

The arrays $\mathbf{p}_{us}$ and $\mathbf{p}_{\theta s}$ indicate the variationally consistent mechanical and thermal loadings, respectively. Along with these governing equations the following boundary conditions on the edge Γ_k of the in-plane integration domain Ω_k hold:

$$\boldsymbol{\Pi}_{uu}^{k\tau s}\ \boldsymbol{u}_\tau^k+\boldsymbol{\Pi}_{u\theta}^{k\tau s}\ \theta_\tau^k=\ \boldsymbol{\Pi}_{uu}^{k\tau s}\ \bar{\boldsymbol{u}}_\tau^k+\boldsymbol{\Pi}_{u\theta}^{k\tau s}\ \bar{\theta}_\tau^k$$

$$\boldsymbol{\Pi}_{\theta u}^{k\tau s}\ \boldsymbol{u}_\tau^k+\boldsymbol{\Pi}_{\theta\theta}^{k\tau s}\ \theta_\tau^k=\boldsymbol{\Pi}_{\theta u}^{k\tau s}\ \bar{\boldsymbol{u}}_\tau^k+\boldsymbol{\Pi}_{\theta\theta}^{k\tau s}\ \bar{\theta}_\tau^k\ . \quad (27)$$

As indicated in [7] , the sovra-temperature θ is a variable of the problem. The displacements **u** can be seen in ESL or LW form. Independently by the choice made for the displacements, the sovra-temperature is always seen in LW form.

As discussed in Altay and Dokmeci [10], [11] and Cannarozzi and Ubertini [12], the variational statement includes only the internal thermal work made by the gradient of temperature in the case of applied temperature at the top and bottom of the structure; it includes only the internal thermal work made by the temperature in the case of applied mechanical load on the structure or free vibration problem.

In the case of temperature imposed at the top and bottom of the structure, in the Eq. 25 does not exist a virtual variation of temperature, and the variational statement is:

$$\sum_{k=1}^{N_l}\int_{\Omega_k}\int_{A_k}\left\{\delta\boldsymbol{\epsilon}_{pG}^{k\ T}\boldsymbol{\sigma}_{pC}^k+\delta\boldsymbol{\epsilon}_{nG}^{k\ T}\boldsymbol{\sigma}_{nC}^k-\delta\boldsymbol{\vartheta}_{pG}^{k\ T}\boldsymbol{h}_{pC}^k-\right.$$

$$\left.\delta\boldsymbol{\vartheta}_{nG}^{k\ T}\boldsymbol{h}_{nC}^k\right\}d\Omega_k dz=\sum_{k=1}^{N_l}\delta L_e^k\ . \quad (28)$$

In the case of mechanical load applied on the structure or the free vibration analysis, in the Eq. 25 does not exist a gradient of temperature variation, and the variational statement is:

$$\sum_{k=1}^{N_l}\int_{\Omega_k}\int_{A_k}\left\{\delta\boldsymbol{\epsilon}_{pG}^{k\ T}\boldsymbol{\sigma}_{pC}^k+\delta\boldsymbol{\epsilon}_{nG}^{k\ T}\boldsymbol{\sigma}_{nC}^k-\delta\theta^k\eta_C^k\right\}$$

$$d\Omega_k dz=\sum_{k=1}^{N_l}\delta L_e^k-\sum_{k=1}^{N_l}\delta L_{in}^k\ . \quad (29)$$

5.3 Navier Solution

Navier-type closed form solution is obtained via substitution of harmonic expressions for the displacements and temperature as well as considering the following material coeffcients equal zero: $Q_{16} = Q_{26} = Q_{36} = Q_{45} = 0$ and $\lambda_6 =$

κ_{12} = 0. The following harmonic assumptions can be made for the variables, which correspond to simply supported boundary conditions:

$$u_\tau^k = \sum_{m,n} (\hat{U}_\tau^k) \cos\left(\frac{m\pi\alpha_k}{a_k}\right) \sin\left(\frac{n\pi\beta_k}{b_k}\right) ,$$
$$v_\tau^k = \sum_{m,n} (\hat{V}_\tau^k) \sin\left(\frac{m\pi\alpha_k}{a_k}\right) \cos\left(\frac{n\pi\beta_k}{b_k}\right) , \quad (30)$$
$$(w_\tau^k, \theta_\tau^k) = \sum_{m,n} (\hat{W}_\tau^k, \hat{\theta}_\tau^k) \sin\left(\frac{m\pi\alpha_k}{a_k}\right) \sin\left(\frac{n\pi\beta_k}{b_k}\right) ,$$

with k=1,….,N_l and τ,s=t,b,r (r=2,…,N). The amplitudes are indicated with the symbol ^.

By starting from the fundamental nuclei described in this section, matrices can be obtained for the considered multilayered plates/shells by simply expanding and assembling via the indexes k,τ,s. By expanding via indexes τ,s the order of expansion N from 1 to 4 in the thickness direction is considered. The matrices are obtained for each considered layer, and the index k permits the multilayer assembling procedure, which can either be ESL or LW.

5.4 Acronyms

A system of acronyms is here given in order to define the several refined two-dimensional models developed for plates and shells. The choice made in this paper is that displacements can be in ESL or LW form, but the temperature is always considered in LW form. Therefore, a two-dimensional model is defined as ESL or LW, depending on the choice made for the displacement. ESL models are indicated as ED1-ED4, where E means the ESL approach, D means that the Principle of Virtual Displacements or their extensions to thermo-mechanical analysis have been employed; the last digit, from 1 to 4, indicates the order of expansion in the thickness direction for both displacements and temperature. In the case of LW models, the letter E is replaced by a letter L, therefore the relative models are indicated as LD1-LD4. In the case of a thermo-mechanical analysis, additional parenthesis are introduced in the acronyms: (θa) is added in the case of partially coupled thermo-mechanical analysis with a linear assumed temperature profile; (θc) is used to indicate the case of partially coupled thermo-mechanical analysis with a calculated temperature profile; (TM) means a fully coupled thermo(T)-mechanical(M) analysis. No parenthesis are added in the case of a pure mechanical problem.

6 Results and Discussion

The results discussed in this section consider three main topics: - evaluation of the thermo-mechanical coupling in the case of free vibration problem; - evaluation of the thermo-mechanical coupling for a static analysis with an applied mechanical load; - static analysis of structures subjected to an imposed temperature at the external surfaces. Further results and comments will be given at the conference.

The free vibration problem considers a square simply supported plate made of two isotropic layers with the same thickness ($h_1=h_2=h_{tot}/2$). The bottom layer is in Al2024 (Young's modulus E = 73GPa, Poisson's ratio ν= 0.3 and mass density ρ = 2800 Kg/m^3. The thermal properties are the specific heat per unit mass Cv = 897 J/KgK and the thermal expansion coefficient α = 25x10^{-6} 1/K) and the top layer is in Ti22 (E = 110GPa, ν = 0.32 and ρ = 4420 Kg/m^3. Its thermal properties are Cv = 560 J/KgK and α = 8.6x10^{-6} 1/K). By imposing the wave numbers m and n in the x and y directions, respectively, the considered two-dimensional theory gives a number of frequencies equal the number of degrees of freedom through the thickness direction. Tab. 1 considers the fundamental frequency f in Hz for m=n=1 for thickness ratios a/h=5, 10, 50 and 100. For each thickness ratio the difference between the pure mechanical frequency and the thermo-mechanical one is less than 0.3%, this means that such an effect can be discarded in the free vibration analysis. The plate is multilayered, so advanced LW models are requested in order to identify the correct values of the frequency and the correct evaluation of the thermo-mechanical coupling. In Fig. 2, the global thermo-mechanical coupling is evaluated for all the frequencies f_i (not only the fundamental one), and for several imposed waves number (m=n) different plate configurations are considered (thick and thin plates, and one-layered and two-layered plates). The global thermo-mechanical

a/h	5	10	50	100
LD4	167.89	45.716	1.8881	0.4725
LD4(TM)	168.31(0.250%)	45.847(0.286%)	1.8938(0.302%)	0.4739(0.296%)
ED4	168.04	45.729	1.8881	0.4725
ED4(TM)	168.46(0.250%)	45.860(0.286%)	1.8938(0.302%)	0.4739(0.296%)
FSDT	169.87	45.881	1.8884	0.4725
FSDT(TM)	170.76(0.524%)	46.149(0.584%)	1.8999(0.609%)	0.4754(0.614%)
CLT	183.36	46.899	1.8902	0.4726
CLT(TM)	184.48(0.611%)	47.185(0.610%)	1.9017(0.608%)	0.4755(0.614%)

Table 1. Free vibrations of the two-layered isotropic plate: fundamental frequency f in Hz for m=n=1. The difference in percentage is put in brackets.

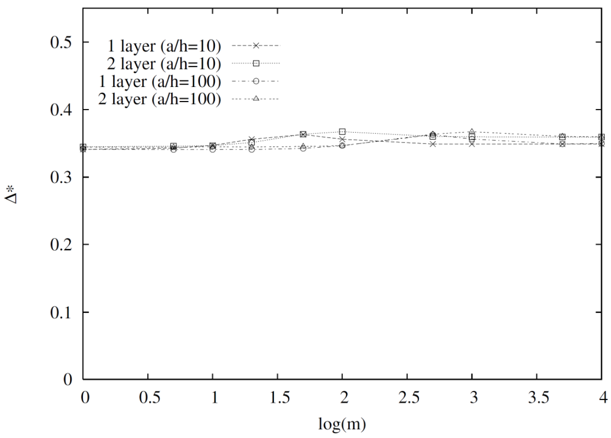


Figure 2. Global energetic thermo-mechanical coupling in one-layered and two-layered isotropic plates for different values of wave number (from m=n=1 to m=n=10000) and thickness ratio (a/h=10 and a/h=100). $\Delta^* = \sqrt{\sum_i \left(\frac{f_{TMi}^2 - f_i^2}{f_i^2}\right)}$ calculated by using LD4 and LD4(TM) theories.

coupling does not depend on the thickness ratio, wave number, lamination sequence and investigated frequency.

The second assessment considers the same lamination sequence of the case 1, but the geometry is a cylindrical shell panel with radius of curvature in β direction $R_\beta=\infty$ and radius in α direction R_α=10m. The in-plane dimensions are $a=\pi/3R_\alpha$ and b=1m, the considered thickness ratio is R_α/h=50 with applied mechanical load at the top in z direction p_z=-200000Pa. In Fig. 3 the differences in terms of transverse displacement w are given through the thickness direction when the thermo-mechanical coupling is considered. If a bending problem is investigated, a temperature profile θ is generated with an increasing of temperature for the compressed part of the shell and a decreasing of temperature for its enlarged part. For this static case the thermo-mechanical coupling is very small (less than 0.5%) and it can be discarded as in the free vibration problem. In these two assessments the use of fully coupled thermo-mechanical models is not mandatory and the use of a refined pure mechanical plate/shell model is enough.

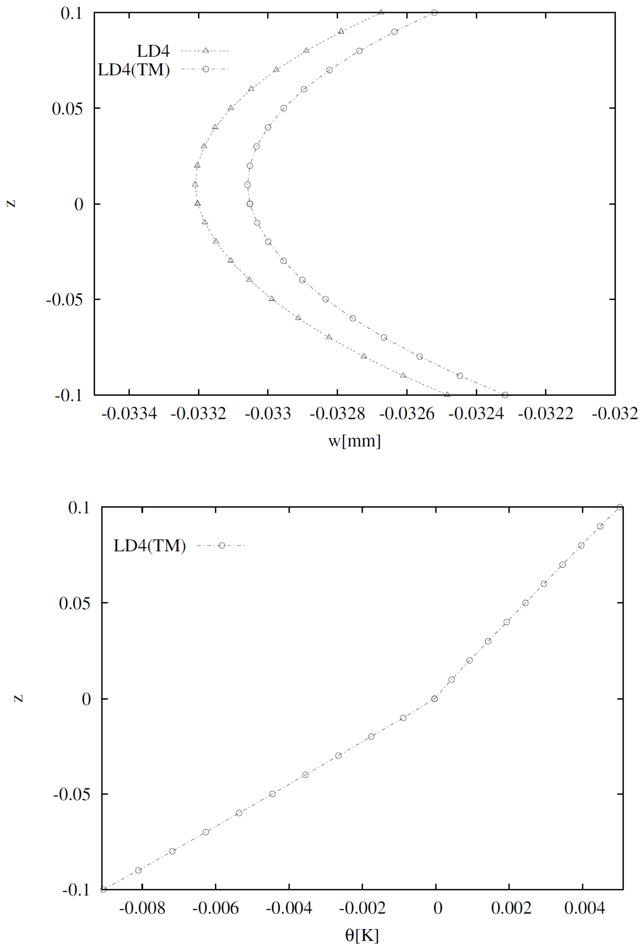


Figure 3. Two-layered isotropic shell with applied mechanical load in the case of R_α/h=50. Transverse displacement through the thickness (top figure) and sovra-temperature through the thickness (bottom figure).

The third assessment considers the same shell of the second case, but with a sovra-temperature θ_t=1.0K at the top and θ_b=0.0K at the bottom. The static case in terms of transverse displacement w is given in Fig. 4, the results can be provided by assuming a linear temperature profile (θ_a), by calculating it via Fourier heat conduction equation (θ_c) or by using a fully coupled thermo-mechanical (TM) model where the temperature profile is a primary variable of the problem. The shell has two layers with different elastic and thermal properties, so the temperature profile is never linear even if the shell is thin (R_α/h=50): the assumed temperature profile gives erroneous results, while (θ_c) and (TM) models gives correct and coincident results. It is evident that TM models are more efficient because in them we do not need to solve the Fourier heat conduction equation.

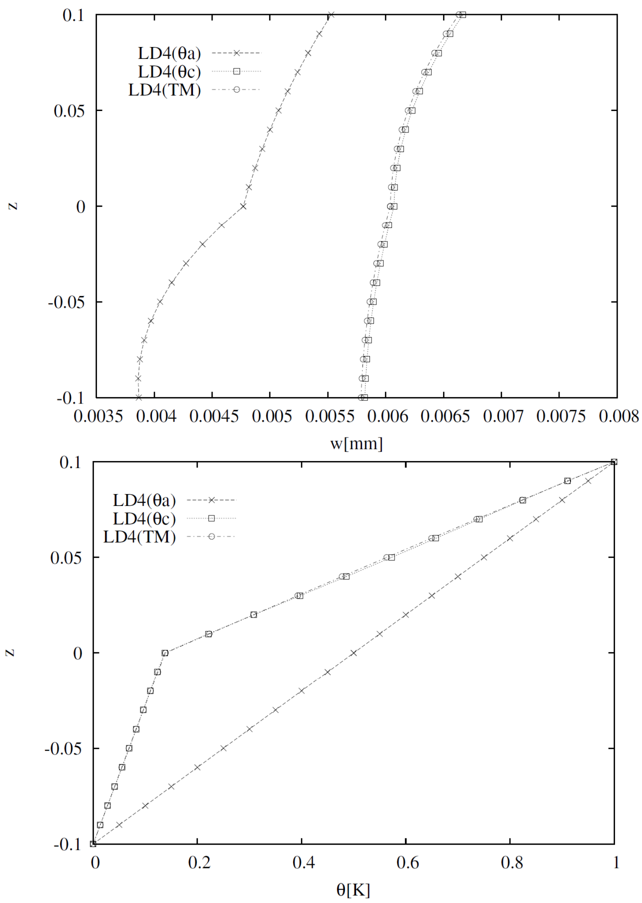


Figure 4. Two-layered isotropic shell with imposed sovra-temperature values in the case of R_α/h=50. Transverse displacement through the thickness (top figure) and sovra-temperature through the thickness (bottom figure).

7 Conclusions

A fully coupled thermo-mechanical analysis has been proposed for one-layered and multilayered plates and shells. Both displacements and temperature are considered as primary variables of the problem, and they can be directly obtained from the solution of the governing equations. These features lead to some advantages:

- the effect of the thermo-mechanical coupling has been evaluated for free vibration analysis; the fully coupled thermo-mechanical analysis permits the frequency values and the vibration modes to be evaluated in terms of the displacement and temperature. The effect of thermo-mechanical coupling has been evaluated for the dynamic case through comparisons with pure mechanical analysis. The coupling effect is very small and it can therefore be discarded in a free vibration analysis;
- in the case of an applied mechanical load to the structure, the fully coupled thermo-mechanical analysis permits the displacement and the temperature generated by the strains to be evaluated. The effect of the thermo-mechanical coupling has been evaluated through comparisons with pure mechanical analysis. The coupling effect is very small and it can therefore be discarded in such an analysis;
- in the case of an applied temperature to the external surfaces of the shell, the fully coupled analysis permits such values to be easily imposed in the governing equations, and the relative displacements and temperature profile are directly obtained from the solutions of such equations. The advantages, with respect to a partially coupled thermo-mechanical analysis, have been clearly indicated. In this latter case, in fact, the temperature profile must be a priori defined (assuming it linear in the thickness direction or calculating it by solving the Fourier heat conduction equation) to determine the thermal load.

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