OPTIMAL CONTROL OF CRUISE FLIGHT AT CONSTANT ALTITUDE

Antonio Franco , Alfonso Valenzuela , Damián Rivas
Department of Aerospace Engineering, Escuela Superior de Ingenieros, Universidad de Sevilla, Seville, 41092, Spain

Keywords: *Optimal Control, Cruise Flight, Flight Mechanics*

Abstract

Cruise optimization at constant altitude is formulated as a singular optimal control problem. The cases of maximum range, minimum direct operating cost and minimum fuel with fixed arrival time are analyzed, providing a comprehensive analysis of optimum cruise at constant altitude. The case of unsteady cruise with variable aircraft mass has been considered. In all cases the singular controls and the optimal speed laws that lead to optimum cruise are obtained, and the corresponding optimal trajectories are analyzed.

1 Introduction

Trajectory optimization is a subject of great importance in air traffic management (ATM), from the operational point of view, that aims at defining optimal flight procedures that lead to energy-efficient flights. In practice, the airlines consider a cost index (*CI*) and define the direct operating cost (DOC) as the combined cost of fuel consumed and flight time, weighted by the cost index. Their goal is to minimize the DOC. A particular case is the problem of minimizing fuel consumption (case *CI*=0); and another related problem is that of maximizing range for a given fuel load. Another important problem in ATM is the design of aircraft trajectories that meet certain arrival time constraints at given waypoints, for instance at the top of descent (TOD), at the initial approach fix (IAF), or at the runway threshold (estimated time of arrival, ETA). These are called four-dimensional (4D) trajectories, which are a key element in the trajectory-based-operations (TBO) concept proposed by NextGen and SESAR for the future ATM system.

In this work we review the problem of cruise optimization at constant altitude, formulated as an optimal control problem, in which the objective is to optimize a given performance index. The following cases are analyzed: maximum range, minimum direct operating cost and minimum fuel with fixed arrival time. In cruise at constant altitude (and constant heading) the only control variable left is thrust, which appears linearly in the equations of motion, as well as on the performance indices to be optimized; as a consequence, the Hamiltonian of the problem is also linear on the control variable, which leads to a *singular optimal control problem* (see Bryson and Ho [1]). Detailed analyses can be found in Rivas and Valenzuela [2], Franco et al. [3] and Franco and Rivas [4].

Trajectory optimization has been studied by different authors. Pargett and Ardema [5] analyze the problem of range maximization in cruise at constant altitude, considering incompressible aerodynamics. The same problem is also analyzed using different approaches by Miele [6] and Torenbeek [7], who consider the case of quasi-steady flight.

Barman and Erzberger [8], Erzberger and Lee [9] and Burrows [10] analyze the minimum-DOC problem for global trajectories (climb, cruise and descent); they consider steady cruise, and take the aircraft mass as constant. Burrows [11] also analyzes the minimum-DOC problem for global trajectories, without the assumption of constant mass, but with the assumption that the cruise segment takes place in the stratosphere. Bilimoria et al. [12] and Chakravarty [13] analyze the minimum-DOC, steady cruise as the outer solution of a singular perturbation approach, where the aircraft mass is taken as constant.

The particular case of minimum-fuel cruise has been considered by others. For example, Schultz and Zagalsky [14], Speyer [15], Schultz [16], Speyer [17], and Menon [18] analyze the optimality of the steady-state cruise, taking the aircraft mass as constant. Fuel-optimal trajectories with fixed arrival times are studied by Sorensen and Waters [19], Burrows [10] and Chakravarty [13], who analyze the 4D minimum-fuel problem as a minimum-DOC problem with free final time, that is, the problem is to find the time cost for which the corresponding free-final-time DOC-optimal trajectory arrives at the assigned time.

Unlike in the works cited above, we consider the unsteady problem, with variable aircraft mass, with compressible aerodynamics, and without any restriction on cruise altitude. The objective is to obtain the optimal controls and the optimal speed laws (speed as a function of aircraft mass) that lead to optimum cruise, and to analyze the corresponding optimal trajectories. Optimum values of the performance indices (maximum range, minimum fuel, minimum cost) are calculated. Results are presented for a model of a Boeing 767-300ER, with realistic aerodynamic and propulsive aircraft models.

2 Equations of motion for cruise at constant altitude

The equations of motion for cruise at constant altitude and constant heading are the following [6]:

$$\begin{aligned} \dot{V} &= \frac{1}{m}(T-D) \\ \dot{m} &= -cT \\ \dot{x} &= V \end{aligned} \tag{1}$$

In these equations, the drag is a general known function $D(V,m)$, which takes into account the remaining equation of motion $L = mg$. The thrust $T(V)$ is given by $T = \pi T_M(V)$, where π models the throttle setting, $0 < \pi \leq 1$, and $T_M(V)$ is a known function. The specific fuel consumption, $c(V)$, is also a known function. Thus, in this problem there are three states, speed (V), aircraft mass (m) and distance (x), and one control (π).

The initial values of speed, aircraft mass and distance (V_i, m_i, x_i), and the final value of speed (V_f) are given. The final values of aircraft mass, distance and time (m_f, x_f, t_f) can be specified or unspecified, depending on the problem under consideration.

The aircraft model considered in this paper for the numerical applications is that of a Boeing 767-300ER (a typical twin-engine, wide-body, long-range transport aircraft) which is described in Ref. [2]. The given cruise altitude is denoted as h_A.

3 Singular Optimal Control

Cruise optimization is formulated as an optimal control problem. The objective is to optimize a given performance index of the form

$$J = \int_0^{t_f} f(V,m,x,\pi)\,\mathrm{d}t \tag{2}$$

subject to the equations of motion (1) as constraints. In this work, the following problems are analyzed: maximum range, minimum direct operating cost, and minimum fuel with fixed arrival time.

In cruise at constant altitude (and constant heading) the control variable π appears linearly in the equations of motion (1), and in all cases considered in this paper the performance indices to be optimized are linear on π as well. As a consequence, the Hamiltonian of the problem is also linear on the control variable, which leads to a singular optimal control problem [1].

Let H be the Hamiltonian of the problem, and λ_V, λ_m and λ_x be the adjoint variables, which are defined by the following equations:

$$\begin{aligned}\dot{\lambda}_V &= -\frac{\partial H}{\partial V}\\ \dot{\lambda}_m &= -\frac{\partial H}{\partial m}\\ \dot{\lambda}_x &= -\frac{\partial H}{\partial x}\end{aligned} \tag{3}$$

These equations are neccesary conditions for optimality.

One also has the following transversality conditions (which are neccesary conditions for optimality as well): first, if the final value of a state variable is not specified, then the corresponding adjoint variable satisfies $\lambda(t_f) = 0$; and, second, if the final time is not specified, then the Hamiltonian satisfies $H(t_f) = 0$.

Moreover, in all cases considered in this paper the Hamiltonian is not an explicit function of time, hence one has the 1st integral $H = constant$ on the optimal path. In those cases in which $H(t_f) = 0$, one has $H = 0$ along the optimal path. Although called optimal paths, they are in fact extremal paths, that is, paths that satisfy the neccesary conditions for optimality.

Since H is linear in the control variable one can write $H = \overline{H} + S\pi$. The function S is called the *switching* function. According to the Minimum Principle of optimal control theory (see Ref. [1]), the optimal control is determined by the condition that the Hamiltonian be a minimum. Hence, since the control variable π is bounded, minimization of H with respect to π establishes that the optimal control is defined by

$$\begin{aligned}\pi &= \pi_{max} \quad \text{if} \quad S < 0\\ \pi &= \pi_{min} \quad \text{if} \quad S > 0\\ \pi &= \pi_{sing} \quad \text{if} \quad S = 0 \text{ over a finite time interval}\end{aligned} \tag{4}$$

where π_{sing} ($\pi_{min} < \pi_{sing} < \pi_{max}$) is the singular control (yet to be determined). Trajectory segments defined by π_{sing} are called singular arcs. The analysis of these singular arcs is the subject of this paper.

In all cases considered in this paper the function $\dot{S}$ does not depend on π, and the function $\ddot{S}$ is linear on π. Note that over the finite time interval where one has $S = 0$ one also has $\dot{S} = 0$ and $\ddot{S} = 0$. The singular control π_{sing} is defined by the condition $\ddot{S} = 0$, and the singular arc is defined by the three equations (see Ref. [1]):

$$\begin{aligned}\overline{H} &= constant\\ S &= 0\\ \dot{S} &= 0\end{aligned} \tag{5}$$

Another necessary condition (the generalized Legendre-Clebsch condition, see Ref. [20]) is that for the singular control to be optimal one must have $-\dfrac{\partial \ddot{S}}{\partial \pi} \geq 0$. This condition is always satisfied in the cases considered in this paper, which can be shown numerically.

In this work, it is assumed that the initial and final cruise points belong to the singular arc. In cases on which they are arbitrarily given, the optimal path is formed, in general, by three arcs: one to go from the initial point to the singular arc, then the singular arc, and a final arc to go from the singular arc to the final point. The initial and final arcs are defined by the control being at its maximum or minimum value (depending on the given initial and final velocities). This type of optimal control is called bang-singular-bang [1]. The analysis of this type of problem is beyond the scope of this paper.

4 Maximum-Range Cruise

In this problem the objective is to maximize the range for a given fuel load, or, equivalently, to minimize the following performance index

$$J = -\int_0^{t_f} V \,\mathrm{d}t \tag{6}$$

subject to the equations of motion (1) as constraints, with the final value of the aircraft mass (m_f) fixed. The final values of distance (x_f) and flight time (t_f) are unspecified.

The Hamiltonian of this problem is given by

$$H = -V + \lambda_V \frac{1}{m}(\pi T_M - D) - \lambda_m c \pi T_M \tag{7}$$

where it has been taken into account that $\lambda_x = 0$ (since $\dot{\lambda}_x = 0$ and $\lambda_x(t_f) = 0$), and it satisfies

$H=0$ along the optimal trajectory. One can write $H=\overline{H}+S\pi$, with

$$\begin{aligned}\overline{H} &= -V-\lambda_V\frac{D}{m}\\ S &= (\lambda_V\frac{1}{m}-\lambda_m c)T_M\end{aligned} \tag{8}$$

One also has the function $\dot{S}$ given by (after taking into account the state equations (1) and the adjoint equations (3)):

$$\begin{aligned}\dot{S}=&-\frac{\mathrm{d}T_M}{\mathrm{d}V}(\frac{\lambda_V}{m}-\lambda_m c)\frac{D}{m}\\ &+\frac{T_M}{m}\left[1+\lambda_V\left(\frac{1}{m}\frac{\partial D}{\partial V}-c\frac{\partial D}{\partial m}+c\frac{D}{m}\right)\right.\\ &\left.+\lambda_m\frac{\mathrm{d}c}{\mathrm{d}V}D\right]\end{aligned} \tag{9}$$

note that the terms in the control variable π have cancelled out of this equation.

4.1 Singular arc

The three equations that define the singular arc ($\overline{H}=0, S=0, \dot{S}=0$) reduce to

$$\begin{aligned}&V+\lambda_V\frac{D}{m}=0\\ &\frac{\lambda_V}{m}-\lambda_m c=0\\ &1+\lambda_V\left(\frac{1}{m}\frac{\partial D}{\partial V}-c\frac{\partial D}{\partial m}+c\frac{D}{m}\right)\\ &\qquad\qquad+\lambda_m\frac{\mathrm{d}c}{\mathrm{d}V}D=0\end{aligned} \tag{10}$$

The singular arc is obtained after eliminating the adjoints λ_V and λ_m from these equations. One obtains the following expression, which generalizes that obtained in Ref. [5],

$$D\left(1-Vc-\frac{V}{c}\frac{\mathrm{d}c}{\mathrm{d}V}\right)-V\frac{\partial D}{\partial V}+Vcm\frac{\partial D}{\partial m}=0 \tag{11}$$

Equation (11) defines a singular line in the $(V,m)-$space, which is in fact the locus of possible points in the state space where optimal paths can lie.

4.2 Optimal singular control

The function $\ddot{S}$ depends linearly on the control variable. Let $\ddot{S}=A(V,m)+B(V,m)\pi$. The singular control is obtained from $A(V,m)+B(V,m)\pi=0$; one gets the following

$$\pi_{sing}=\frac{D}{T_M}\left(1+Vc\frac{A_1(V,m)}{B_1(V,m)}\right) \tag{12}$$

where $A_1(V,m)$ and $B_1(V,m)$ are given by

$$\begin{aligned}A_1(V,m)=&m\frac{\partial^2 D}{\partial m\partial V}-m^2c\frac{\partial^2 D}{\partial m^2}-mc\frac{\partial D}{\partial m}\\ &-\frac{m}{D}\frac{\partial D}{\partial m}\left(\frac{\partial D}{\partial V}-mc\frac{\partial D}{\partial m}\right)\\ B_1(V,m)=&DV\left(c^2+3\frac{\mathrm{d}c}{\mathrm{d}V}+\frac{1}{c}\frac{\mathrm{d}^2c}{\mathrm{d}V^2}\right)\\ &+2\frac{\partial D}{\partial V}\left(Vc+\frac{V}{c}\frac{\mathrm{d}c}{\mathrm{d}V}\right)\\ &-mV\left(c^2+3\frac{\mathrm{d}c}{\mathrm{d}V}\right)\frac{\partial D}{\partial m}+V\frac{\partial^2 D}{\partial V^2}\\ &+m^2c^2V\frac{\partial^2 D}{\partial m^2}-2Vcm\frac{\partial^2 D}{\partial m\partial V}\end{aligned} \tag{13}$$

As indicated in Ref. [5], in general one has $Vc \ll 1$, in which case Eq. (12) reduces to $\pi_{sing}\approx\frac{D}{T_M}$, that is, the singular thrust is very close to the steady-cruise value $T=D$.

4.3 Results

4.3.1 Singular arc

The singular arc defined by Eq. (11) is plotted in Fig. 1 for different values of altitude in terms of the dimensionless variables Mach number M and $\omega=W/(\frac{1}{2}\gamma p_A S_W)$, where W is the aircraft weight, γ=1.4 is the ratio of specific heats, p_A the pressure at the given altitude (h_A), and S_W the reference wing surface. The figure shows that there is a maximum value of the Mach number that can be obtained, namely M =0.7673 (for our aircraft model). Notice that the singular arc in the $(\omega,M)-$plane is practically independent of the flight altitude. Hence, the maximum value of M is practically the same for all altitudes. This is analyzed in more detail in Ref. [2].

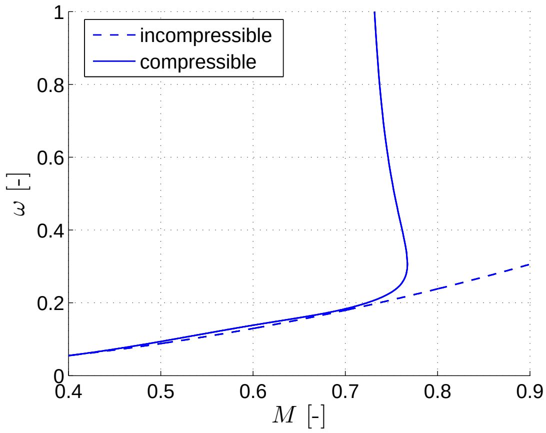


Fig. 1 Singular arc in the $(\omega, M)-$plane.

4.3.2 *Optimal paths*

As one can see in Fig. 1, for low values of ω, until the turning point is reached, the singular arc defines an increase of M with W, whereas for large values of ω, after the turning point, it defines a decrease. Notice that for given aircraft weight, low values of ω correspond to low altitudes, and viceversa. Therefore, for given initial and final values of W, at low altitudes one has the well-known behavior of M decreasing as fuel is consumed, whereas at high altitudes one can have the opposite. This behavior is shown in Fig. 2 for different values of h_A, where the optimal paths are represented by thick lines superposed on the singular arcs. In this simulation we have taken W_i =1600 kN and W_f =1100 kN, that is a fuel load for cruise of 500 kN (approximately 27% of MTOW). Thus, we can conclude that the cruise altitude has a qualitative influence on the results.

Implicit in the results presented in Fig. 2 is that the entire cruise can follow the singular arc. The inequality constraint $0 < \pi \leq 1$ is satisfied for those optimal paths. The optimal singular control for the optimal paths represented in Fig. 2 is depicted in Fig. 3. Notice that the required thrust decreases as fuel is consumed.

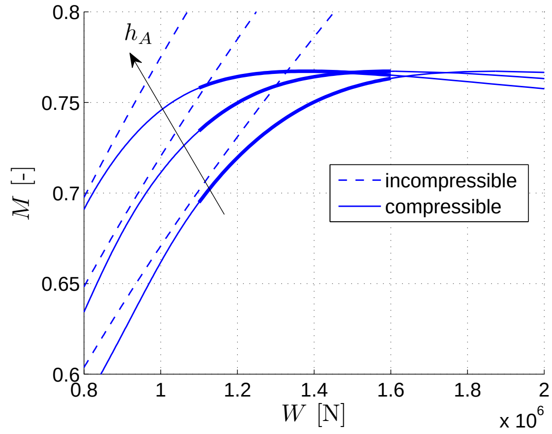


Fig. 2 Optimal paths (h_A = 9000, 10000, 11000 m)

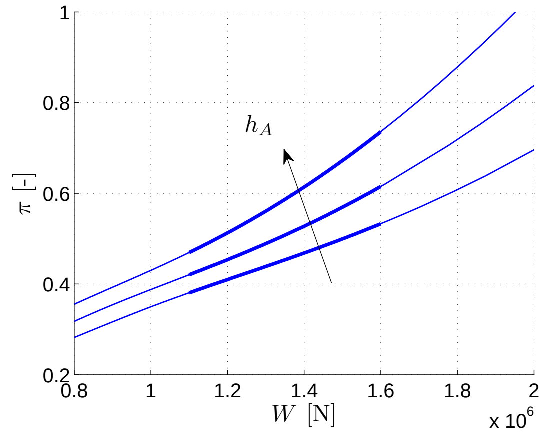


Fig. 3 Optimal singular control (h_A = 9000, 10000, 11000 m)

4.3.3 *Maximum range*

Once the optimal singular control is determined, integration of the state equations yields the maximum range, R_{max}, which is plotted in Fig. 4 as a function of cruise altitude, h_A. The results show that there is a "best" altitude where the maximum range is largest, namely $(h_A)_{best}$ = 10034 m, with $(R_{max})_{best}$ = 10705 km.

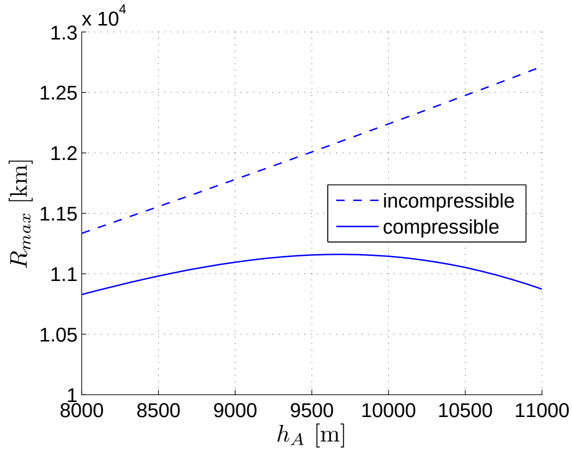


Fig. 4 Maximum range.

4.3.4 *Compressibility effects*

To analyze compressibility effects we consider an incompressible drag polar (see Ref. [2]). The incompressible singular arcs have been plotted in Figs. 1 and 2, and the corresponding maximum range in Fig. 4.

One obtains, first, the result that the Mach number increases strongly with aircraft weight (this is also the result obtained in Ref. [5] for an incompressible model of a Boeing 747-400). Comparing with the compressible results we can see that the incompressible drag polar overestimates the optimal Mach number, which can become quite large (even supersonic), whereas, as shown before, a compressible drag polar defines a maximum value of M, in the subsonic region. Thus, compressibility effects prevent the Mach number from increasing unrealistically. Also, one has that the range results are in complete disagreement with the compressible case; the incompressible drag polar overestimates the value of R_{max}, especially at large altitudes.

We conclude that between the compressible and incompressible results there is no agreement, neither quantitative nor qualitative.

5 Minimum-Cost Cruise

The direct operating cost (DOC) is defined as $J = m_F + CI\, t_f$, where $m_F = m_i - m_f$ is the mass of fuel consumed (m_i and m_f are the initial and final values of aircraft mass), and t_f is the flight time. The cost index is defined as CI=cost of time/cost of fuel. For simplicity, without loss in generality, we have not included the actual fuel cost, so that J is scaled to the units of fuel mass; thus, using SI units of measure, J is measured in kg, and CI in kg/s. Now the goal is to minimize the DOC for a given range, that is, to minimize the following performance index

$$
\begin{aligned}
J &= -\int_{m_i}^{m_f} \mathrm{d}m + CI \int_0^{t_f} \mathrm{d}t \\
&= \int_0^{t_f} (cT + CI)\, \mathrm{d}t
\end{aligned}
\tag{14}
$$

subject to the equations of motion (1) as constraints, with the final value of distance (x_f) fixed. The final value of aircraft mass (m_f) and the final time (t_f) are unspecified.

The Hamiltonian is now given by

$$
\begin{aligned}
H =& (c\pi T_M + CI) + \frac{\lambda_V}{m}(\pi T_M - D) \\
& - \lambda_m c\pi T_M + \lambda_x V
\end{aligned}
\tag{15}
$$

and satisfies $H = 0$ along the optimal trajectory. One has $\lambda_x = constant$ (since $\dot{\lambda}_x = 0$) and $\lambda_m(t_f) = 0$. One can write $H = \overline{H} + S\pi$, with

$$
\begin{aligned}
\overline{H} &= CI - \lambda_V \frac{D}{m} + \lambda_x V \\
S &= \left[\frac{\lambda_V}{m} - (\lambda_m - 1)c\right] T_M
\end{aligned}
\tag{16}
$$

One also has the function $\dot{S}$, given by (after taking into account Eqs. (1) and (3)):

$$
\begin{aligned}
\dot{S} =& -\left[\frac{\lambda_V}{m} - (\lambda_m - 1)c\right] \frac{D}{m}\frac{\mathrm{d}T_M}{\mathrm{d}V} \\
& + \frac{T_M}{m}\left[\frac{\lambda_V}{m}\left(\frac{\partial D}{\partial V} + cD - mc\frac{\partial D}{\partial m}\right)\right. \\
& \left. - \lambda_x + (\lambda_m - 1)D\frac{\mathrm{d}c}{\mathrm{d}V}\right]
\end{aligned}
\tag{17}
$$

note again that the terms in the control variable π have cancelled out of this equation.

5.1 Singular arc

The three equations that define the singular arc ($\overline{H}=0, S=0, \dot{S}=0$) reduce to

$$\begin{aligned}
&CI-\lambda_V\frac{D}{m}+\lambda_x V=0\\
&\frac{\lambda_V}{m}-(\lambda_m-1)c=0\\
&\frac{\lambda_V}{m}\left(\frac{\partial D}{\partial V}+cD-mc\frac{\partial D}{\partial m}\right)-\lambda_x\\
&\qquad\qquad+(\lambda_m-1)D\frac{\mathrm{d}c}{\mathrm{d}V}=0
\end{aligned}\tag{18}$$

The singular arc is obtained after eliminating the adjoints λ_V and λ_m from these equations. One obtains the following expression,

$$\begin{aligned}
&D\left[\left(1-\frac{\Omega}{\Omega+V}\right)-Vc-\frac{V}{c}\frac{\mathrm{d}c}{\mathrm{d}V}\right]\\
&\qquad -V\frac{\partial D}{\partial V}+Vcm\frac{\partial D}{\partial m}=0
\end{aligned}\tag{19}$$

where $\Omega=\dfrac{CI}{\lambda_x}$; this equation defines a family of singular arcs of parameter Ω. The constant λ_x is defined by the condition $\lambda_m(t_f)=0$, and the corresponding singular arc is the solution to the problem. In such case Eq. (19) defines a singular line in the $(V,m)-$space, which again is the locus of possible points in the state space where optimal paths can lie.

Therefore, in this case of minimum-DOC cruise, to impose the condition $\lambda_m(t_f)=0$, one must integrate the adjoint equations to solve the problem, which makes it more involved mathematically.

The case CI=0 corresponds to the problem of minimum-fuel cruise. In this case one has the same singular arc as the one obtained in the maximum-range cruise, given by Eq. (11). Hence, one has the same optimal speed laws to maximize range for a given fuel load and to minimize fuel for a given range.

5.2 Optimal singular control

As in Section 4, the singular control is obtained from $\ddot{S}=A(V,m)+B(V,m)\pi=0$. One gets the same optimal control given by Eqs. (12) and (13). This expression for the optimal singular control depends *implicitly* on the parameter of the family of singular arcs, since V and m are related by the singular arc equation (19) which includes the dependence on Ω.

5.3 Results

Only one value of cruise range is considered: x_f = 10000 km, and the initial weight at the start of the cruise is the same in all cases: $W_i=1600$ kN.

In the definition of the direct operation cost, it is implicitly assumed that the cost index is positive (representative values of CI are in the range 0.5 to 1.5 kg/s). However, negative cost indices can be used to model the problem of minimum fuel with fixed arrival time, as shown in Section 6, hence, in the following, we include results for negative values of CI.

The resolution algorithm used to solve the state and adjoint equations is given in Ref. [4].

5.3.1 Singular arc

The singular arcs in the (ω,M)-plane defined by Eq. (19) are plotted in Fig. 5, for different values of the parameter Ω, and for a representative altitude h_A =10000 m.

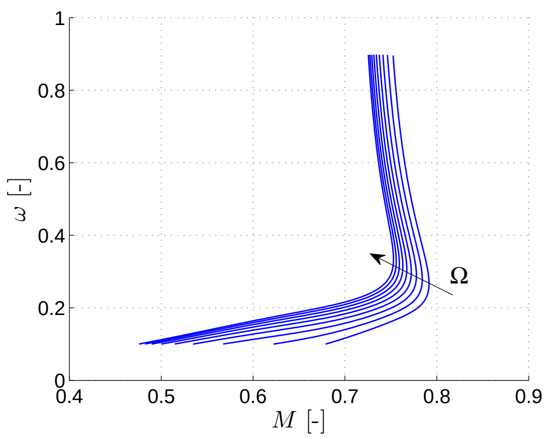
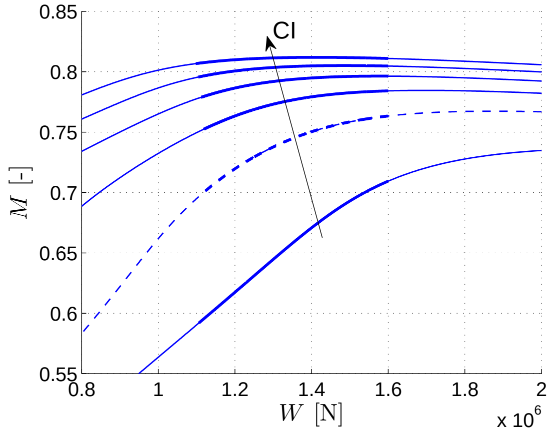


Fig. 5 Singular arcs in the (ω,M)-plane, for h_A =10000 m ($\Omega=-100,-75,-50,-25,0,25,50,75,100$ m/s).

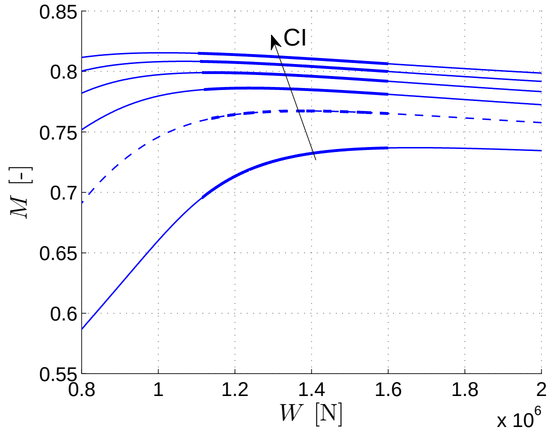
Note that these curves present a maximum value of the Mach number, as discussed in Section 4. Regarding the effect of the cruise altitude in these dimensionless arcs, it can be shown that it is very small, as already seen in Fig. 1.

5.3.2 *Optimal paths*

The optimal paths are represented in Fig. 6, for two values of cruise altitude ($h_A = 9000$ and 11000 m); they are represented by thick lines superposed on the singular arcs.



(a)



(b)

Fig. 6 Optimal paths for various values of h_A: (a) 9000, (b) 11000 m ($CI = -0.5, 0, 0.5, 1, 1.5, 2$ kg/s; dashed lines corresponds to CI=0).

In each graph, optimal paths for different values of the cost index CI are plotted. Note that depending on the value of CI the optimal procedure changes: for small values, the speed decreases as fuel is consumed, whereas for large values the speed increases (although slightly); also, for certain values of CI, the optimal Mach number is roughly constant.

Implicit in the results presented in Fig. 6 is that the entire cruise can follow the singular arc. The inequality constraint $0 < \pi < 1$ is satisfied for those optimal paths. The optimal singular control for the optimal paths that correspond to $h_A =$ 9000, 10000, 11000 and 12000 m is depicted in Fig. 7. Note that the optimal control decreases as fuel is consumed, and increases as cruise altitude increases.

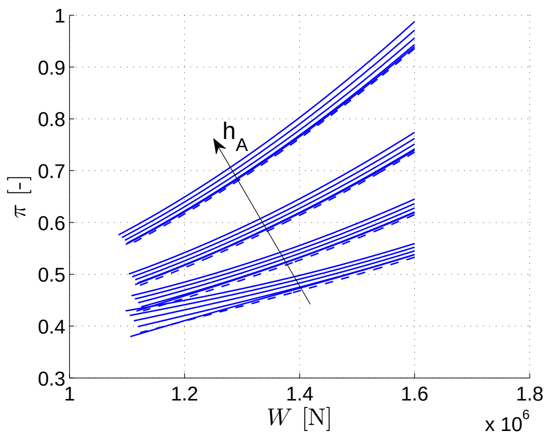


Fig. 7 Optimal singular control for h_A = 9000, 10000, 11000, 12000 m ($CI = -0.5, 0, 0.5, 1, 1.5, 2$ kg/s; dashed lines corresponds to CI=0).

5.3.3 *Minimum cost*

For each value of CI, the corresponding optimal path leads to minimum cost, associated to an optimal flight time and an optimal fuel consumption. A trade off between these flight time and fuel consumption is represented in Fig. 8 (in this figure, on each curve the cost index ranges from -0.7 to 2 kg/s, in the counterclockwise direction). The point of vertical tangent corresponds to CI=0 (minimum fuel). The optimal flight time decreases as CI increases (that is, as the time cost increases); the optimal fuel consumption as a function of CI presents a minimum at CI=0, as it corresponds to the minimum-fuel problem. For positive cost indices (the lower part of the curves), decreasing the flight time requires increasing the fuel consumption.

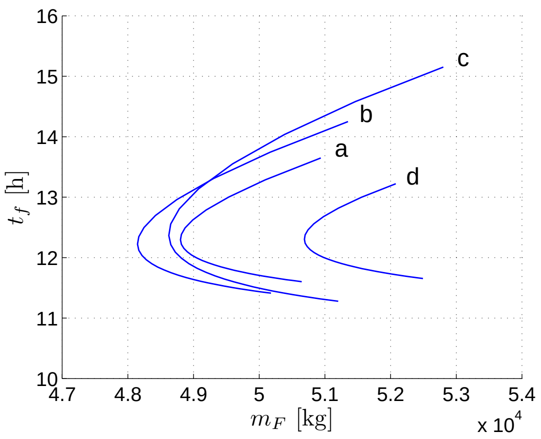


Fig. 8 Optimal flight time vs. optimal fuel consumption, for various values of h_A: (a) 9000, (b) 10000, (c) 11000, (d) 12000 m (on each curve, the cost index ranges from -0.7 to 2 kg/s).

The minimum direct operating cost is represented in Fig. 9 as a function of CI. It can be seen that the minimum DOC increases with CI (in fact, the corresponding fuel and time costs both increase with CI).

6 Minimum-Fuel Cruise with Fixed Arrival Time

In this problem the objective is to minimize fuel consumption for a given range and a given final time, that is, to minimize the following performance index

$$J = \int_0^{t_f} cT\,dt \tag{20}$$

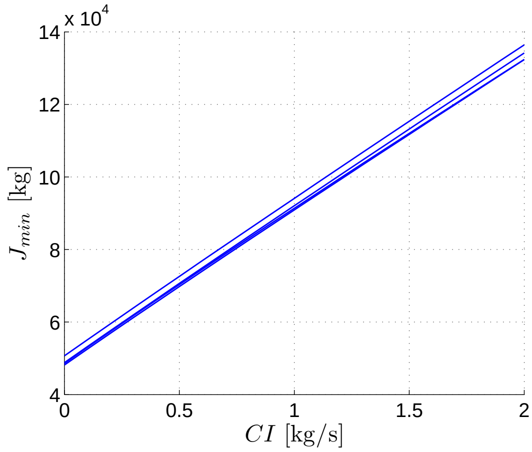


Fig. 9 Minimum DOC as a function of CI (h_A= 9000, 10000, 11000, 12000 m).

subject to Eqs. (1) as constraints, with the final flight distance (x_f) and the flight time (t_f) fixed. The final aircraft mass (m_f) is unspecified.

The Hamiltonian is given by

$$\begin{aligned} H =& c\pi T_M + \frac{\lambda_V}{m}(\pi T_M - D) \\ & - \lambda_m c\pi T_M + \lambda_x V \end{aligned} \tag{21}$$

and satisfies $H = constant$ along the optimal trajectory. One has $\lambda_x = constant$ (since $\dot{\lambda}_x = 0$) and $\lambda_m(t_f) = 0$. One can write $H = \overline{H} + S\pi$, with

$$\begin{aligned} \overline{H} &= \lambda_x V - \lambda_V \frac{D}{m} \\ S &= \left[\frac{\lambda_V}{m} - (\lambda_m - 1)c\right] T_M \end{aligned} \tag{22}$$

The function $\dot{S}$ is now identical to Eq. (17).

6.1 Singular arc

The three equations that define the singular arc ($\overline{H}$ =constant, $S = 0, \dot{S} = 0$) reduce to

$$\begin{aligned} & \lambda_x V - \lambda_V \frac{D}{m} = H \\ & \frac{\lambda_V}{m} - (\lambda_m - 1)c = 0 \\ & \frac{\lambda_V}{m}\left(\frac{\partial D}{\partial V} + cD - mc\frac{\partial D}{\partial m}\right) - \lambda_x \\ & \qquad\qquad + (\lambda_m - 1)D\frac{dc}{dV} = 0 \end{aligned} \tag{23}$$

where H is the constant value of the Hamiltonian.

Eliminating the adjoint variables λ_V and λ_m from these equations and making $\omega_t = \dfrac{H}{\lambda_x}$, one obtains the following expression

$$D\left[\left(1-\frac{\omega_t}{\omega_t - V}\right) - Vc - \frac{V}{c}\frac{dc}{dV}\right] - V\frac{\partial D}{\partial V} + Vmc\frac{\partial D}{\partial m} = 0 \tag{24}$$

which is a family of singular arcs of parameter ω_t. The value of this constant is determined by the constraint $t(x_f) = t_f$. The corresponding singular arc is the solution to the problem, in which case Eq. (24) defines a singular line in the (V,m) space, which is, again, the locus of possible points where optimal paths can lie.

The constant λ_x (or H) is defined by the condition $\lambda_m(t_f) = 0$. As long as the values of λ_x or H are not needed, there is no need to integrate the adjoint equations.

The problem of *free* final time corresponds to the case $\omega_t = 0$; in this case one has the same singular arc obtained in Section 4 for the problem of maximum-range cruise for a given fuel load.

Note that the family of singular arcs (24) is the same as that obtained in Section 5 for the problem of minimum-cost cruise, except that the family parameter is now $-\omega_t$ instead of Ω. Therefore, as indicated in Refs. [10,13], for given t_f (or, equivalently, for given ω_t), the problem is equivalent to solving a minimum-DOC problem with $\Omega = -\omega_t$, or, equivalently, with a cost index such that the corresponding free-final-time, DOC-optimal trajectory arrives at the assigned time.

6.2 Optimal singular control

As in the previous cases, the singular control is obtained from $\ddot{S} = A(V,m) + B(V,m)\pi = 0$, and one gets the same optimal control given by Eqs. (12) and (13). This expression again depends *implicitly* on the parameter of the family of singular arcs, since V and m are related by the singular arc equation (24) which includes the dependence on ω_t.

6.3 Results

Only one value of cruise range is considered: $x_f = 10000$ km, and the initial weight at the start of the cruise is the same in all cases: $W_i = 1600$ kN.

The iterative procedure used to solve the state equations is described in Ref. [3]. The relationship between ω_t and t_f is represented in Fig. 10, where $t_{f,0}$ is the flight time in the problem of *free* final time, that corresponds to ω_t =0 (note that $t_{f,0}$ depends on cruise altitude). One has that positive values of ω_t correspond to flight times smaller than $t_{f,0}$, and vice versa.

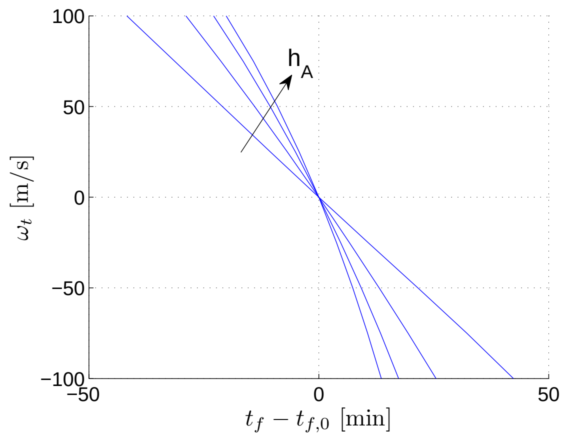


Fig. 10 Parameter ω_t vs. flight time ($h_A = 9000$, 10000, 11000, 12000 m).

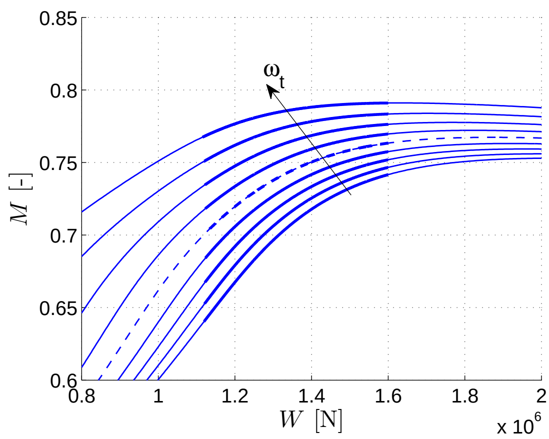
6.3.1 Singular arc

The singular arcs in the (ω, M)-plane defined by Eq. (24) are the same as those corresponding to the minimum-DOC problem plotted in Fig. 5, with $\omega_t = -\Omega$.

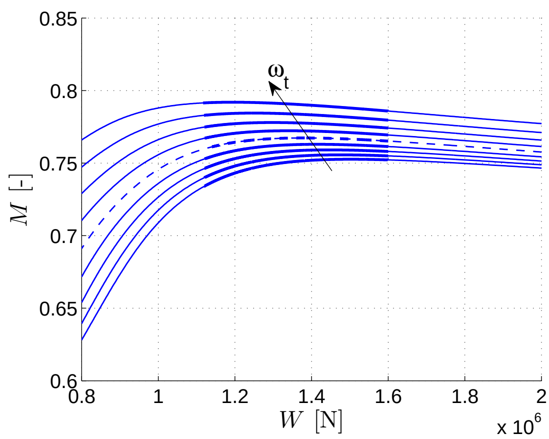
6.3.2 Optimal paths

The optimal paths are represented in Fig. 11, for two values of cruise altitude ($h_A = 9000$ and 11000 m); they are represented by thick lines superposed on the singular arcs. In each graph, optimal paths for different values of the parameter ω_t are plotted. They are completely similar to those of the minimum-DOC problem, represented in Fig. 6.

Again, implicit in the results presented in Fig. 11 is that the entire cruise can follow the singular arc. The optimal singular control is represented in Fig. 12, where one can see that the inequality constraint $0 < \pi \leq 1$ is satisfied for the optimal paths that correspond to $h_A = 9000$, 10000, 11000 and 12000 m. As in the previous section, the optimal control decreases as fuel is consumed, and increases as cruise altitude increases.



(a)



(b)

Fig. 11 Optimal paths for various values of h_A: (a) 9000, (b) 11000 m ($\omega_t = -100, -75, -50, -25, 0, 25, 50, 75, 100$ m/s; dashed lines corresponds to ω_t=0).

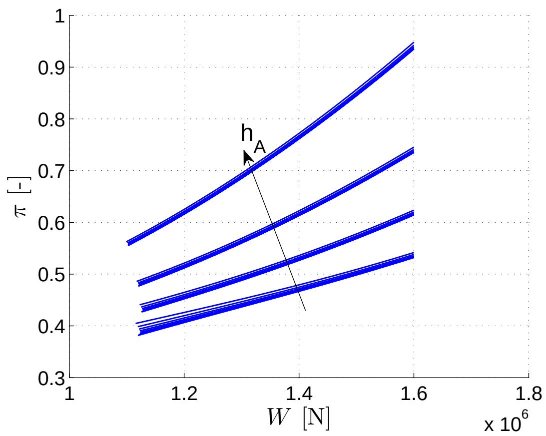


Fig. 12 Optimal singular control for h_A = 9000, 10000, 11000, 12000 m ($\omega_t = -100, -75, -50, -25, 0, 25, 50, 75, 100$ m/s).

6.3.3 Minimum fuel consumption

For each value of the flight time t_f, the corresponding optimal path leads to minimum fuel consumption m_F. Let $m_{F,0}$ be the minimum fuel consumption in the problem of *free* final time (note that $m_{F,0}$ varies with cruise altitude). The increment in fuel consumption $m_F - m_{F,0}$ is represented in Fig. 13 as a function of the increment in flight time $t_f - t_{f,0}$, for different cruise altitudes. One can see that fuel consumption always increases, both when the flight time is larger and when it is smaller than the reference flight time $t_{f,0}$. The increment in fuel consumption increases with altitude, quite strongly in the case $t_f < t_{f,0}$ and moderately in the case $t_f > t_{f,0}$.

In actual operational practice an aircraft may be required to absorb a given flight delay along the cruise (reaching the TOD point at a given time), or to make an early arrival (which may be required, for instance, to resolve a conflict). Delays correspond to positive values of $t_f - t_{f,0}$, and early arrivals correspond to negative values. Figure 13 clearly indicates that fuel consumption always increases, both for early arrivals and for delays.

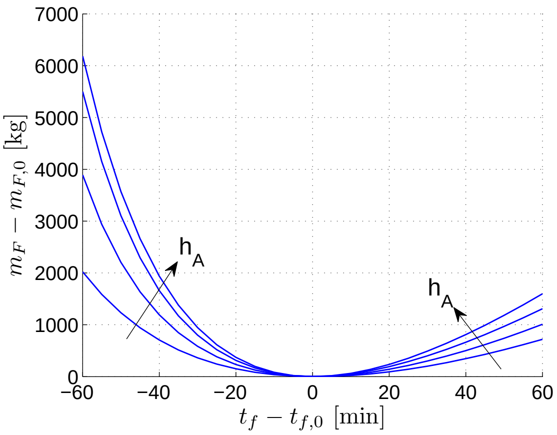


Fig. 13 Minimum fuel consumption vs. flight time (h_A = 9000, 10000, 11000, 12000 m).

7 Conclusions

Cruise optimization at constant altitude has been formulated as a singular optimal control problem. The cases of maximum range, minimum direct operating cost and minimum fuel with *fixed* arrival time have been analyzed, providing a comprehensive analysis of optimum cruise at constant altitude. The case of unsteady cruise with variable aircraft mass has been considered. In all cases the singular arcs, the optimal control and the optimal paths have been studied. It has been shown that the singular arcs in the three problems belong to a same family, and that the singular optimal control is always given by the same function (in terms of the state variables).

The influence of cruise altitude has been shown to be qualitatively important: cruise at low altitudes requires that the Mach number decrease as fuel is consumed, whereas at high altitudes one has that the optimal Mach number is roughly constant. It has been also shown that compressibility effects must be taken into account in order to properly describe the behavior of modern, high-speed, subsonic transport aircraft (comparison with an incompressible drag polar has shown large differences, both quantitative and qualitative). In the analysis of the minimum-DOC problem a trade off between flight time and fuel consumption has been made: the increase in fuel consumption required by a decrease in flight time has been quantified. In the analysis of minimum-fuel cruise with fixed final time, the cost of having a flight time longer or shorter than that of the reference free-time problem has been quantified as well: the fuel consumption is always larger, increasing with cruise altitude. It has been shown that the problem of minimum-fuel cruise with fixed arrival time can be formulated as one of minimum DOC: the problem is to find the cost index for which the corresponding DOC-optimal trajectory arrives at the assigned final time.

It must be emphasized that in this work a constrained regime has been considered, namely, flight at constant altitude (of interest from the Air Traffic Control point of view). The optimization problem, then, has defined a constrained maximum (it is well known that improved performance is obtained flying, for instance, a *cruise climb*, where altitude slightly increases).

From the operational point of view, the optimal variable-Mach solutions obtained in the paper present the drawback of its flyability. However, they can be used to define flyable procedures close to optimal: for example, a cruise formed by constant-Mach segments, defined so as to approximate the theoretical variable-Mach curves, that is, a *stepped Mach cruise* (similarly to the way the stepped climb cruise approximates the optimal cruise climb solution).

References

[1] Bryson, A.E., and Ho, Yu-Chi. *Applied Optimal Control*. Hemisphere Publishing Corporation, Washington, DC, 1975, pp. 55, 252.

[2] Rivas, D., and Valenzuela, A. Compressibility effects on maximum range cruise at constant altitude, *Journal of Guidance, Control, and Dynamics*, Vol. 32, No. 5, 2009, pp. 1654-1658. doi: 10.2514/1.44039

[3] Franco, A., Rivas, D., and Valenzuela, A. Minimum-fuel cruise at constant altitude with fixed arrival time, *Journal of Guidance, Control, and Dynamics*, Vol. 33, No. 1, 2010, pp. 280-285. doi: 10.2514/1.46465

[4] Franco, A., and Rivas, D. An analysis of minimum-cost cruise at constant altitude includ-

ing wind effects, to be submitted to *Journal of Guidance, Control, and Dynamics*, 2010.

[5] Pargett, D.M., and Ardema, M.D. Flight path optimization at constant altitude, *Journal of Guidance, Control, and Dynamics*, Vol. 30, No. 4, 2007, pp. 1197-1201. doi: 10.2514/1.28954

[6] Miele, A. *Flight Mechanics. Theory of Flight Paths*. Addison-Wesley Publishing Company, Reading, Massachusetts, 1962, pp. 107, 225.

[7] Torenbeek, E. Cruise performance and range prediction reconsidered, *Progress in Aerospace Sciences*, Vol. 33, 1997, pp. 285-321.

[8] Barman, J.F., and Erzberger, H. Fixed-range optimum trajectories for short-haul aircraft, *Journal of Aircraft*, Vol. 13, No.10, 1976, pp. 748-754.

[9] Erzberger, H., and Lee, H. Constrained optimum trajectories with specified range, *Journal of Guidance, Control, and Dynamics*, Vol. 3, No. 1, 1980, pp. 78-85.

[10] Burrows, J.W. Fuel-optimal aircraft trajectories with fixed arrival times, *Journal of Guidance, Control, and Dynamics*, Vol. 6, No. 1, 1983, pp. 14-19.

[11] Burrows, J.W. Fuel optimal trajectory computation, *Journal of Aircraft*, Vol. 19, no. 4, 1982, pp. 324-329.

[12] Bilimoria, K.D., Cliff, E.M. and Kelley, H.J. Classical and neo-classical cruise-dash optimization, *Journal of Aircraft*, Vol. 22, no. 7, 1985, pp. 555-560.

[13] Chakravarty, A. Four-dimensional fuel-optimal guidance in the presence of winds, *Journal of Guidance, Control, and Dynamics*, Vol. 8, No. 1, 1985, pp. 16-22.

[14] Schultz, R.L. and Zagalsky, N.R. Aircraft performance optimization, *Journal of Aircraft*, Vol. 9, no. 2, 1972, pp. 108-114.

[15] Speyer, J.L. On the fuel optimality of cruise, *Journal of Aircraft*, Vol. 10, no. 12, 1973, pp. 763-765.

[16] Schultz, R.L. Fuel optimality of cruise, *Journal of Aircraft*, Vol. 11, no. 9, 1974, pp. 586-587.

[17] Speyer, J.L. Nonoptimality of the steady-state cruise for aircraft, *AIAA Journal*, Vol. 14, no. 11, 1976, pp. 1604-1610.

[18] Menon, P.K.A. Study of aircraft cruise, *Journal of Guidance, Control, and Dynamics*, Vol. 12, no. 5, 1989, pp. 631-639.

[19] Sorensen, J.A., and Waters, M.H. Airborne method to minimize fuel with fixed time-of-arrival constraints, *Journal of Guidance, Control, and Dynamics*, Vol. 4, No. 3, 1981, pp. 348-349.

[20] Bell, D.J., and Jacobson, D.H. *Singular Optimal Control Problems*. Academic Press, New York, 1975, p. 12.

Copyright Statement