SCALED DOWN MODELS FOR FREE VIBRATION ANALYSIS OF ORTHOGONALLY STIFFENED CYLINDRICAL SHELLS USING SIMILITUDE THEORY

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Abstract

In this paper, the necessary similarity conditions, or scaling laws, for free vibrations of orthogonally stiffened cylindrical shells are developed using similitude theory. The Donnell-type nonlinear strain-displacement relations along with smearing theory are used to model the structure. Then the principle of virtual work is used to analyze the free vibration of the stiffened shell. After nondimensionalizing the derived formulations, the scaling laws are developed, using similitude theory. Then, different examples are solved to validate the scaling laws numerically and experimentally. The obtained results show the effectiveness of the derived formulations.

1. Introduction

Although designers employ most powerful analysis tools, using the most elaborate electronic computers, actual testing is required in order to extract some input design parameters and also to ensure the proper functioning of the designed system [1]. Structural dynamic properties of aerospace structures are one of fundamental requirements in designing such structures and evaluating their control systems effectiveness. Finding structural dynamic properties-often including natural frequencies, mode shapes, and damping ratios- through performing ground vibration tests in case of heavy and huge aerospace structures is very difficult to do and also requires advanced and huge test instrumentations and considerable expense and time. But through testing a small-scale model of the structure, which simulates the behavior of its prototype exactly, not only modal parameters can be extracted, but also the reliability and accuracy of assumptions used in numerical and analytical models can be verified. Besides, the designers can access some useful data during designing and pre-manufacturing processes. Through fabricating and testing such small-scale models, design modifications and revisions will be possible without costly and time consuming full-scale fabrication and tests. Scaling laws provide the relationship between a full scale structure and its scale models and can be used to extrapolate the experimental data of a small, inexpensive, and easily tested model into design information for the large prototype [2]. Meanwhile, considering the significant roll of stiffened cylindrical shells in various types of aerospace structures, practical applications of scaled down stiffened cylindrical shells and the importance of establishing a similarity and finding proper scaling laws will be evident. Due to the large number of design parameters in stiffened cylindrical shells, the identification of the principal scaling laws through conventional method of dimensional analysis and pi-theorem is tedious. While similitude theory based on the governing equations of the structural system is more direct and simpler in execution, but some limitations may be encountered during the designing of small-scale stiffened shells, both from a fabrication and an economic viewpoint. Some of these difficulties and limitations are discussed in this paper and some solutions are proposed to solve them.

Some studies particularly concerning the use of scaled down shell structures have been conducted in the past. In 1962, Ezra [3] presented a study based on dimensional analysis, for the buckling behavior of scaled down models of shell structures subjected to impulsive loads. A similar investigation was presented in 1964 by Morgen[4] for an orthotropic cylinder subjected to different static loads. In 1971, Soedel[5] investigated similitude requirements for vibrating thin shells. In 1994, Buckling stability prediction of laminated cylindrical shells using scaled down models and based on partial similitude was done in an investigation by Rezaeepazhand et al. [6]. In 1996, prediction of shell vibration response in cross-ply cylindrical shells using scaled down models, based on partial similitude was accomplished by the same authors [7]. In a similar investigation presented in 1997 by the same authors, vibration response of laminated cylindrical shells with double curvature is investigated based on structural similitude [8]. In 2005, effect of extensive use of welding on buckling behavior of large cylindrical shells constructed from a large number of curved panels, is investigated experimentally using scale models by Teng and Lin[9]. In 2007, a procedure is presented by R. Oshiro and M. alves for correction of distortion due to material strain rate sensitivity in the scaling laws of cylindrical shells under axial impact [10].

Many research activities have been conducted on scale-down modeling of dynamic and static behavior of other structural systems. In most of these researches similitude theory is discussed based on dimensional analysis. Among the most recent of these works is an investigation done in 2005 by P. Singhatanadgid and V. Ungbhakorn in which similitude invariants and scaling laws for buckling of polar orthotropic annular plates subjected to radial compressive load and torsional load is derived [11]. The similitude transformation is applied to the governing differential equation directly resulting in a scaling law for buckling load of annular plates and the similarity conditions between a model and a prototype. In 2006, J. J. Wu predicted the vibration characteristics of an elastically supported full-size flat plate subjected to circular-moving loads from those of its scale model using the associated scaling laws [12]. The similarity conditions between the full-size system and its complete-similitude scale model were derived from their equations of motion. In another work done in 2007 by the same author the lateral vibration characteristics of the full-size rotor-bearing system is predicted by using the scale rotor-bearing model and the associated scaling laws [13]. The scaling laws are derived both from the equations of motion and the theory of dimensional analysis.

In the current investigation, scaling laws in free vibration of orthogonally stiffened cylindrical shells are developed through direct use of the governing equations of system. The Donnel-type nonlinear strains along with Hamilton's principle are used to derive equations of motion of the shell. Then the nondimensional solution is developed and used to derive the scaling laws. In order to overcome the difficulties encountered in fabricating small scaled orthogonally stiffened cylindrical shells, designing equivalent sections for stiffeners and some other approaches, based on similitude theory were taken into consideration. Finally, a typical stiffened cylindrical shell and its equivalent scale model are fabricated and tested. Results comparison indicates that the small-scale equivalent stiffened cylindrical shell can predict the behavior of its prototype accurately.

2. Derivation of Equations of Motion Using Smearing

There are two main approaches in vibration analysis of stiffened cylindrical shells; in the first approach, stiffened structure is replaced by an equivalent continuum and the effect of stiffeners is averaged or "smeared out" over the shell. Then one should just find a method for averaging the effect of discrete elements. Through this method, when the wave length of vibration is larger enough than the distance between stiffeners, very accurate results are achieved. But for vibrations having short wave length the second approach should be used, in which stiffener elements are treated as discrete elements [14]. Here, the stiffeners are not considered as discrete elements, but their effects are averaged over the shell and the equilibrium equations and boundary conditions (BC's) then are determined by formulating the potential energy of the system and applying the principle of minimum potential energy. The nonlinear equations of motion of stiffened cylinder are obtained as

$$\begin{aligned}
&N_{x,x}+N_{xy,x}=0 \\
&N_{y,y}+N_{xy,y}=0 \\
&-M_{x,xx}+M_{xy,xy}-M_{yx,xy}-M_{y,yy}+N_y/R-N_x w_{,xx} \\
&\qquad -N_y w_{,yy}-2N_{xy}w_{,xy}-\bar{M}\omega^2 w+p=0,
\end{aligned} \tag{1}$$

and a set of BC's (essential and natural BC's) to be satisfied at each end of the cylinder are

$$\begin{aligned}
&M_{x,x}-\left(M_{xy,y}-M_{yx,y}\right)+N_x w_{,x}+N_{xy}w_{,y}=0 && or && w=0 \\
&M_x=0 && or && w_{,x}=0 \\
&N_x+\bar{N}_x=0 && or && u=0 \\
&N_{xy}=0 && or && v=0,
\end{aligned} \tag{2}$$

where M_x, M_y, M_{xy} are the moment resultants per unit length and N_x, N_y, N_{xy} are the stress resultants per unit length[15]. $\bar{M}$ is the averaged or smeared out mass per unit area of the stiffened cylinder. $\bar{N}_x$ is a constant axial load resultant applied along the middle surface of isotropic shell, and p is a constant hydrostatic pressure applied over the shell. For further information on derivation of equations of motion and definition of parameters, refer to the Appendix.

3. Solution of the Equations of Motion

The deformation associated with the vibration of a prestressed cylinder are divided into two parts; the first part denoted by a subscript A is an axisymmetric, static prestressed deformation which occurs prior to excitation of one of the natural frequencies; the second part denoted by a subscript B is a small additional deformation which occurs as a result of the excitation. In general, the equations of motion have variable coefficients and would be quite difficult to solve analytically for many cases, but if we assume that the prestress deformation w_A is constant prior to the excitation and if we neglect the nonlinear terms in the additional deformations (subscript B), the solution to the equation will be greatly simplified. Therefore, the solution to the axisymmetric, static prestress equations are $N_{yA}=-\bar{N}_y=-pR$ and $N_{xA}=-\bar{N}_x$. Then, the linear equations of motion of the stiffened cylinder will be written as

$$\begin{aligned}
&N_{xB,x}+N_{xyB,y}=0 \\
&N_{yB,y}+N_{xyB,x}=0 \\
&-M_{xB,xx}+M_{xyB,xy}-M_{yxB,xy}-M_{yB,yy}+ \\
&N_{yB}/R+\bar{N}_x w_{B,xx}+\bar{N}_y w_{B,yy}-\bar{M}\omega^2 w_B=0
\end{aligned} \tag{3}$$

where the moment resultants M_{xB}, M_{yB}, and M_{xyB} and the stress resultants N_{xB}, N_{yB}, and N_{xyB} are obtained through substituting displacements u_B, v_B, w_B into Eqs.(A-4), and neglecting the nonlinear terms. The simply supported BC's to be satisfied at each end $x=0,L$ are

$$w_B=M_{xB}=v_B=N_{xB}=0 \tag{4}$$

Expressions for the displacements u_B, v_B, and w_B which satisfy these BC's are given as

$$\begin{aligned}
u_B&=\bar{u}\,Cos\left(m\pi x/L\right)Cos\left(ny/R\right), \\
v_B&=\bar{v}\,Sin\left(m\pi x/L\right)Sin\left(ny/R\right), \\
w_B&=\bar{w}\,Sin\left(m\pi x/L\right)Cos\left(ny/R\right),
\end{aligned} \tag{5}$$

where m is longitudinal wave number, which is the number of axial half-waves and n is circumferential wave number, which is the number of circumferential full waves. If Eqs. .(5) are substituted into Eqs.(3) and the determinent of the coefficients of $\bar{u}, \bar{v}$, and $\bar{w}$ is set equal to zero for a non-trivial solution, the equation of free vibrations of orthogonally stiffened cylindrical shell or frequancy equation will be obtained.

4. Non-Dimensionalizing

Performing the necessary transformations, the characteristic equation or the frequency equation so obtained, in terms of non-dimensional parameters may be written as

$$\begin{aligned}
\bar{\Omega}^2 &= m^4\left[\left(1+\frac{n^2}{\Lambda^2}\right)^2+\Xi_s+\frac{n^2}{\Lambda^2}\left(\Gamma_s+\Gamma_r\right)+\frac{n^4}{\Lambda^4}\Xi_r\right] \\
&-m^2\left(\bar{F}_x+\bar{F}_y\frac{n^2}{\Lambda^2}\right)+\Pi^2\left(\frac{1+\bar{S}\Sigma_s+\bar{R}\Sigma_r+\bar{R}\bar{S}\Sigma_{rs}}{\Upsilon}\right),
\end{aligned} \tag{6}$$

where $\bar{\Omega}$ is the non-dimensional frequency. Other non-dimensional parameters used above, some encompass shell, stringers, and rings characteristics, and some others represent the mode shapes of the stiffened shell, and some express static loading of the stiffened shell. These non-dimensional parameters are defined below.

The non-dimensional frequency, $\bar{\Omega}$, is written as

$$\bar{\Omega}^2 = \frac{\bar{M}L^4\omega^2}{D} \tag{7}$$

where ω is the natural frequancy of the stiffened cylinder and L is the length of the stiffened cylinder. The non-dimensional wave length, Λ, is defined as follows [16]

$$\Lambda = \frac{m\pi R}{L} \tag{8}$$

The non-dimensional parameter Π, expresses the geometry of the shell and is defined as follows

$$\Pi = \frac{L^2}{Rh}. \tag{9}$$

The non-dimensional parameter $\bar{S}$, is the ratio of the longitudinal stiffness of the stringers to the longitudinal stiffness of the shell element between the stringers. Similarly, the non-dimensional parameter $\bar{R}$ corresponds to the rings and is written as

$$\begin{aligned} \bar{S} &= \frac{E_s A_s}{Ehd} \\ \bar{R} &= \frac{E_r A_r}{Ehl}. \end{aligned} \tag{10}$$

The $\bar{F}_x$ and $\bar{F}_y$ non-dimensional parameters denote the external static loading of the shell along x and y directions, respectively and are defined as

$$\begin{aligned} \bar{F}_x &= \frac{L^2\bar{N}_x}{D} \\ \bar{F}_y &= \frac{L^2\bar{N}_y}{D}, \end{aligned} \tag{11}$$

where $\bar{N}_x$ is the constant axial force per unit length and $\bar{N}_y$ is the constant circumferential force per unit length, applied along middle surface of the shell. The non-dimensional parameters Ξ_r and Ξ_s express the ratio of the averaged bending stiffness of the rings and the stringers, respectively, to the bending stiffness of the shell. The expression for these parameters can be stated as

$$\begin{aligned} \Xi_r &= \frac{E_r I_r}{Dl} \\ \Xi_s &= \frac{E_s I_s}{Dd}. \end{aligned} \tag{12}$$

In a similar way, the non-dimensional parameters Γ_r and Γ_s express the ratio of the averaged torsional stiffness of the rings and the stringers, respectively, to the bending stiffness of the shell and are defined as

$$\begin{aligned} \Gamma_r &= \frac{G_r J_r}{Dl} \\ \Gamma_s &= \frac{G_s J_s}{Dd}. \end{aligned} \tag{13}$$

The parameters $\bar{E}_r$ and $\bar{E}_s$ are the non-dimensional eccentricity of the rings and the stringers, respectively and can be stated as

$$\begin{aligned} \bar{\mathrm{E}}_r &= \frac{\bar{z}_r}{R} \\ \bar{\mathrm{E}}_s &= \frac{\bar{z}_s}{R}. \end{aligned} \tag{14}$$

The remaining non-dimensional parameters used in the Eq.(6), i.e. Υ, Σ_r, Σ_s, and Σ_{rs} are somehow functions of the non-dimensional parameters defined above. Disregarding the functions φ, ψ_r, ψ_s, and ψ_{rs}, the functional equations of these parameters can be written as

$$\begin{aligned} \Upsilon &= \varphi\left(n, \upsilon, \Lambda, \bar{R}, \bar{S}\right) \\ \Sigma_r &= \psi_r\left(n, \upsilon, \Lambda, \bar{\mathrm{E}}_r\right) \\ \Sigma_s &= \psi_s\left(n, \upsilon, \Lambda, \bar{\mathrm{E}}_s\right) \\ \Sigma_{rs} &= \psi_{rs}\left(n, \upsilon, \Lambda, \bar{\mathrm{E}}_r, \bar{\mathrm{E}}_s\right) \end{aligned} \tag{15}$$

5. Similarity Conditions for Free Vibrations of Stiffened Shells

The frequency equation (Eq.(6)) may be written for model and prototype. By defining scale factor λ_i, the variables of the prototype can be written as

$$x_{ip} = \lambda_i x_{im} \tag{16}$$

where subscripts m and p refer to model and prototype, respectively. The similarity conditions between model and prototype are determined by substitution of $\lambda_i x_{im}$ into the frequency equation of the prototype and by requiring that the result be the frequency equation of the model (complete similarity)[17]. Performing the necessary manipulations, and omitting the redundant conditions, the necessary scaling laws (similarity conditions) for free vibrations of orthogonally stiffened cylindrical shells may be obtained as

$$
\begin{aligned}
&\lambda_n = \lambda_m = \lambda_\upsilon = 1 \\
&\lambda_m \lambda_R \lambda_L^{-1} = 1 \\
&\lambda_{E_s} \lambda_{A_s} \lambda_E^{-1} \lambda_h^{-1} \lambda_d^{-1} = 1 \\
&\lambda_{\bar{M}} \lambda_L^4 \lambda_\omega^2 \lambda_D^{-1} = 1 \\
&\lambda_L^2 \lambda_R^{-1} \lambda_h^{-1} = 1 \\
&\lambda_{E_r} \lambda_{A_r} \lambda_E^{-1} \lambda_h^{-1} \lambda_l^{-1} = 1 \\
&\lambda_L^2 \lambda_{\bar{N}_x} \lambda_D^{-1} = 1 \\
&\lambda_{E_s} \lambda_{I_s} \lambda_D^{-1} \lambda_d^{-1} = 1 \\
&\lambda_{G_s} \lambda_{J_s} \lambda_D^{-1} \lambda_d^{-1} = 1 \\
&\lambda_{\bar{z}_r} \lambda_R^{-1} = 1 \\
&\lambda_L^2 \lambda_{\bar{N}_y} \lambda_D^{-1} = 1 \\
&\lambda_{E_r} \lambda_{I_r} \lambda_D^{-1} \lambda_l^{-1} = 1 \\
&\lambda_{G_r} \lambda_{J_r} \lambda_D^{-1} \lambda_l^{-1} = 1 \\
&\lambda_{\bar{z}_s} \lambda_R^{-1} = 1.
\end{aligned}
\tag{17}
$$

Eqs. (17) are necessary and sufficient conditions for complete similarity between model and prototype. When at least one of the similarity conditions cannot be satisfied, partial similarity is achieved and the model which has some relaxation in similarity conditions is called a distorted model. It is worth noting that the presented form of arranging the scaling laws is not unique.

6. Applying Scaling Laws

6.1. Replica Scaling

Replica scaling means that all geometrical parameters of the prototype are exactly and precisely scaled by a same scale factor; in other words, a replica model is a physical model of the prototype which is geometrically similar in all aspects to the prototype and employs identically the same materials at similar locations [18]. Considering this, and taking the geometrical scale factor equal to k, and by assuming that the BC's and static loading conditions (σ_y, p) are similar for model and prototype, the inputs to the similarity problem are

$$
\begin{aligned}
&\lambda_E = \lambda_{E_s} = \lambda_{E_r} = \lambda_\upsilon = 1 - \\
&\lambda_L = \lambda_R = \lambda_h = \lambda_l = \lambda_d = \lambda_{\bar{z}_r} = \lambda_{\bar{z}_s} = k \\
&\lambda_{\bar{N}_x} = \lambda_{\bar{N}_y} = \lambda_{\bar{M}} = k \\
&\lambda_{A_r} = \lambda_{A_s} = k^2 \\
&\lambda_D = k^3 \\
&\lambda_{I_r} = \lambda_{J_r} = \lambda_{I_s} = \lambda_{J_s} = k^4.
\end{aligned}
\tag{18}
$$

Substituting Eqs. (18) into the similarity conditions, Eqs.(17), yields

$$
\lambda_n = \lambda_m = 1 \ , \ \lambda_\omega = k^{-1}. \tag{19}
$$

It means that if a stiffened cylindrical shell, having uniformly distributed stiffeners is geometrically scaled by a specific scale factor denoted by k, then the ratio of the frequency of any particular mode shape of the model to that of corresponding mode shape of the prototype will be equal to the inverse of geometrical scale factor, i.e., $\frac{1}{k}$.

6.2. Design of Equivalent Cross Section for Stiffeners

The purpose of utilizing the similarity conditions in this section is to design equivalent cross sections for stiffeners of a stiffened shell in such a way that without shell material and geometry, and stiffeners material and distribution, and also without BC's and loadings being changed, the similar mode shapes in model and prototype have the same frequencies. Thus, the inputs to the similarity problem are

$$\begin{aligned} &\lambda_n = \lambda_m = \lambda_\omega = 1 \\ &\lambda_R = \lambda_L = \lambda_h = \lambda_d = \lambda_l = 1 \\ &\lambda_E = \lambda_{E_s} = \lambda_{E_r} = 1 \\ &\lambda_G = \lambda_{G_s} = \lambda_{G_r} = 1 \\ &\lambda_{\bar{N}_y} = \lambda_{\bar{N}_x} = 1 \\ &\lambda_\upsilon = \lambda_D = 1. \end{aligned} \tag{20}$$

For the similarity conditions of Eqs. (17) to hold, the following relations must be satisfied

$$\begin{aligned} &\lambda_{\bar{M}} = 1 \\ &\lambda_{I_s} = \lambda_{J_s} = \lambda_{\bar{z}_s} = \lambda_{A_s} = 1 \\ &\lambda_{I_r} = \lambda_{J_r} = \lambda_{\bar{z}_r} = \lambda_{A_r} = 1. \end{aligned} \tag{21}$$

Considering the expression for $\bar{M}$ in Eqs.(A-4), and for all the relations of Eqs. (21) to be satisfied, it is sufficient that the stiffeners in model and prototype have the same cross section area, moment of inertia, polar moment of inertia, and eccentricity. Therefore, when fabricating the elaborations of the scaled stiffeners' cross section profiles is difficult or impossible, equivalent stiffeners can be used that in spite of having simpler geometry, counterbalance the vibrational effects of the original stiffeners. The simplest geometry that can be used as equivalent cross section is a general T-shaped cross. The geometry of suchlike cross sections is specified by 4 parameters, demonstrated in Fig. 1. Therefore, parameters t, b, w, and h can be obtained through solving a nonlinear system of 4 equations and 4 unknowns.

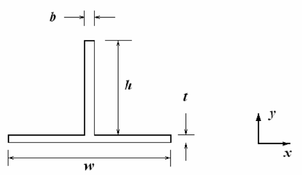


Fig. 1. T-Shaped Equivalent Cross section

6.3. Using Dissimilar Material in Fabricating Both Shell and Stiffeners of the Scale Model

Another problem encountered in fabricating a scaled down stiffened shell arises from stiffeners thickness being extremely small. approach to the problem of small thickness is that dissimilar material having better formability be used in fabricating scale model's shell and stiffeners. Through using such materials, smaller thicknesses will be achievable. The purpose of utilizing the similarity conditions in this section is to change the scale model material in such a way that shell and stiffeners geometry, and stiffeners distribution and also BC's do not undergo any change. Assuming the model and the prototype material having the same Poisson's ratio and also assuming not to have any static loadings, then the inputs to the similarity problem are

$$\begin{aligned} &\lambda_E = \lambda_G = \lambda_{E_r} = \lambda_{G_r} = \lambda_{E_s} = \lambda_{G_s} = k \\ &\lambda_R = \lambda_L = \lambda_h = \lambda_d = \lambda_l = 1 \\ &\lambda_{I_s} = \lambda_{J_s} = \lambda_{\bar{z}_s} = \lambda_{A_s} = 1 \\ &\lambda_{I_r} = \lambda_{J_r} = \lambda_{\bar{z}_r} = \lambda_{A_r} = 1 \\ &\lambda_D = k \\ &\lambda_{\bar{M}} = k' \\ &\lambda_\upsilon = 1, \end{aligned} \tag{22}$$

where k is the ratio of modulus of elasticity in model to that of the prototype, and k' is the ratio of model density (or average mass per area) to that of the prototype. Considering the assumptions above and for the similarity conditions of Eqs. (17) to hold, the following relations must be satisfied

$$\begin{aligned} &\lambda_n = \lambda_m = 1 \\ &\lambda_\omega = k^{\frac{1}{2}} k'^{-\frac{1}{2}}. \end{aligned} \tag{23}$$

Thus, in the cases where fabricating the scaled model is impossible due to extremely small thicknesses, it is possible to use dissimilar materials having better formability in fabricating the model.

7. Validation of the Scaling Laws through Fabricating and Modal Testing of a Stiffened Shell and Its Equivalent Scale Model

For the purpose of experimental validation of the methodology as applied here, two cylindrical shells are fabricated. The first shell denoted as prototype is a stiffened cylindrical shell with the diameter being $1.25m$ and the length of $1.03m$ and the thickness of $1.5mm$, made of aluminum alloy, having 2 equally spaced internal Z-shaped ribs and 8 equally spaced internal Ω-shaped stringers. The second shell denoted as model is a $\frac{1}{3}$-scale model of the prototype, made of alloy steel, having 2 equivalent internal T-shaped ribs and 8 equivalent internal T-shaped stringers. Modal tests are performed both on the model and prototype (Fig. 2). A three cable soft suspension system is used to provide free-free boundary condition for model and prototype. An electromagnetic shaker driven by a power amplifier, and generating a random excitation, is used for exciting the model and prototype into vibration. The response is measured using piezoelectric accelerometers in the radial direction on 68 points of the prototype and the corresponding points of the model. The excitation force is measured by a force transducer and a spectrum analyzer is used to extract frequency response functions (FRFs). The form of measured FRFs is accelerance.

a) Modal testing of the model   b) Modal testing of the prototype

Fig. 2. Modal testing using an electrodynamic shaker

Modal parameters of 6 primary modes are extracted through curve-fitting the set of 68 measured FRFs for either of model and prototype. Having the natural frequencies of the model obtained, those of the prototype are predicted using the similarity relations. The predicted natural frequencies using similarity conditions are compared with those extracted experimentally through modal testing of the prototype and the results for the 6 primary modes are listed in Table 1. It is clear from Table 1 that similitude theory as used here yields an excellent accuracy. Therefore it is possible to predict the natural frequencies of a stiffened shell with a good accuracy through designing a small scale stiffened shell, having equivalent stiffeners with simpler cross section profile, made of dissimilar material by means of similitude theory. Referring to Table 1, it can be seen that there exist modes having the same wave numbers, but different frequencies, both in model and prototype. The reason is that for the mode shapes where n=4 the frequency of symmetric and antisymmetric mode shapes differs as a result of the number of stringers being 8 [19] [20]. As seen before, in the case of complete similarity conditions being satisfied, the non-dimensional frequency parameter defined as below

$$\bar{\Omega} = \omega\sqrt{\frac{\bar{M}L^4}{D}} \tag{24}$$

will have the same value in model and prototype. Thus if the frequency response functions of model and prototype is plotted versus non dimensional frequency, both of the frequency response functions should yield similar behavior. Therefore the vibration response of the model and the prototype can be compared throughout the frequency range of excitation for each of measuring points. One of such plots, comparing the frequency response functions of a distinct points on the prototype and its corresponding points on the model, is presented in Fig. 3. It is worth noting that this point on the prototype and its counterpoint on the model are located at similar positions with respect to the excitation point.

Table 1. Results of prototype modal testing compared to those predicted by similarity.

Mode No.	n	m	Prototype Freq. by Test	Model Freq. by Test	Prototype Freq. by Similarity	Error
			(Hz)	(Hz)	(Hz)	(%)
1	3	2	20.8223	61.8732	20.201	-2.98
2	3	1	35.5957	109.8861	35.877	0.79
3	4	2	57.269	163.0589	53.238	-7.03
4	4	2	62.9106	190.0987	62.066	-1.34
5	4	1	91.7844	272.0469	88.822	-3.22
6	4	1	100.5191	301.168	98.330	-2.17

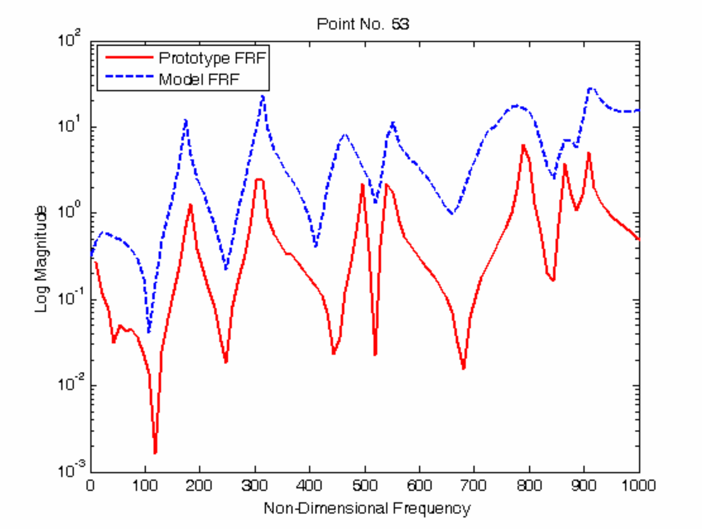
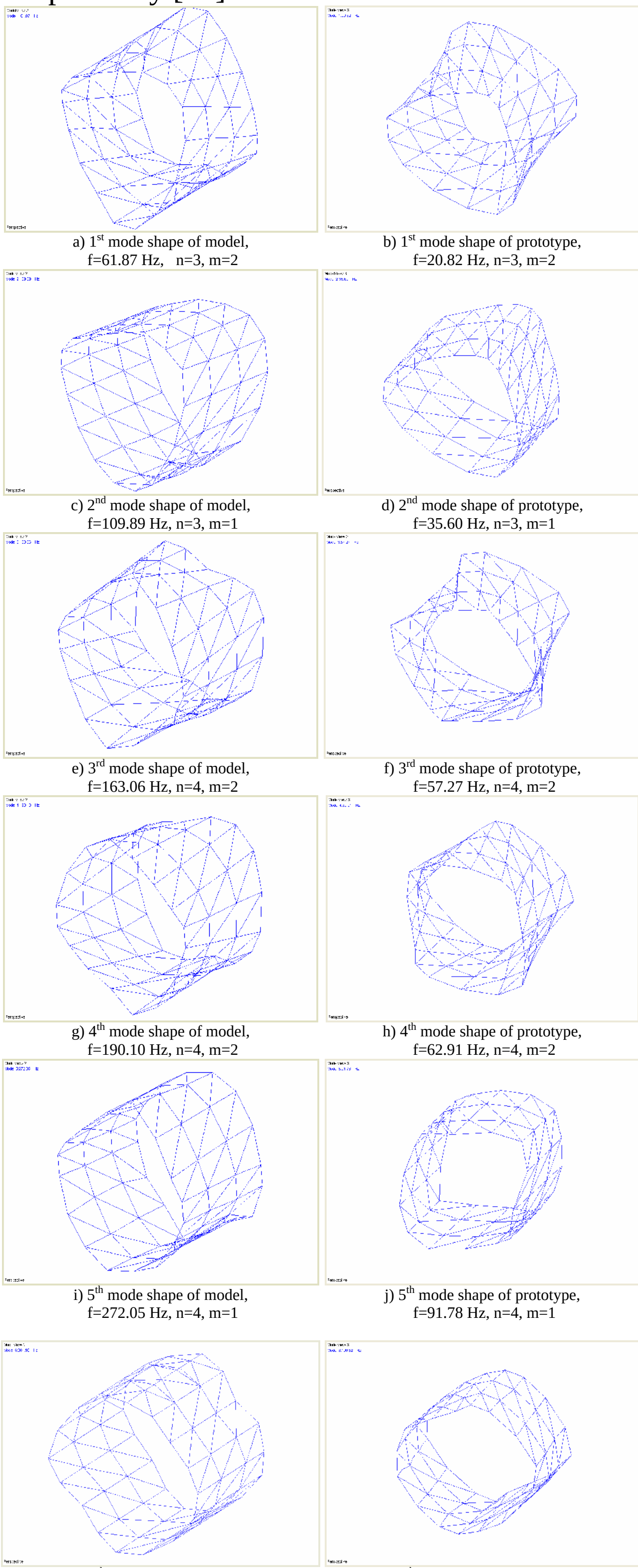


Fig. 3. Frequency response functions of a sample point (No. 53) on the prototype and its counterpoint on the Model.

As shown in Figs. 3, the overall behaviour of the model and the prototype closely resemble each other, but in some cases such as third mode there exists a little inconsistency in the location of modal peaks. The first 6 mode shapes of model and prototype are compared qualitatively in Fig. 4. In order to quantify the comparison between measured mode shapes of model and prototype, "Modal Assurance Criterion" (MAC) is used. Originally designed as a means of quantifying the degree of correlation between mode shapes from experimental measurement and computer simulation [21], MAC can serve for this purpose. For comparing measured mode shapes of model and prototype, MAC can be written as

$$\left[MAC\left(\{\phi_M\}_r,\{\phi_P\}_s\right)\right]=\left[\frac{\left|\{\phi_M\}_r^T\{\phi_P\}_s\right|^2}{\{\phi_M\}_r^T\{\phi_M\}_r\{\phi_P\}_s^T\{\phi_P\}_s}\right]_{m\times m} \quad (23)$$

where m is the number of vibration modes measured and $\{\phi_M\}_r$ is the rth mode shape of the model and $\{\phi_P\}_s$ is the sth mode shape of the prototype. Although no precise value is prescribed in references which the assurance criterion should take in order to guarantee good results, but it is found that a value in excess of 0.9 should be attained for well correlated mode shapes and a value of less than 0.1 for uncorrelated modes. In some situations the boundaries for "accepted" and "non" correlation are quoted as above 0.8 and less than 0.2, respectively [22].



a) 1st mode shape of model, f=61.87 Hz, n=3, m=2
b) 1st mode shape of prototype, f=20.82 Hz, n=3, m=2
c) 2nd mode shape of model, f=109.89 Hz, n=3, m=1
d) 2nd mode shape of prototype, f=35.60 Hz, n=3, m=1
e) 3rd mode shape of model, f=163.06 Hz, n=4, m=2
f) 3rd mode shape of prototype, f=57.27 Hz, n=4, m=2
g) 4th mode shape of model, f=190.10 Hz, n=4, m=2
h) 4th mode shape of prototype, f=62.91 Hz, n=4, m=2
i) 5th mode shape of model, f=272.05 Hz, n=4, m=1
j) 5th mode shape of prototype, f=91.78 Hz, n=4, m=1
k) 6th mode shape of model, f=301.16 Hz, n=4, m=1
l) 6th mode shape of prototype, f=100.52 Hz, n=4, m=1

Fig. 4. Mode shape qualitative comparison.

The MAC matrix so obtained, is presented in Table 2. As cleared from this table, the first 6 mode shape of the model are most correlated with the first 6 mode shapes of the prototype, respectively and the rate of correlation is about 90 percent except for the first mode that this rate is about 85 percent. As mentioned in reference [20], there exist some causes of less-than-perfect results other than uncorrelated mode shapes. One of them is the existence of nonlinearity in the test structure. In this case, since the less-than-perfect value is occurred in the first mode, the reason probably is the existence of nonlinearity as a result of large amplitude.

Table 2. MAC as a means of comparing mode shapes of model and prototype.

		Prototype Mode Shapes					
		1	2	3	4	5	6
Model Mode Shapes	1	0.8536	0.0041	0.0115	0.0115	0.0074	0.0032
	2	0.0021	0.9242	0.0020	0.0020	0.0013	0.0021
	3	0.0009	0.0011	0.9296	0.0060	0.0376	0.0140
	4	0.0082	0.0007	0.0060	0.9296	0.0112	0.0110
	5	0.0031	0.0004	0.0237	0.0237	0.8976	0.0039
	6	0.0059	0.0109	0.0214	0.0214	0.0019	0.9123

8. Conclusions

In this paper, free vibrations of stiffened shells having longitudinal and circumferential stiffeners, is investigated using scaled down equivalent model of the structure and through applying similitude theory. The necessary similarity conditions, or scaling laws, in free vibrations of orthogonally stiffened cylindrical shells are developed through nondimensionalizing the governing equations. Based on the similarity condition, the relation between natural frequencies of a replica scale model and those of its prototype is obtained. In order to overcome the difficulties encountered in fabricating small scaled stiffeners, design of equivalent cross section is explored that in spite of having much simpler profile, offer same vibrational effect in free vibration of stiffened shell. The results are also confirmed experimentally by designing and fabricating a scaled model of a typical prototype, having equivalent stiffeners, made of dissimilar material and performing modal tests to measure modal parameters of the model and the prototype. It is finally concluded that, scaling laws provide precise relationship between a full-scale structure and its small-scale equivalent model, and can be used to extrapolate the experimental data of a small inexpensive and testable model into design information for a large prototype.

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Appendix: Derivation of Nonlinear Equations of Motion and BC’s

Consider a thin-walled circular cylinder, stiffened by evenly spaced uniform rings and/or stringers. The only applied loadings considered in the analysis are an axial end load and a constant internal or external pressure load. It is assumed that the stiffeners spacing is small, so that the stiffener effect on the behavior of the structure may be averaged or smeared out.

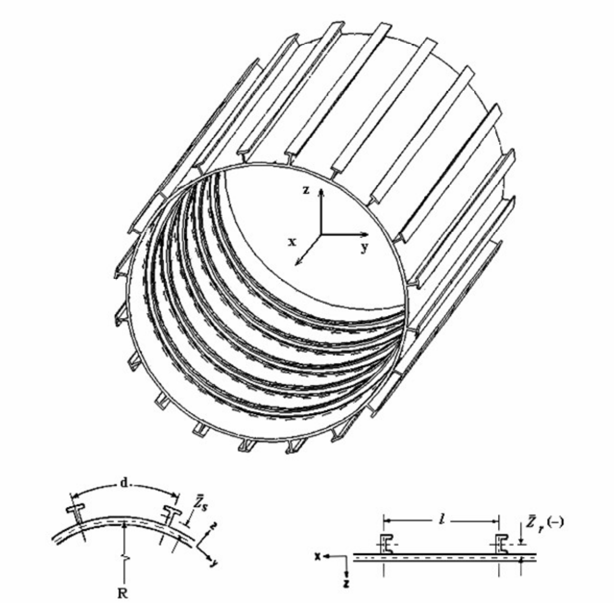

Fig. 5. Geometry of orthogonally stiffened cylindrical shell

The Donnell-type nonlinear strain-displacement relations are [23]

$$\begin{aligned}\varepsilon_{xT} &= u_{,x} + \frac{1}{2}w_{,x}^2 - zw_{,xx}\\ \varepsilon_{yT} &= v_{,y} + \frac{w}{R} + \frac{1}{2}w_{,y}^2 - zw_{,yy}\\ \gamma_{xyT} &= u_{,y} + v_{,x} + w_{,x}w_{,y} - 2zw_{,xy}\end{aligned} \tag{A-1}$$

where $\varepsilon_{xT}, \varepsilon_{yT}$, and γ_{xyT} are the total strains, comparing to $\varepsilon_x, \varepsilon_y$, γ_{xy} which are the strains in the middle surface $(z = 0)$ of the cylinder wall. The quantities u, v, w are the displacements of the middle surface in x, y, z directions, respectively and R is the cylinder radius (Fig. 5). The total strain energy of the stiffened cylinder, U, consists of the following terms

$$U = U_c + U_s + U_r + U_L + U_\omega \tag{A-2}$$

where U_c is the strain energy of a non-stiffened thin-walled isotropic cylinder and is obtained through substituting the above nonlinear strain-displacement relations into the general strain energy equation and using Hook's law. U_s is the strain energy due to extension, bending, and torsion of stringers of spacing d attached to the shell and is obtained from the assumption of continuity of displacement in the cylinder and stringers and through averaging the effect of stringers over the circumference. U_r is the strain energy of rings spacing l and is obtained in a similar way as U_s. U_L is the potential energy due to a constant axial load resultant $\bar{N}_x$ (positive in compression) applied along the middle surface of isotropic shell, and a constant hydrostatic pressure p (positive externally) applied over the shell. If u, v, and w are amplitudes of a simple harmonic motion with circular frequency ω, then U_ω is the potential energy of inertia loading at maximum deflection and neglecting in-plane inertia. Substituting terms of Eq. (A-2), the total energy at maximum deflection of the stiffened shell can be written in terms of stress and moment resultants as

$$\begin{aligned}U = \frac{1}{2}\int_0^{2\pi R}\int_0^L (&N_x\varepsilon_x + N_y\varepsilon_y + N_{xy}\gamma_{xy} - M_x w_{,xx}\\ &- M_y w_{,yy} + M_{xy}w_{,xy} - M_{yx}w_{,xy})\,dxdy\\ &+ \int_0^{2\pi R}\int_0^L \left(pw - \frac{1}{2}\bar{M}\omega^2 w^2\right)dxdy\\ &+ \int_0^{2\pi R}\bar{N}_x\,(u_{x=L} - u_{x=0})\,dy\end{aligned} \tag{A-3}$$

where M_x, M_y, and M_{xy} are the moment resultants per unit length and N_x, N_y, and N_{xy} are the stress resultants per unit length, defined as follows [24]

$$\begin{aligned}N_x &= \left[Eh/(1-\upsilon^2)\right]\left[u_{,x} + \frac{1}{2}w_{,x}^2 + \upsilon\left(v_{,y} + \frac{w}{R} + \frac{1}{2}w_{,y}^2\right)\right]\\ &\quad + \left(E_s A_s/d\right)\left(u_{,x} + \frac{1}{2}w_{,x}^2 - \bar{z}_s w_{,xx}\right)\\ N_y &= \left[Eh/(1-\upsilon^2)\right]\left[v_{,y} + \frac{w}{R} + \frac{1}{2}w_{,y}^2 + \upsilon\left(u_{,x} + \frac{1}{2}w_{,x}^2\right)\right]\\ &\quad + \left(E_r A_r/l\right)\left(v_{,y} + \frac{w}{R} + \frac{1}{2}w_{,y}^2 - \bar{z}_r w_{,yy}\right)\\ N_{xy} &= Gh\left(u_{,y} + v_{,x} + w_{,x}w_{,y}\right)\\ M_x &= -\Big[D\left(w_{,xx} + \upsilon w_{,yy}\right) + \left(E_s I_s/d\right)w_{,xx}\\ &\quad - \bar{z}_s\left(E_s A_s/d\right)\left(u_{,x} + \frac{1}{2}w_{,x}^2 - \bar{z}_s w_{,xx}\right)\Big]\\ M_y &= -\Big[D\left(w_{,yy} + \upsilon w_{,xx}\right) + \left(E_r I_r/l\right)w_{,yy}\\ &\quad - \bar{z}_r\left(E_r A_r/l\right)\left(v_{,y} + \frac{w}{R} + \frac{1}{2}w_{,y}^2 - \bar{z}_r w_{,yy}\right)\Big]\\ M_{xy} &= \left(Gh^3/6 + G_s J_s/d\right)w_{,xy}\\ M_{yx} &= -\left(Gh^3/6 + G_r J_r/l\right)w_{,xy}\\ I_s &= I_{0s} - \bar{z}_s^2 A_s\\ I_r &= I_{0r} - \bar{z}_r^2 A_r\\ D &= \frac{Eh^3}{12\left(1-\upsilon^2\right)}\\ \bar{M} &= \rho_c h + \rho_s\left(A_s/d\right) + \rho_r\left(A_r/l\right)\end{aligned} \tag{A-4}$$

where J_s and I_s are respectively moment of inertia and polar moment of inertia of stringers with respect to an axis parallel to the middle surface of the shell passing through its centroid. A_s is cross sectional area and $G_s J_s$ is torsional stiffness of stringers. D is flextural stiffness of the cylinder shell. $\bar{z}_s$ is the distance from shell middle surface to the centroid of stringers and I_{0s} is the moment of inertia of stringers with respect to the shell middle surface. $\bar{M}$ is the averaged or smeared out mass per unit area of the stiffened cylinder. h is the thickness of the shell and ρ_c and ρ_s are mass density of the stringers and the shell, respectively. The parameters having subscript r are similar parameters corresponding to the rings. It is worth noting that parameters $\bar{z}_r$ and $\bar{z}_s$ can have negative values, too (for stiffeners on the inner surface of the shell).

Using Eq.(A-3) and by application of the principle of minimum potential energy

$\left(\delta U = 0\right)$, the nonlinear equations of motion of stiffened cylinder and BC's are obtained as

$$\begin{aligned} &N_{x,x} + N_{xy,x} = 0 \\ &N_{y,y} + N_{xy,y} = 0 \\ &-M_{x,xx} + M_{xy,xy} - M_{yx,xy} - M_{y,yy} + N_y/R - N_x w_{,xx} \\ &\qquad - N_y w_{,yy} - 2N_{xy} w_{,xy} - \bar{M}\omega^2 w + p = 0. \end{aligned} \qquad \text{(A-4)}$$

A set of BC's (essential and natural BC's) to be satisfied at each end of the cylinder are

$$\begin{aligned} &M_{x,x} - \left(M_{xy,y} - M_{yx,y}\right) + N_x w_{,x} + N_{xy} w_{,y} = 0 && or \;\; w = 0 \\ &M_x = 0 && or \;\; w_{,x} = 0 \\ &N_x + \bar{N}_x = 0 && or \;\; u = 0 \\ &N_{xy} = 0 && or \;\; v = 0 \end{aligned} \qquad \text{(A-5)}$$

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