THROUGH OPTIMIZATION OF BRANCHED TRAJECTORIES WITH RANDOM DISTURBANCES BY THE PONTRYAGIN MAXIMUM PRINCIPLE

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Abstract

The problem of through optimization of branched spatial trajectories of space transportation systems (STS) is considered in view of atmospheric disturbances (a wind and variations of thermodynamic parameters) and restrictions on admissible areas of dispersion of separated parts fall points. The maximized criterion is the mass injected into an orbit. The Pontryagin maximum principle is applied to solve the problem. The influence of stochastic disturbances is defined on the basis of the "minimax" approach at which the worst random parameters values of the set, corresponding to a considered probability level, are determined, maximizing the dispersion of fall points. The disturbances influence on the trajectories is supposed small so that the Bliss formula is applicable for its estimation. In view of complexity of the used mathematical model of motion the much importance is attached to a multilevel system of verification of the developed program complex and assurance of objective control of accuracy of obtained optimal solutions. Described ways of the verification have the universal character as they base on the use of known integrals of the motion and physical sense of conjugate variables as influence functions. All stages of the carried out research, from the verification up to the problem solution, are illustrated by corresponding numerical results.

1 Introduction

The necessity of considering branched trajectories appears in researching the motion of compound aircraft (staged rockets, airplanes-carriers etc.), airdrops of loads or missile starts. At the same time, branched trajectories can be inserted artificially as a convenient virtual image to take into account simultaneously all phases of multi-purpose flying missions, different scripts of rescue in emergency situations, or a set of variants subjected to random disturbances, etc.

Among optimization methods the Pontryagin maximum principle (PMP) [1] seems preferential for the effective solution of this type of practical problems due to following essential advantages over direct optimization methods and approximate engineering approaches:

- The PMP does not require a priori definition of the optimal control law structure, as it results from the problem solution;
- The PMP determines the optimal control in a function space with a high accuracy solving the two-point boundary value problem with a small number of parameters (does not exceed the state space dimension);
- First integrals of state and conjugate systems as well as the physical meaning of conjugate variables as influence functions allow to form criteria of the objective software test, calculation accuracy control and, if necessary, to reveal the source of errors in blocks forming the optimal control, boundary conditions, and conjugate system;
- The PMP solution of the conjugate system allows defining an influence on the functional of small variations of problem parameters (characterizing aerodynamic layout, propulsion system, boundary conditions, constraints, etc.) and small systematic and random external disturbances

(variations of atmosphere thermodynamic parameters, a wind, etc.) practically without additional computing expenditures.

Branched trajectories optimization methods using the PMP are based in [2-4]. The automated program package realizing this approach both for qualitative studies of the optimal solutions and for the practical analysis of particular aircraft is submitted in [5]. Its efficiency for the optimal reduction of fall zones for aerospace transport systems is demonstrated in [6]. In [7] the PMP is used for synthesis of STS optimal fail-safety trajectories. In this case the trajectory side branches represent virtual images of aircraft return trajectories in emergency situations. Since failures can happen in any point of the main trajectory, side branches generate the continuum.

This paper considers two groups of branched trajectories. The first group corresponds to real physical processes, associated, for example, with separation of some parts from aircraft. The second is virtual one including possible trajectories under the influence of different random factors, for example, atmospheric disturbances. The second group of branches can contain either a finite set of trajectories, if just extreme disturbances appropriated to an assigned level of the event probability (the “minimax” approach) are taken into account, or the continuum (tubes of trajectories), if all random realizations are considered (the Monte-Carlo technique).

Numerical results are demonstrated below in the framework of the "minimax" approach. Thus the optimal control provides the maximum of the functional (the injected mass). Atmospheric disturbances are set on unguided segments of separated parts return branches to maximize the area of fall points dispersion, thus creating the worst conditions to meet the relevant boundary restrictions: the dispersion area has to place into admissible fields (see Figure 1).

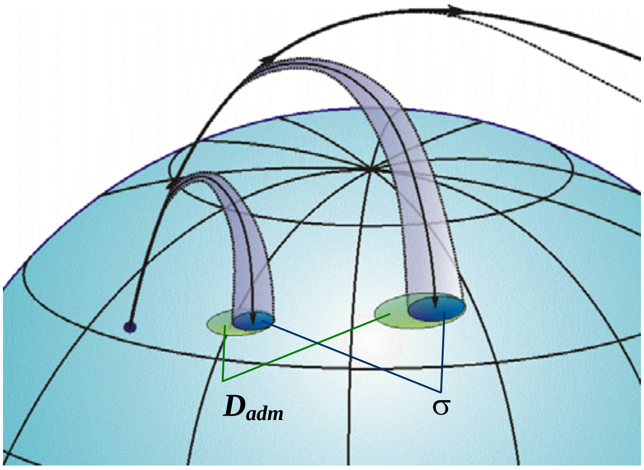

Figure 1. The scheme of considered optimal branched trajectories.

2 Problem Statement

It is supposed, that the aircraft motion on each trajectory branch is described by a normal system of ordinary differential equations [13]:

$$\frac{d\mathbf{x}}{dt}=\mathbf{f}(\mathbf{x},\mathbf{u},\boldsymbol{\varepsilon},t),\ \mathbf{f}=\left\{\mathbf{v},\ \frac{\mathbf{T}+\mathbf{A}}{m}+\mathbf{g}+\boldsymbol{\Omega},\ -\mu\right\},(1)$$

where $\mathbf{x}=\{\mathbf{r},\mathbf{v},m\}^{\mathrm{T}}\in\mathbf{X}$ is the state vector, $\mathbf{r}$ is the radius vector, $\mathbf{v} = V\mathbf{e}_{\mathrm{v}}$ is the velocity vector, $\mathbf{e}_{\mathrm{v}}$ is the unit velocity vector, m is the mass, $\mathbf{f}$ is the right part vector, $\mathbf{u}\in\mathbf{U}$ is the control vector, $\boldsymbol{\varepsilon}\in\mathbf{D}$ is the disturbance vector, $t\in\mathbf{T}^{j}=\left(t_i,t_f\right)$ is the time, $\mathbf{g}$ is the gravity acceleration vector, $\boldsymbol{\Omega}=(\boldsymbol{\omega}\times\mathbf{R}+2\mathbf{v})\times\boldsymbol{\omega}$ is the acceleration vector due to coordinate system noninertiality, $\boldsymbol{\omega}$ is the Earth rate vector, $\mathbf{R}=\mathbf{r}+\mathbf{R}_{\mathrm{o}}$ is the vector from Earth center, $\mathbf{R}_{\mathrm{o}}$ is the vector from Earth center to the origin point, μ is the mass flow rate, $\mathbf{T}=T\cdot\mathbf{e}_{\mathrm{T}}$ is the thrust vector, $\mathbf{e}_{\mathrm{T}}$ is the thrust unit vector, $\mathbf{A}$ is the aerodynamic force vector:

$$\mathbf{A}=a_r\rho V^2\left(C_L^{\alpha}\mathbf{e}_{\tau}-\left(D_0+\left(C_L^{\alpha}+D_{\alpha}\right)(\mathbf{e}_{\tau},\mathbf{e}_{\mathrm{v}})\right)\mathbf{e}_{\mathrm{v}}\right),(2)$$

a_r is the parameter of disdimensiality, ρ is the atmosphere density, $\mathbf{e}_{\tau}$ is the vehicle longitudinal unit vector. The following form for aerodynamic coefficients is used [2]:

$$C_L = C_L^{\alpha}\sin\alpha,\quad C_D = D_0 + D_{\alpha}\cos\alpha\,, \quad (3)$$

where α is the angle of attack. The coefficients C_L^{α}, D_0 and D_{α} depend on the Mach number M and the altitude h.

The boundary conditions as well as matching conditions are specified in initial, intermediate and final points of the branches [4]. Boundary conditions also can depend on random factors.

It is required to find such control functions $\mathbf{u}_{opt} = \{\mathbf{e}_{\tau}, \mathbf{T}\}$ on all branches that the functional of the problem, the inserted-into-orbit mass, reaches the maximum at all limitations, including boundary conditions, on all set of perturbed trajectories. In particular, it is required that the areas σ of dispersion of separated part fall points belong to a set of admissible fields D_{adm} (see Figure 1):

$$\sigma \subset D_{\text{adm}}\,. \quad (4)$$

The Pontryagin maximum principle for branching processes is used to solve this problem. The state and conjugate systems are written taking into account only systematic disturbances, such as seasonal deviations of atmosphere thermodynamic parameters and a systematic latitude wind. Random disturbances are used at formation of terminal requirements and limitations on the control and flight regimes.

It is supposed, that random disturbances of the trajectory are small, so their influence on some function $L(\mathbf{x}_f)$ of the state vector $\mathbf{x}_f = \mathbf{x}(t_f)$ at the branch right end can be estimated by the Bliss formula [8]:

$$\delta L = (\boldsymbol{\psi}, \delta\mathbf{x})_i + \int_{t_i}^{t_f} (\boldsymbol{\psi}, \delta_{\varepsilon}\mathbf{f})dt, \qquad \delta_{\varepsilon}\mathbf{f} = \mathbf{f}(\mathbf{x},\mathbf{u},\varepsilon,t) - \mathbf{f}(\mathbf{x},\mathbf{u},0,t) \quad (5)$$

Here, $\boldsymbol{\psi}$ is the conjugate vector obtained from the solution of the Cauchy problem:

$$\dot{\boldsymbol{\psi}} = -\frac{\partial \mathbf{f}}{\partial \mathbf{x}}^{\mathrm{T}}\boldsymbol{\psi},\quad \boldsymbol{\psi}^{\mathrm{T}}(t_f) = \nabla L(\mathbf{x}_f) - \frac{\dot{L}(\mathbf{x}(t_f))}{\dot{G}(\mathbf{x}(t_f))}\nabla G(\mathbf{x}_f), \quad (6)$$

where $G(\mathbf{x}(t_f)) = 0$ is the condition of implicit determination of the right end of the trajectory (the branch). The coordinates of the conjugate vector corresponding to the radius vector, velocity and mass are denoted $\mathbf{P}$, $\mathbf{S}$ and P_m, so $\boldsymbol{\psi} = \{\mathbf{P}, \mathbf{S}, P_m\}^{\mathrm{T}}$. The vectors $\mathbf{x}, \mathbf{u}$ in (5), (6) correspond to the optimal solution of the problem (1) - (4) without random disturbances ($\boldsymbol{\varepsilon} = 0$).

3 Models of Disturbances

The following atmosphere disturbances are considered: the density ρ, pressure p and sound speed a_s, and a horizontal wind with velocity vector $\mathbf{W}$.

In indignations we shall allocate regular and casual components:

$$\rho = \rho_{sys} + \Delta\rho\,,\ a_s = a_{sys} + \Delta a\,,$$
$$\mathbf{W} = \mathbf{W}_{sys} + \Delta\mathbf{W}\,,\ \Delta\mathbf{W} = \Delta W_{\lambda}\mathbf{e}_{\lambda} + \Delta W_{\varphi}\mathbf{e}_{\varphi}\,, \quad (7)$$

where ΔW_{λ} is the longitude projection (in the east direction), ΔW_{φ} is the meridional projection (to the north), $\mathbf{e}_{\lambda}$ and $\mathbf{e}_{\varphi}$ are unit vectors in corresponding directions.

The random components are defined in the form of a canonical decomposition:

$$\begin{aligned}
\frac{\Delta\rho}{\rho} &= \kappa_1(\varphi)\kappa_2(T)\sum_{j=1}^{k}\xi_j b_j^{\rho}(h),\\
\frac{\Delta a}{a_s} &= \kappa_1(\varphi)\kappa_2(T)\sum_{j=1}^{k}\xi_j b_j^{a}(h),\\
\Delta W_{\lambda} &= \kappa_3(\varphi)\kappa_4(T)\sum_{j=k+1}^{k+m}\xi_j b_j^{W}(h),\\
\Delta W_{\varphi} &= \kappa_3(\varphi)\kappa_4(T)\sum_{j=k+1}^{k+m}\xi_{j+m} b_j^{W}(h),
\end{aligned} \quad (8)$$

where $\xi = (\xi_1,\ldots,\xi_k,\ldots,\xi_{k+2m})^{\mathrm{T}}$ is a vector of independent random numbers distributed under the central normal law with the unit dispersion, φ is the latitude, T is the day of year.

To describe the disturbances of aerodynamic coefficients they usual use the uniform distribution law. Variations of the state vector at separation points (just after), apparently, can be described by the normal distribution law.

4 Separated Part Trajectories

Atmospheric disturbances of the right parts of the equations (1) for passive motion of aircraft can be presented as follows:

$$\delta_\varepsilon \mathbf{A} = \mathbf{A}_\rho \frac{\Delta\rho}{\rho} + \mathbf{A}_a \frac{\Delta a}{a} + \mathbf{A}_W \Delta\mathbf{W}\,, \tag{9}$$

where

$$\mathbf{A}_\rho = \mathbf{A},\ \mathbf{A}_a = a_r \rho W^2 M \Bigg\{ -\frac{\partial C_L^\alpha}{\partial M}\mathbf{e}_\tau + \left[\frac{\partial D_0}{\partial M} + (\mathbf{e}_\tau, \mathbf{e}_W)\frac{\partial (C_L^\alpha + D_\alpha)}{\partial M}\right]\mathbf{e}_W \Bigg\},$$

$$\mathbf{A}_W = -a_r \rho W^2 \Big\{ \big[D_0 + (C_L^\alpha + D_\alpha)(\mathbf{e}_\tau, \mathbf{e}_W)\big]\mathbf{E} + (C_L^\alpha + D_\alpha)\mathbf{e}_W \mathbf{e}_\tau^T + \left[\left[D_0 + \left(\frac{\partial D_0}{\partial M} + (\mathbf{e}_\tau, \mathbf{e}_W)\frac{\partial (C_L^\alpha + D_\alpha)}{\partial M}\right)M\right]\mathbf{e}_W \mathbf{e}_W^T - \left(2C_L^\alpha + \frac{\partial C_L^\alpha}{\partial M}M\right)\mathbf{e}_\tau \mathbf{e}_W^T\right]\Big\}.$$

Variations of aerodynamic parameters C_L^α D_0, D_α cause the following change of the aerodynamic vector:

$$\delta_C \mathbf{A} = \mathbf{A}_{C_L^\alpha}\frac{\Delta C_L^\alpha}{C_L^\alpha} + \mathbf{A}_{D_0}\frac{\Delta D_0}{D_0} + \mathbf{A}_{D_\alpha}\frac{\Delta D_\alpha}{D_\alpha}, \tag{10}$$

$$\mathbf{A}_{C_L^\alpha} = a_r \rho W^2 C_L^\alpha \left[\mathbf{e}_\tau - (\mathbf{e}_\tau, \mathbf{e}_W)\mathbf{e}_W\right],$$

$$\mathbf{A}_{D_0} = -a_r \rho W^2 D_0 \mathbf{e}_W,$$

$$\mathbf{A}_{D_\alpha} = -a_r \rho W^2 D_\alpha (\mathbf{e}_\tau, \mathbf{e}_W)\mathbf{e}_W.$$

A projection of displacement of the fall point from nominal one onto a unit vector $\mathbf{e}_L$, belonging the local horizontal plane, is determined by the Bliss formula (5) for random disturbances (8), (9):

$$\Delta L = \sum_{j=1}^{k+2m} L_{\xi j}\xi_j,$$

$$L_{\xi j} = \frac{\kappa_2(T)}{m}\int_{t_i}^{t_f}\left[(\mathbf{S}, \mathbf{A}_\rho)b_j^\rho(h) + (\mathbf{S}, \mathbf{A}_a)b_j^a(h)\right]\kappa_1(\varphi)dt,$$

$$j = 1, \ldots, k, \tag{11}$$

$$L_{\xi j} = \frac{\kappa_4(T)}{m}\int_{t_i}^{t_f}(\mathbf{S}, \mathbf{A}_W \mathbf{e}_\lambda)b_j^W(h)\kappa_3(\varphi)dt,$$

$$j = k+1, \ldots, k+m,$$

$$L_{\xi j+m} = \frac{\kappa_4(T)}{m}\int_{t_i}^{t_f}(\mathbf{S}, \mathbf{A}_W \mathbf{e}_\varphi)b_j^W(h)\kappa_3(\varphi)dt,$$

$$j = k+1, \ldots, k+m.$$

The influence of variations of aerodynamic parameters C_L^α, D_0 and D_α on fall point dispersion is defined by (5), (10) as follows:

$$\Delta_C L = \frac{1}{m}\left[\int_{t_i}^{t_f}(\mathbf{S}, \mathbf{A}_{C_L^\alpha})\frac{\Delta C_L^\alpha}{C_L^\alpha}dt + \int_{t_i}^{t_f}(\mathbf{S}, \mathbf{A}_{D_0})\frac{\Delta D_0}{D_0}dt + \int_{t_i}^{t_f}(\mathbf{S}, \mathbf{A}_{D_\alpha})\frac{\Delta D_\alpha}{D_\alpha}dt\right]. \tag{12}$$

The variation of the initial state vector (at the separation point) causes the following deviation of fall point in the chosen direction $\mathbf{e}_L$:

$$\Delta_i L = (\mathbf{S}, \Delta\mathbf{V})_i + (\mathbf{P}, \Delta\mathbf{r})_i + (P_m \Delta m)_i\,. \tag{13}$$

In formulae (11) - (13) the solution of the conjugate system (6) is used with the following transversality conditions on the right end (at the fall point):

$$\mathbf{P}_f = \mathbf{e}_L - \frac{(\mathbf{e}_L, \mathbf{e}_V)}{(\mathbf{e}_R, \mathbf{e}_V)}\mathbf{e}_R,\ \mathbf{S}_f = 0, \tag{14}$$

where $\mathbf{e}_R$ is the unit vector along $\mathbf{R}$.

To obtain the longitudinal range increment it is necessary to set in (14)

$$\mathbf{e}_L = \frac{(\mathbf{e}_{Ro} \times \mathbf{e}_{Rf}) \times \mathbf{e}_{Rf}}{\left|(\mathbf{e}_{Ro} \times \mathbf{e}_{Rf}) \times \mathbf{e}_{Rf}\right|},$$

and for definition of an increment of the lateral range:

$$\mathbf{e}_L = \frac{\mathbf{e}_{Ro} \times \mathbf{e}_{Rf}}{\left|\mathbf{e}_{Ro} \times \mathbf{e}_{Rf}\right|}.$$

Let us define the maximal deviation from the nominal fall point in the direction $\mathbf{e}_L$ as

$$\Delta L_{\max} = \max_{\xi \in D \subset R^{k+2m}} \Delta L(\xi)\,. \qquad (15)$$

If the set D is a hypersphere $D = D_s = \left\{ \xi : \|\xi\| \overset{\Delta}{=} \sqrt{\sum_{j=1}^{k+2m} \xi_j^2} \le \kappa_\sigma \right\}$ limited to a surface with equal probability density, equations (11), (15) lead us to the “worst” (according to the “minimax” approach) value of the random vector and the maximum fall point deviation:

$$\xi_{\max} = \frac{\kappa_\sigma}{\|L_\xi\|} L_\xi\,, \quad \Delta L_{\max} = \Delta L(\xi_{\max}) = \kappa_\sigma \|L_\xi\|\,. \qquad (16)$$

If the set D is a hypercube

$$D = D_c = \{\xi : |\xi_j| \le \kappa_\sigma,\ j = 1, \ldots, k+2m\},$$

then

$$\xi_{j\max} = \kappa_\sigma \operatorname{sign} L_{\xi j}\,, \quad \Delta L_{\max} = \kappa_\sigma \sum_{j=1}^{k+2m} |L_{\xi j}|\,. \qquad (17)$$

In Figure 2 calculation results of the maximum fall point deviation of the first-stage booster due to the random wind by the above mentioned technique are shown at $\kappa_\sigma = 1$. At the moment of the booster separation the following trajectory parameters were set: $V = 1669$ m/s, $h = 49468$ m, $\theta = 33.57°$ for the optimal trajectory with the gravitational turn control program on the first-stage atmospheric flight and $V = 1780.1$ m/s, $h = 39362$ m, $\theta = 25.25°$ for the optimal atmospheric flight. Spread of trajectory parameters at the separation moment and aerodynamic coefficients was not taken into account here.

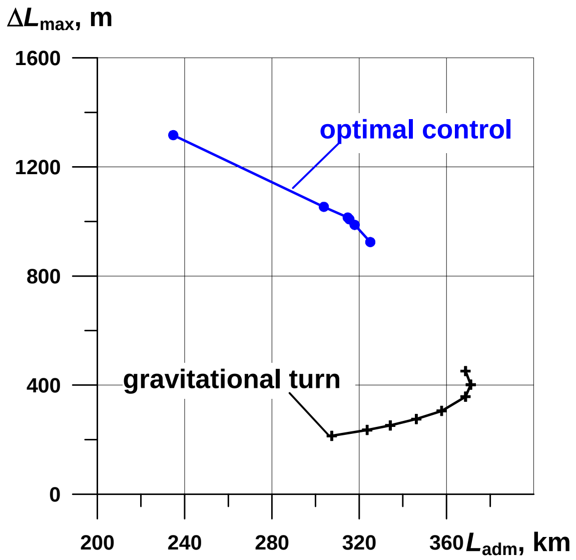

Figure 2. The maximum deviation of first-stage booster fall points due to the random wind as a function of admissible fall area range L_{adm} for two first stage ascent programs: the gravitational turn and the optimal.

Let the nominal motion of a separated part occurs with zero effective lift. Then, in a case when ΔL is the increment of the longitudinal range the conjugated vector $\mathbf{S}$ in (11) can be approximately determined by known integrals of motion in the Newtonian central gravitational field [9] (in the inertial coordinate system):

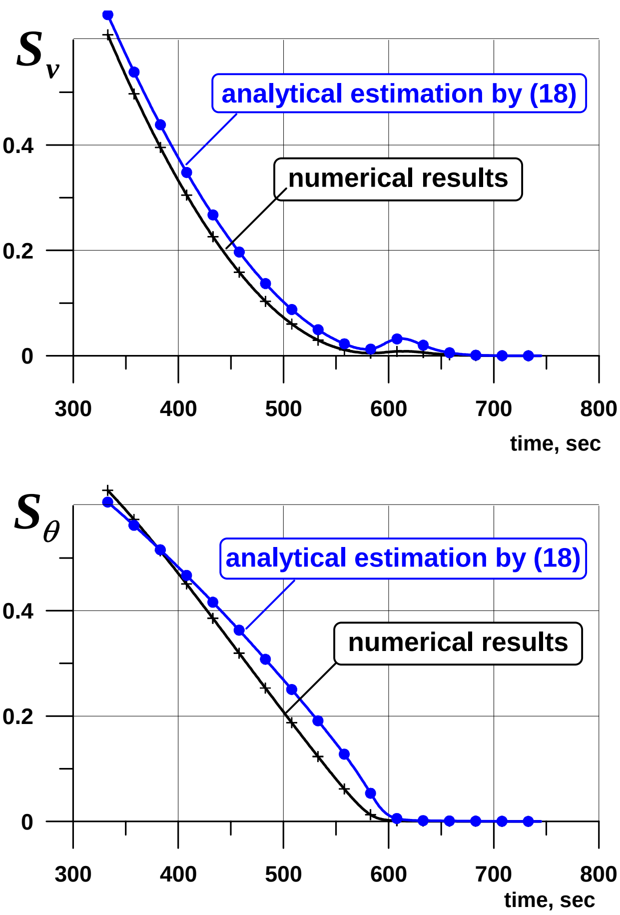

Figure 3. The comparison of the analytical estimation (18) of the conjugate variables with the precise numerical results.

$$S_V = \frac{\partial L}{\partial V} = \frac{4R_f\left(1+\mathrm{tg}^2\theta\right)\sin^2\phi\,\mathrm{tg}\,\phi}{V^3R\left(R-R_f+R_f\,\mathrm{tg}\,\theta\,\mathrm{tg}\,\phi\right)},$$

$$S_\theta = \frac{\partial L}{\partial \theta} = \frac{2R_f\left(1+\mathrm{tg}^2\theta\right)\left[V^2R-2\,\mathrm{tg}\,\theta\,\mathrm{tg}\,\phi\right]\mathrm{tg}^2\phi}{V^2R\left(R-R_f+R_f\,\mathrm{tg}\,\theta\,\mathrm{tg}\,\phi\right)\left(1+\mathrm{tg}^2\phi\right)}, \quad (18)$$

$$\mathrm{tg}\,\phi = \frac{V^2RR_f\,\mathrm{tg}\,\theta}{2R_f\left(1+\mathrm{tg}^2\theta\right)-\left(R+R_f\right)V^2R}\left[-1+\right.$$

$$\left.+\sqrt{1+\frac{\left[2R_f\left(1+\mathrm{tg}^2\theta\right)-\left(R+R_f\right)V^2R\right]\left(R-R_f\right)}{V^2RR_f^2\,\mathrm{tg}^2\theta}}\right].$$

The integrals (18) of the conjugate system allow calculating quadratures in (11). For practical tasks the formulae (18), as a rule, provide accuracy sufficient for an estimation of dispersion in a plane of separated part falling. In Figure 3 the estimation (18) is compared to the conjugate variables obtained on the basis of numerical integration of the conjugate system for typical ballistic trajectories of launcher separated parts.

5 Testing Technique

The optimization technique stated above is realized in updating the program complex ASTER [5]. To solve complex applied tasks of launcher ascent optimization in view of features of a thrust profile and loading restriction at a powered ascent leg, of admissible fall fields of separated boosters and a nose fairing, etc., a verification of the model and program is of principal importance.

Before calculating it is necessary to be convinced of correctness of the program work. After that the accuracy of optimum trajectory determination should be estimated. As a rule, the trajectory optimization error should not result in relative mistake of definition of the maximal inserted mass more than 10^{-5}. It means, for example, if the inserted mass is 20 tons (launching mass is 600-800 tons) the mistake has not to exceed a tenth part of kg. It is necessary to emphasize, that here the question is not only and not so much the accuracy of trajectory integration (as a rule, it is 5-7 orders higher), but a total error of determination of the *optimum* solution. The total error depends appreciably on such factors, as an accuracy of the multipoint boundary value problem solution, a smoothness of the right parts of the motion equations, an accuracy of definition of discontinuous moments of these functions and their derivatives, and many others.

The fulfillment of these requirements is possible when using indirect methods of optimization. In the complex ASTER the special procedures of all-round testing of the program are provided. It is important, that they are based on objective criteria, such as conservation of the first integrals, conformity of the conjugate vector to the physical sense as a function of influence of current state vector variations on the optimum functional, etc.

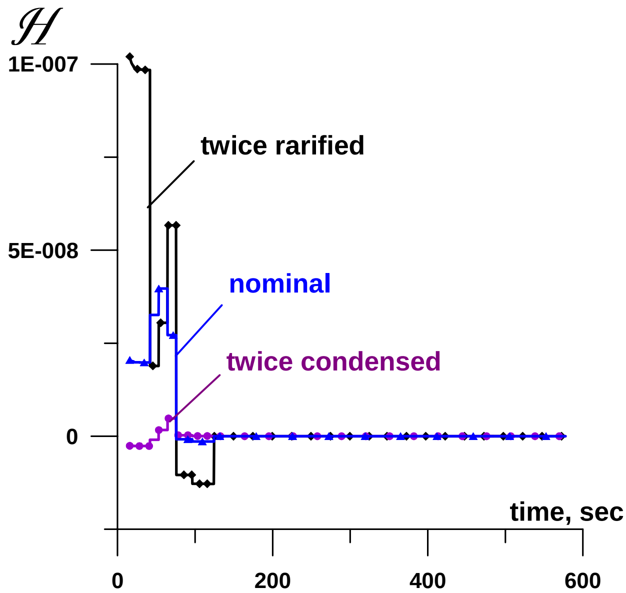

Figure 4. The Hamiltonian time-histories on the optimal branched ascent trajectory for three integration time-grids: nominal, twice rarefied and twice condensed.

The changes of the Hamiltonian $\mathcal{H} \overset{\Delta}{=} \boldsymbol{\psi}^{\mathrm{T}}\mathbf{f}\left(\mathbf{x},\mathbf{u},t\right)$ on the optimal branched ascent trajectory of the launcher are demonstrated in Figure 4. Calculations are carried out for three integration grids: nominal, twice rarefied and twice condensed. It is visible, that fluctuations of the Hamiltonian do not exceed 10^{-7} -10^{-8}, and reduction of the integration step by half decrease it in one-two orders. Similar dependence on the integration step is more reliable indication of correctness of the program, than a Hamiltonian value because the Hamiltonian is necessary to be compared to zero. Nevertheless, the following reasons can be useful for Hamiltonian value comparison. The physical sense of the Hamiltonian is a partial derivative of the optimum functional (the maximum inserted mass m_f) on the current time, i.e.

$$\delta m_f = \int_{t_i}^{t_f} \mathscr{H}\, dt \approx \mathscr{H}\left(t_f - t_i\right). \tag{19}$$

Let us δm_f in (19) is the admissible error of calculation of the functional, then the estimation of the permissible deviation of the Hamiltonian can be represented as

$$\left|\mathscr{H}\right| < \frac{\left|\delta m_f\right|}{\left(t_f - t_i\right)}. \tag{20}$$

If calculations are carried out in a dimensionless form (the characteristic time is the satellite period on the conventional circular orbit of the Earth radius, the characteristic mass is the start mass), than equation (20) has the following numerical estimation:

$$\left|\mathscr{H}\right| < 10^{-7} - 10^{-8}.$$

The second stage of the verification of the program is connected to check of conformity of the conjugate variables to their physical sense as influence functions of the current state variables on the maximum inserted mass. For example, in Table 1 coordinates of the conjugate vector $\boldsymbol{\psi} = \left\{\mathbf{P}, \mathbf{S}\right\}$ are compared to corresponding coordinates of the gradient $\frac{\partial m_{f\,max}}{\partial \mathbf{x}} = \left\{\frac{\partial m_{f\,max}}{\partial \mathbf{r}}, \frac{\partial m_{f\,max}}{\partial \mathbf{V}}\right\}$ in characteristic points of the optimum trajectory. The gradient is calculated approximately by numerical differentiation on the basis of the central differences:

$$\frac{\partial m_{f\,max}}{\partial x_j} \approx \frac{m_{f\,max}\left(x_j + \hbar\right) - m_{f\,max}\left(x_j - \hbar\right)}{2\hbar},$$

where x_j is a "disturbed" j- coordinate of the state vector $\mathbf{x}$. A step value $\hbar$ is selected from a condition of a minimum total (methodical and rounding) errors of the numerical differentiation [12].

6 Numerical Results

The developed optimization technique has been applied to the calculation of the optimal branched trajectories of the three-stage launcher of Proton type inserted the payload into the low Earth orbit. Four admissible fall areas for three separated boosters and the fair are assigned downrange.

The calculations of the optimal branched trajectories have been carried out taking into account the systematic season (December) wind and without wind.

Table 1. The comparison of the numerical influence functions with conjugate variables on the optimal branched trajectory just after first staging.

$\mathbf{P}$	0.542e-02	0.453	-0.400e-03
$\frac{\partial m_{f\,max}}{\partial \mathbf{r}}$	0.543e-02	0.454	-0.416e-03
$\mathbf{S}$	0.141	0.938e-01	-0.435e-03
$\frac{\partial m_{f\,max}}{\partial \mathbf{V}}$	0.141	0.940e-01	-0.436e-03

Figure 5 shows differences of state variables (the speed, trajectory angle and altitude) along the optimal trajectories due to the wind in a case of the gravitational turn program for the first stage. The optimal trajectory without the wind has the functional in 0.4% more then with it.

Table 2 demonstrates the influence of the fall point range deviation due to the random disturbances on the functional and optimal state variables at the fist-stage separation point.

Table 2.

ΔL_I, km	$\overline{m}_{f\,max}$	h_{sep}, km	V_{sep}, m/sec	θ_{sep}, deg
-15	0.936	55.66	1603.6	45.8
-10	0.9453	55.01	1610.3	44.32
-5	0.9564	54.17	1619.1	42.49
-0	1	49.47	1669	33.57
2.5	0.9869	51.26	1649.8	36.73

7 Conclusions

The experience of development and practical use of the technique and automated complex of programs of through optimization of branched trajectories on the basis of the Pontryagin maximum principle, including above mentioned results of research on optimization of branched trajectories of compound space transport systems in view of atmospheric disturbances and restrictions on separated parts fall areas, confirms broad opportunities of such approach for complex research of optimal motion of complex dynamic systems, including:

- supplying with objective multilevel verification of the software,
- the control of accuracy of the optimum solution obtained,
- high self-descriptiveness,
- automation of the procedure of solution of multipoint boundary value problems of the considered class without heavy requirements to the choice of an initial conjugate variable values,
- ability of a further evolution of methodical and information opportunities, etc.

Acknowledgments

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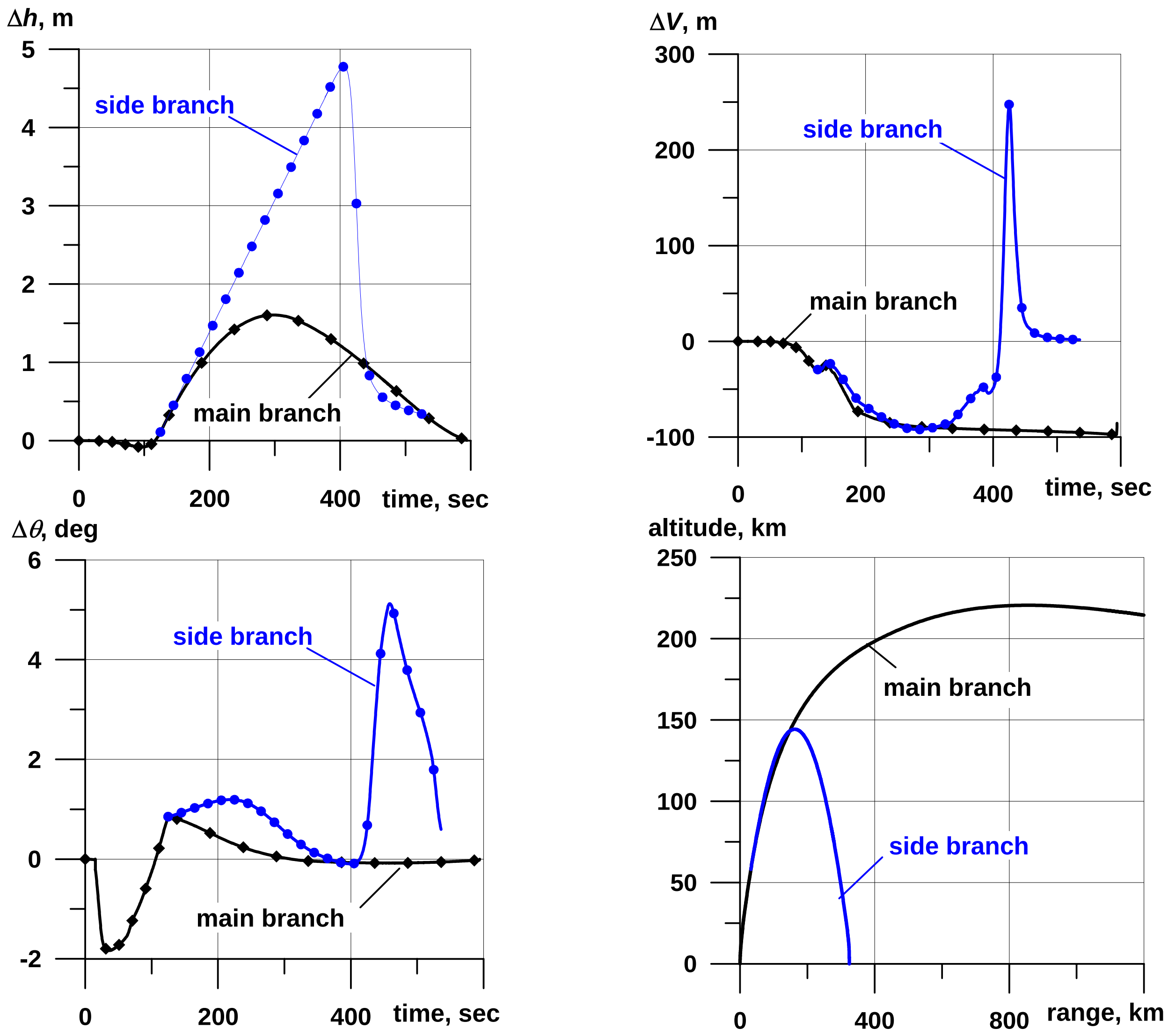

Figure 5. The systematic wind influence on the state variables of the optimal branched space launcher trajectories.