AN ANALYTICAL METHOD IN COMPUTATIONAL AEROELASTICITY BASED ON WAGNER FUNCTION

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Abstract

This paper presents an analytical method in computational aeroelasticity for an airfoil considering two degrees of freedom (heaving and pitching) based on the Wagner integral function. In the obtained aeroelastic equations of motion, there are some integral parts that give an integro-differential system of equations. Using appropriate approximation for the Wagner function, a new form of equations can be obtained by derivation from mentioned equations. These equations are in the form of ordinary differential equations. Using the obtained equations, the flutter speed is predicted for a given airfoil and the results are compared with the results of other investigators. Also, the dynamic responses of the airfoil to a sharp-edged gust are shown in pre-flutter regime for different cases.

Nomenclature

c, b	airfoil chord and semi chord length
m, μ	mass per unit length and reduced mass ratio $m/\pi\rho b^2$
$I_{e.a.}$	moment of inertia about elastic axis per unit length
r_θ	nondimensional radius of gyration about elastic axis $(I_{e.a}/mb^2)^{1/2}$
a	nondimensional distance between elastic axis and midchord
x_θ	nondimensional distance between elastic axis and center of mass
w, θ	airfoil motion in heave and pitch directions
k_w , k_θ	heave and pitch stiffness coefficients
$\omega, \omega_w, \omega_\theta$	circular frequency and plunging $(k_w/m)^{1/2}$ and pitching frequencies $(k_\theta/I_{e.a})^{1/2}$
ρ	density of air
U_f	flutter speed of airfoil
U, V	freestream velocity and its nondimensional value $U/b\omega_\theta$
t, σ	time
$\varphi(t)$	the Wagner function
$\psi(t)$	the Kussner function

Introduction

The indicial function was introduced by Wagner to describe the lift response of a two dimensional flat plate in incompressible flow [1]. Some years latter, Theodorsen [2] introduced the frequency response of a two dimensional flat plate airfoil in incompressible flow. The use of Laplace transformation was suggested by Jones [3], and Sears [4] applied this method to solve some problems. Garrick [5] used a convenient approximation according to the Fourier integral transform for the Wagner function. Garrick [6] and Miles [7] used Duhamel superposition formula on a simple harmonic motion of an airfoil that leads to arbitrary equations of motion. Marzocca et.al [8] used a two dimensional rigid/elastic lifting surface in unsteady incompressible flow. The Wagner's function was used to describe the time domain unsteady aerodynamic lift and moment. Also, the Kussner's function was applied for gust loads modeling.
In recent years (Over the past two decades) two different approaches are developed for unsteady

aerodynamic modeling for aeroelastic application which are known as Peters' aerodynamics and reduced order modeling (ROM). Peters et.al [9] offers a new type of finite state aerodynamic model. This model offers finite state equations for the induced flow field which are derived directly from the potential flow. The resultant equations can be exercised in the frequency-domain, Laplace-domain or time-domain and have capability to apply the two or three dimensional problems. Also ROM was introduced, developed and used for aeroelastic problems by many authors [10-12]. Both of the Peters' finite state aerodynamic model and ROM describes unsteady aerodynamics in a state space form. But the use of the Wagner function seems to be most appealing for the researchers to develop simple and exact model for unsteady flow analysis.

In this regard, transformation of the aeroelastic equations in differentials form provides a good physical interpretation of the different terms in these equations. Transforming the integral terms into differentials with the addition of two new second order differential equations and corresponding augmented states were presented in Poirel and Price study [13]. Details of this process are given in Dinyavari and Friedmann work [14].

This study, presents an analytical approach for calculating the aeroelastic response of a two-dimensional airfoil (typical section) in time-domain. In this method, the resulted integral parts from the Duhamel integral part of the Wagner's function in aeroelastic equations will be omitted by using an appropriate approximation of the Wagner's function and by-part integral method and therefore a set of two fourth-order differential equations with corresponding initial conditions will be obtained. The present formulation will be examined in the time and frequency domain and the obtained results will be compared with those of other investigations.

1 Structural Modeling

The airfoil structure is modeled as a rigid flat plate, with two degrees of freedom in heave and pitch directions (Fig1). The structural stiffness is provided by translational and torsional springs.

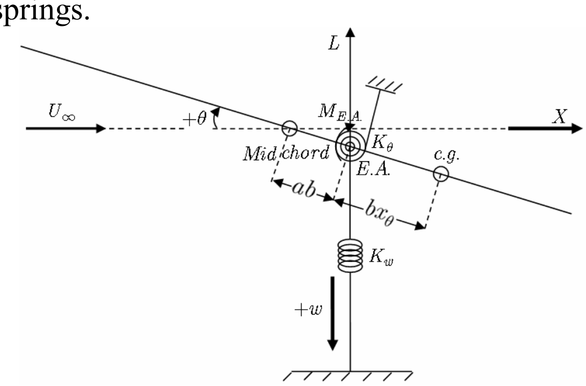


Figure 1. Schematic of the two-dimensional airfoil model.

The linear structural equations for this model by neglecting structural damping can be stated as follows [13]:

$$m\ddot{w} + mbx_\theta\ddot{\theta} + k_w w = -L(t) \quad (1)$$

$$mbx_\theta\ddot{w} + I_{e.a}\ddot{\theta} + k_\theta\theta = M_{e.a}(t) \quad (2)$$

where the positive direction of (w,θ) is shown in Fig. 1.

2 Aerodynamic Loading

Assuming the subsonic incompressible, irrotational unsteady potential flow, the aerodynamic lift and moment about elastic axis can be modeled as [15]:

$$\begin{aligned} L(t) &= \pi\rho b^2\left[\ddot{w} - ab\ddot{\theta} + U\dot{\theta}\right] \\ &+ 2\pi\rho Ub\left[\dot{w}(0) + b(\tfrac{1}{2} - a)\dot{\theta}(0) + U\theta(0)\right]\varphi(t) \\ &+ 2\pi\rho Ub\int_0^t \varphi(t-\sigma)\left[\ddot{w} + b(\tfrac{1}{2} - a)\ddot{\theta} + U\dot{\theta}\right]d\sigma \end{aligned} \quad (3)$$

$$\begin{aligned} M_{e.a.}(t) &= \pi\rho b^3\left[a\ddot{w} - b(\tfrac{1}{8} + a^2)\ddot{\theta} - U(\tfrac{1}{2} - a)\dot{\theta}\right] \\ &+ 2\pi\rho Ub^2(\tfrac{1}{2} + a)\left[\dot{w}(0) + b(\tfrac{1}{2} - a)\dot{\theta}(0) + U\theta(0)\right]\varphi(t) \\ &+ 2\pi\rho Ub^2(\tfrac{1}{2} + a)\int_0^t \varphi(t-\sigma)\left[\ddot{w} + b(\tfrac{1}{2} - a)\ddot{\theta} + U\dot{\theta}\right]d\sigma \end{aligned} \quad (4)$$

The unsteady aerodynamic loads can be computed using appropriate approximation of the Wagner function. Therefore, the Wagner function is approximated by [16]:

$$\varphi(t) = 1 - c_1 e^{-\varepsilon_1 t} - c_2 e^{-\varepsilon_2 t} \quad (5)$$

Where $c_1 = 0.165$, $c_2 = 0.335$, $\varepsilon_1 = 0.0455 U/b$ and $\varepsilon_2 = 0.3 U/b$.

3 Aeroelastic Modeling

Combining the structural and aerodynamic equations (Eqs. 1-4), the aeroelastic equations of motion can be obtained as follows:

$$\begin{aligned}
&(m + \pi\rho b^2)\ddot{w} + (mbx_\theta - \pi\rho a b^3)\ddot{\theta} \\
&+\pi\rho b^2 U\dot{\theta} + k_w w \\
&+2\pi\rho U b\varphi(t)\left[\dot{w}(0) + b(1/2 - a)\dot{\theta}(0) + U\theta(0)\right] \\
&= -2\pi\rho U b\int_0^t \varphi(t-\sigma)\left[\ddot{w} + b(1/2 - a)\ddot{\theta} + U\dot{\theta}\right]
\end{aligned} \quad (6)$$

$$\begin{aligned}
&(mbx_\theta - \pi\rho a b^3)\ddot{w} + (I_{e.a.} + \pi\rho b^4(1/8 + a^2))\ddot{\theta} \\
&+\pi\rho b^3 U(1/2 - a)\dot{\theta} + k_\theta\theta \\
&-2\pi\rho U b^2(1/2 + a)\varphi(t)\left[\dot{w}(0) + b(1/2 - a)\dot{\theta}(0) + U\theta(0)\right] \\
&= 2\pi\rho U b^2(1/2 + a)\int_0^t \varphi(t-\sigma)\left[\ddot{w} + b(1/2 - a)\ddot{\theta} + U\dot{\theta}\right]
\end{aligned} \quad (7)$$

In order to eliminate the integral parts, using by-part integral method and some simplification along with using Eq (5) instead of the Wagner function, equations (6) and (7) will lead to equations (8, 9). Equations (8) and (9) which are ordinary differential equations of the aeroelastic system.

$$\begin{aligned}
&(m + \pi\rho b^2)\ddot{w} + (mbx_\theta - \pi\rho a b^3)\ddot{\theta} + 2\pi\rho U b\varphi(0)\dot{w} \\
&+(\pi\rho b^2 U + 2\pi\rho U b^2(1/2 - a)\varphi(0))\dot{\theta} \\
&+(k_w + 2\pi\rho U b\dot{\varphi}(0))w \\
&+(2\pi\rho U^2 b\varphi(0) + 2\pi\rho U b^2(1/2 - a)\dot{\varphi}(0))\theta \\
&-2\pi\rho U b\dot{\varphi}(t)(w(0) + b(1/2 - a)\theta(0)) \\
&= 2\pi\rho U b(-\lambda_{\theta_1} e^{-\varepsilon_1 t} I_{1\theta} - \lambda_{\theta_2} e^{-\varepsilon_2 t} I_{2\theta} \\
&\qquad + \lambda_{w_1} e^{-\varepsilon_1 t} I_{1w} + \lambda_{w_2} e^{-\varepsilon_2 t} I_{2w})
\end{aligned} \quad (8)$$

$$\begin{aligned}
&(mbx_\theta - \pi\rho b^3 a)\ddot{w} + (I_{e.a.} + \pi\rho b^4(1/8 + a^2))\ddot{\theta} \\
&-2\pi\rho U b^2(1/2 + a)\varphi(0)\dot{w} \\
&+(\pi\rho b^3 U(1/2 - a) - 2\pi\rho U b^3(1/4 - a^2)\varphi(0))\dot{\theta} \\
&-2\pi\rho U b^2(1/2 + a)\dot{\varphi}(0)w \\
&+(k_\theta - 2\pi\rho U^2 b^2(1/2 + a)\varphi(0) \\
&-2\pi\rho U b^3(1/4 - a^2)\dot{\varphi}(0))\theta \\
&+2\pi\rho U b^2(1/2 + a)(w(0) + b(1/2 - a)\theta(0))\dot{\varphi}(t) \\
&= 2\pi\rho U b^2(1/2 + a)(\lambda_{\theta_1} e^{-\varepsilon_1 t} I_{1\theta} + \lambda_{\theta_2} e^{-\varepsilon_2 t} I_{2\theta} \\
&\qquad -\lambda_{w_1} e^{-\varepsilon_1 t} I_{1w} - \lambda_{w_2} e^{-\varepsilon_2 t} I_{2w})
\end{aligned} \quad (9)$$

Where $\lambda_{\theta_i} = c_i\varepsilon_i\left[U - \varepsilon_i b(1/2 - a)\right]$ and $\lambda_{w_i} = c_i{\varepsilon_i}^2$ and $I_{ix} = \int_0^t e^{\varepsilon_i\sigma} x(\sigma)\, d\sigma$.

Using the coefficients defined in table (1):

Table (1) : Define Coefficients for equations (10,11)

$A = m + \pi\rho b^2$	$A' = mbx_\theta - \pi\rho b^3 a$
$B = mbx_\theta - \pi\rho a b^3$	$B' = I_{e.a.} + \pi\rho b^4(1/8 + a^2)$
$C = 2\pi\rho U b\varphi(0)$	$C' = -2\pi\rho U b^2(1/2 + a)\varphi(0)$
$D = \pi\rho U b^2 + 2\pi\rho U b^2(1/2 - a)\varphi(0)$	$D' = \pi\rho b^3 U(1/2 - a) - 2\pi\rho b^3 U(1/4 - a^2)\varphi(0)$
$E = k_w + 2\pi\rho U b\dot{\varphi}(0)$	$E' = -2\pi\rho b^2 U(1/2 + a)\dot{\varphi}(0)$
$F = 2\pi\rho U^2 b\varphi(0) + 2\pi\rho U b^2(1/2 - a)\dot{\varphi}(0)$	$F' = k_\theta - 2\pi\rho b^2 U^2(1/2 + a)\varphi(0) - 2\pi\rho b^3 U(1/4 - a^2)\dot{\varphi}(0)$
$G = -2\pi\rho U b(w(0) + b(1/2 - a)\theta(0))$	$G' = 2\pi\rho b^2 U(1/2 + a)(w(0) + b(1/2 - a)\theta(0))$

A simple form of the aeroelastic equations can be written as follows:

$$\begin{aligned} & A\ddot{w} + B\ddot{\theta} + C\dot{w} + D\dot{\theta} \\ & + Ew + F\theta + G\dot{\varphi} \\ & = -\beta_{\theta_1} e^{-\varepsilon_1 t} I_{1\theta} - \beta_{\theta_2} e^{-\varepsilon_2 t} I_{2\theta} \\ & + \beta_{w_1} e^{-\varepsilon_1 t} I_{1w} + \beta_{w_2} e^{-\varepsilon_2 t} I_{2w} \end{aligned} \quad (10)$$

$$\begin{aligned} & A'\ddot{w} + B'\ddot{\theta} + C'\dot{w} + D'\dot{\theta} \\ & + E'w + F'\theta + G'\dot{\varphi} \\ & = \beta'_{\theta_1} e^{-\varepsilon_1 t} I_{1\theta} + \beta'_{\theta_2} e^{-\varepsilon_2 t} I_{2\theta} \\ & - \beta'_{w_1} e^{-\varepsilon_1 t} I_{1w} - \beta'_{w_2} e^{-\varepsilon_2 t} I_{2w} \end{aligned} \quad (11)$$

Where $\beta_{x_i} = 2\pi\rho U b \lambda_{x_i}$, $\beta'_{x_i} = 2\pi\rho U b^2 (1/2 + a)\lambda_{x_i}$ and x_i denote θ_i and w_i.

Omitting the integral parts $(I_{iw}, I_{i\theta})$, equations will transform the Eqs(10), (11) into ordinary differential equations, those are so friendly to solve. So, multiplying equations (10, 11) by $e^{\varepsilon_1 t}$ and differentiation with respect to t (time) will yield to following equations:

$$\begin{aligned} & Ae^{\varepsilon_1 t}(\varepsilon_1 \ddot{w} + \dddot{w}) + Be^{\varepsilon_1 t}(\varepsilon_1 \ddot{\theta} + \dddot{\theta}) \\ & + Ce^{\varepsilon_1 t}(\varepsilon_1 \dot{w} + \ddot{w}) + De^{\varepsilon_1 t}(\varepsilon_1 \dot{\theta} + \ddot{\theta}) \\ & + Ee^{\varepsilon_1 t}(\varepsilon_1 w + \dot{w}) + Fe^{\varepsilon_1 t}(\varepsilon_1 \theta + \dot{\theta}) \\ & + Ge^{\varepsilon_1 t}(\varepsilon_1 \dot{\varphi} + \ddot{\varphi}) \\ & = -\beta_{\theta_1} e^{\varepsilon_1 t}\theta - \beta_{\theta_2} e^{(\varepsilon_1 - \varepsilon_2)t}\left[(\varepsilon_1 - \varepsilon_2) I_{2\theta} + \theta e^{\varepsilon_2 t}\right] \\ & + \beta_{w_1} e^{\varepsilon_1 t} w + \beta_{w_2} e^{(\varepsilon_1 - \varepsilon_2)t}\left[(\varepsilon_1 - \varepsilon_2) I_{2w} + w e^{\varepsilon_2 t}\right] \end{aligned} \quad (12)$$

$$\begin{aligned} & A'e^{\varepsilon_1 t}(\varepsilon_1 \ddot{w} + \dddot{w}) + B'e^{\varepsilon_1 t}(\varepsilon_1 \ddot{\theta} + \dddot{\theta}) \\ & + C'e^{\varepsilon_1 t}(\varepsilon_1 \dot{w} + \ddot{w}) + D'e^{\varepsilon_1 t}(\varepsilon_1 \dot{\theta} + \ddot{\theta}) \\ & + E'e^{\varepsilon_1 t}(\varepsilon_1 w + \dot{w}) + F'e^{\varepsilon_1 t}(\varepsilon_1 \theta + \dot{\theta}) \\ & + G'e^{\varepsilon_1 t}(\varepsilon_1 \dot{\varphi} + \ddot{\varphi}) \\ & = \beta'_{\theta_1} e^{\varepsilon_1 t}\theta + \beta'_{\theta_2} e^{(\varepsilon_1 - \varepsilon_2)t}\left[(\varepsilon_1 - \varepsilon_2) I_{2\theta} + \theta e^{\varepsilon_2 t}\right] \\ & - \beta'_{w_1} e^{\varepsilon_1 t} w - \beta'_{w_2} e^{(\varepsilon_1 - \varepsilon_2)t}\left[(\varepsilon_1 - \varepsilon_2) I_{2w} + w e^{\varepsilon_2 t}\right] \end{aligned} \quad (13)$$

Similarly, multiplying left and right hand sides of equations (12, 13) by $e^{(\varepsilon_2 - \varepsilon_1)t}$ factor and differentiating with respect to t will omit I_{2w} and $I_{2\theta}$. After simplifying and rearranging, the following equations we arrive at following equations:

$$\begin{aligned} & Aw^{(4)} + B\theta^{(4)} + [A(\varepsilon_1 + \varepsilon_2) + C]\,\dddot{w} \\ & + [B(\varepsilon_1 + \varepsilon_2) + D]\,\dddot{\theta} \\ & + [A\varepsilon_1\varepsilon_2 + C(\varepsilon_1 + \varepsilon_2) + E]\,\ddot{w} \\ & + [B\varepsilon_1\varepsilon_2 + D(\varepsilon_1 + \varepsilon_2) + F]\,\ddot{\theta} \\ & + [C\varepsilon_1\varepsilon_2 + E(\varepsilon_1 + \varepsilon_2) - \beta_{w_1} - \beta_{w_2}]\,\dot{w} \\ & + [D\varepsilon_1\varepsilon_2 + F(\varepsilon_1 + \varepsilon_2) + \beta_{\theta_1} + \beta_{\theta_2}]\,\dot{\theta} \\ & + [E\varepsilon_1\varepsilon_2 - \beta_{w_1}\varepsilon_2 - \beta_{w_2}\varepsilon_1]\,w \\ & + [F\varepsilon_1\varepsilon_2 + \beta_{\theta_1}\varepsilon_2 + \beta_{\theta_2}\varepsilon_1]\,\theta \\ & + G\,[\dddot{\varphi} + (\varepsilon_1 + \varepsilon_2)\ddot{\varphi} + \varepsilon_1\varepsilon_2\dot{\varphi}] = 0 \end{aligned} \quad (14)$$

$$\begin{aligned} & A'w^{(4)} + B'\theta^{(4)} + [A'(\varepsilon_1 + \varepsilon_2) + C']\,\dddot{w} \\ & + [B'(\varepsilon_1 + \varepsilon_2) + D']\,\dddot{\theta} \\ & + [A'\varepsilon_1\varepsilon_2 + C'(\varepsilon_1 + \varepsilon_2) + E']\,\ddot{w} \\ & + [B'\varepsilon_1\varepsilon_2 + D'(\varepsilon_1 + \varepsilon_2) + F']\,\ddot{\theta} \\ & + [C'\varepsilon_1\varepsilon_2 + E'(\varepsilon_1 + \varepsilon_2) + \beta'_{w_1} + \beta'_{w_2}]\,\dot{w} \\ & + [D'\varepsilon_1\varepsilon_2 + F'(\varepsilon_1 + \varepsilon_2) - \beta'_{\theta_1} - \beta'_{\theta_2}]\,\dot{\theta} \\ & + [E'\varepsilon_1\varepsilon_2 + \beta'_{w_1}\varepsilon_2 + \beta'_{w_2}\varepsilon_1]\,w \\ & + [F'\varepsilon_1\varepsilon_2 - \beta'_{\theta_1}\varepsilon_2 - \beta'_{\theta_2}\varepsilon_1]\,\theta \\ & + G'\,[\dddot{\varphi} + (\varepsilon_1 + \varepsilon_2)\ddot{\varphi} + \varepsilon_1\varepsilon_2\dot{\varphi}] = 0 \end{aligned} \quad (15)$$

The required initial conditions for the new equations of motion can be obtained from the old I.Cs and by putting them into Eqs.(10-13), the following additional I.Cs will be obtained.

$$\begin{aligned} \ddot{w}(0) = & \frac{1}{A'B - AB'}[(B'C - BC')\dot{w}(0) \\ & + (B'D - BD')\dot{\theta}(0) + (B'E - BE')w(0) \\ & + (B'F - BF')\theta(0) + (B'G - BG')\dot{\varphi}(0)] \end{aligned} \quad (16)$$

$$\begin{aligned} \ddot{\theta}(0) = & \frac{1}{AB' - A'B}[(A'C - AC')\dot{w}(0) \\ & + (A'D - AD')\dot{\theta}(0) + (A'E - AE')w(0) \\ & + (A'F - AF')\theta(0) + (A'G - AG')\dot{\varphi}(0)] \end{aligned} \quad (17)$$

$$\begin{aligned}
&\ddot{w}(0) = \\
&\frac{1}{A'B - AB'}[((A\varepsilon_1 + C)B' - (A'\varepsilon_1 + C')B)\ddot{w}(0) \\
&+((B\varepsilon_1 + D)B' - (B'\varepsilon_1 + D')B)\ddot{\theta}(0) \\
&+((C\varepsilon_1 + E)B' - (C'\varepsilon_1 + E')B)\dot{w}(0) \\
&+((D\varepsilon_1 + F)B' - (D'\varepsilon_1 + F')B)\dot{\theta}(0) \\
&+((E\varepsilon_1 - \beta_{w_1} - \beta_{w_2})B' \\
&-(E'\varepsilon_1 + \beta'_{w_1} + \beta'_{w_2})B)w(0) \\
&+((F\varepsilon_1 + \beta_{\theta_1} + \beta_{\theta_2})B' \\
&-(F'\varepsilon_1 - \beta'_{\theta_1} - \beta'_{\theta_2})B)\theta(0) \\
&+(B'G - BG')(\varepsilon_1\dot{\varphi}(0) + \ddot{\varphi}(0))]
\end{aligned} \qquad (18)$$

$$\begin{aligned}
&\ddot{\theta}(0) = \\
&\frac{1}{AB' - A'B}[((A\varepsilon_1 + C)A' - (A'\varepsilon_1 + C')A)\ddot{w}(0) \\
&+((B\varepsilon_1 + D)A' - (B'\varepsilon_1 + D')A)\ddot{\theta}(0) \\
&+((C\varepsilon_1 + E)A' - (C'\varepsilon_1 + E')A)\dot{w}(0) \\
&((D\varepsilon_1 + F)A' - (D'\varepsilon_1 + F')A)\dot{\theta}(0) \\
&+((E\varepsilon_1 - \beta_{w_1} - \beta_{w_2})A' - (E'\varepsilon_1 + \beta'_{w_1} + \beta'_{w_2})A)w(0) \\
&+((F\varepsilon_1 + \beta_{\theta_1} + \beta_{\theta_2})A' - (F'\varepsilon_1 - \beta'_{\theta_1} - \beta'_{\theta_2})A)\theta(0) \\
&+(A'G - AG')(\varepsilon_1\dot{\varphi}(0) + \ddot{\varphi}(0))]
\end{aligned} \qquad (19)$$

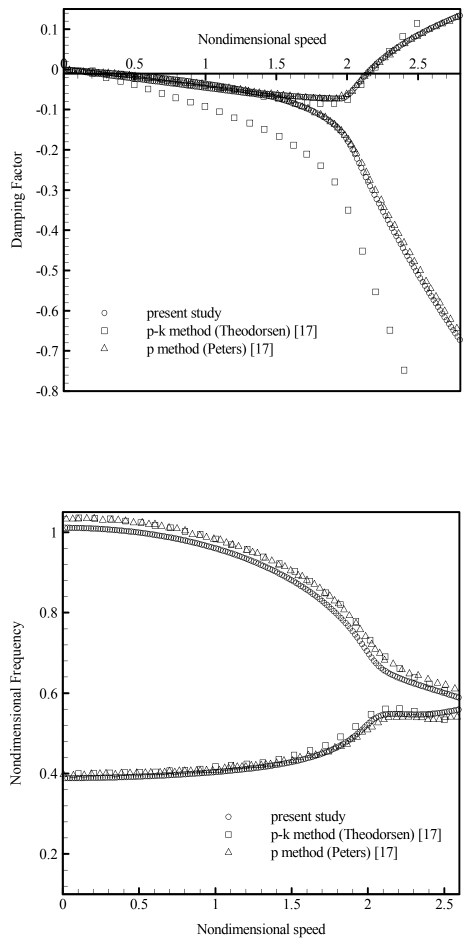
4 Case study

Verification of introduced formulation is carried out by considering a two dimensional linear airfoil with specifications as shown in table (2). First the damping and frequency parts of the eigenvalues of the aeroelastic system are compared with the results that obtained based on the Theodorsen and Peters theories. The results of present formulation which are obtained using the P method are compared with the results of the Theodorsen using P-k method and Peters using P method, respectively. Figures (2a, 2b) show the good correspondence between the different theories. The difference between the results of the present theory and Peters' theory with those of the Theodorsen theory is due to using P and P-k methods. Also from these figures, we can determine the dynamic instability speed of aeroelastic system (flutter speed). All of the mentioned theories and present method predict the same value for flutter speed for the given airfoil.

Table (2) : Given airfoil specifications [17]

ρ	0.002378 Sluges/ft^3
$I_{e.a.}$	1.606 Sluges.ft^2
k_θ	1003.75 ft.lb/rad
$c = 2b$	5.18 ft
m	1 Sluges
k_w	100 lb/ft
x_θ	0.1
a	-0.2

In following figures, $\frac{U}{b\omega_\theta}$, $\frac{\sigma}{\omega_\theta}$ and $\frac{\omega}{\omega_\theta}$ are nondimensional speed, damping factor, and nondimensional frequency, respectively.



(a)

(b)

Figure 2. damping(a); and frequency (b).of the aeroelastic system versus nondimensional airspeed

In order to show the applicability of the present formulation, the response of the aeroelastic system to a sharp-edged gust is determined. The sharp-edged gust is the most critical case for gust modeling. It can change the angle of attack of the system largely and rapidly and can induce large lift and load factor beyond the structural strength. In order to account the gust response, first it is necessary to develop the formulation and to find the higher order initial conditions to cover the sharp-edged gust effects. In the forwarding section the dynamic responses of the typical section due to a sharp-edged gust will be investigated. Those equations can be driven by adding one part to equations (14, 15) as sharp-edged gust effects. These parts are based on Kussner's function and obtained in a similar manner which was introduced in present study [18].

$$\begin{aligned}
&Aw^{(4)} + B\theta^{(4)} + [A(\varepsilon_1+\varepsilon_2)+C]\,\dddot{w}\\
&+[B(\varepsilon_1+\varepsilon_2)+D]\,\dddot{\theta}\\
&+[A\varepsilon_1\varepsilon_2+C(\varepsilon_1+\varepsilon_2)+E]\,\ddot{w}\\
&+[B\varepsilon_1\varepsilon_2+D(\varepsilon_1+\varepsilon_2)+F]\,\ddot{\theta}\\
&+[C\varepsilon_1\varepsilon_2+E(\varepsilon_1+\varepsilon_2)-\beta_{w_1}-\beta_{w_2}]\,\dot{w}\\
&+[D\varepsilon_1\varepsilon_2+F(\varepsilon_1+\varepsilon_2)+\beta_{\theta_1}+\beta_{\theta_2}]\,\dot{\theta}\\
&+[E\varepsilon_1\varepsilon_2-\beta_{w_1}\varepsilon_2-\beta_{w_2}\varepsilon_1]\,w\\
&+[F\varepsilon_1\varepsilon_2+\beta_{\theta_1}\varepsilon_2+\beta_{\theta_2}\varepsilon_1]\,\theta\\
&+G\,[\dddot{\varphi}+(\varepsilon_1+\varepsilon_2)\ddot{\varphi}+\varepsilon_1\varepsilon_2\dot{\varphi}]\\
&+H\,\left[\ddot{\psi}+(\varepsilon_1+\varepsilon_2)\dot{\psi}+\varepsilon_1\varepsilon_2\psi\right]=0
\end{aligned} \tag{20}$$

$$\begin{aligned}
&A'w^{(4)} + B'\theta^{(4)} + [A'(\varepsilon_1+\varepsilon_2)+C']\,\dddot{w}\\
&+[B'(\varepsilon_1+\varepsilon_2)+D']\,\dddot{\theta}\\
&+[A'\varepsilon_1\varepsilon_2+C'(\varepsilon_1+\varepsilon_2)+E']\,\ddot{w}\\
&+[B'\varepsilon_1\varepsilon_2+D'(\varepsilon_1+\varepsilon_2)+F']\,\ddot{\theta}\\
&+[C'\varepsilon_1\varepsilon_2+E'(\varepsilon_1+\varepsilon_2)+\beta'_{w_1}+\beta'_{w_2}]\,\dot{w}\\
&+[D'\varepsilon_1\varepsilon_2+F'(\varepsilon_1+\varepsilon_2)-\beta'_{\theta_1}-\beta'_{\theta_2}]\,\dot{\theta}\\
&+[E'\varepsilon_1\varepsilon_2+\beta'_{w_1}\varepsilon_2+\beta'_{w_2}\varepsilon_1]\,w\\
&+[F'\varepsilon_1\varepsilon_2-\beta'_{\theta_1}\varepsilon_2-\beta'_{\theta_2}\varepsilon_1]\,\theta\\
&+G'\,[\dddot{\varphi}+(\varepsilon_1+\varepsilon_2)\ddot{\varphi}+\varepsilon_1\varepsilon_2\dot{\varphi}]\\
&+H'\,\left[\ddot{\psi}+(\varepsilon_1+\varepsilon_2)\dot{\psi}+\varepsilon_1\varepsilon_2\psi\right]=0
\end{aligned} \tag{21}$$

So, the second order initial conditions would be found as following

$$\begin{aligned}
\ddot{w}(0) &= \frac{1}{A'B-AB'}[(B'C-BC')\dot{w}(0)\\
&+(B'D-BD')\dot{\theta}(0)+(B'E-BE')w(0)\\
&+(B'F-BF')\theta(0)+(B'G-BG')\dot{\varphi}(0)\\
&+(B'H-BH')\dot{\psi}(0)]
\end{aligned} \tag{22}$$

$$\begin{aligned}
\ddot{\theta}(0) &= \frac{1}{AB'-A'B}[(A'C-AC')\dot{w}(0)\\
&+(A'D-AD')\dot{\theta}(0)+(A'E-AE')w(0)\\
&+(A'F-AF')\theta(0)+(A'G-AG')\dot{\varphi}(0)\\
&+(A'H-AH')\dot{\psi}(0)]
\end{aligned} \tag{23}$$

Also, the third order initial conditions in sharp-edged gust case are written as

$$\begin{aligned}
&\dddot{w}(0)=\\
&\frac{1}{A'B-AB'}[((A\varepsilon_1+C)B'-(A'\varepsilon_1+C')B)\ddot{w}(0)\\
&+((B\varepsilon_1+D)B'-(B'\varepsilon_1+D')B)\ddot{\theta}(0)\\
&+((C\varepsilon_1+E)B'-(C'\varepsilon_1+E')B)\dot{w}(0)\\
&((D\varepsilon_1+F)B'-(D'\varepsilon_1+F')B)\dot{\theta}(0)\\
&+((E\varepsilon_1-\beta_{w_1}-\beta_{w_2})B'-(E'\varepsilon_1+\beta'_{w_1}+\beta'_{w_2})B)w(0)\\
&+((F\varepsilon_1+\beta_{\theta_1}+\beta_{\theta_2})B'-(F'\varepsilon_1-\beta'_{\theta_1}-\beta'_{\theta_2})B)\theta(0)\\
&+(B'G-BG')(\varepsilon_1\dot{\varphi}(0)+\ddot{\varphi}(0))\\
&+(B'H-BH')(\varepsilon_1\dot{\psi}(0)+\ddot{\psi}(0))]
\end{aligned} \tag{24}$$

$$\begin{aligned}
&\dddot{\theta}(0)=\\
&\frac{1}{AB'-A'B}[((A\varepsilon_1+C)A'-(A'\varepsilon_1+C')A)\ddot{w}(0)\\
&+((B\varepsilon_1+D)A'-(B'\varepsilon_1+D')A)\ddot{\theta}(0)\\
&+((C\varepsilon_1+E)A'-(C'\varepsilon_1+E')A)\dot{w}(0)\\
&((D\varepsilon_1+F)A'-(D'\varepsilon_1+F')A)\dot{\theta}(0)\\
&+((E\varepsilon_1-\beta_{w_1}-\beta_{w_2})A'-(E'\varepsilon_1+\beta'_{w_1}+\beta'_{w_2})A)w(0)\\
&+((F\varepsilon_1+\beta_{\theta_1}+\beta_{\theta_2})A'-(F'\varepsilon_1-\beta'_{\theta_1}-\beta'_{\theta_2})A)\theta(0)\\
&+(A'G-AG')(\varepsilon_1\dot{\varphi}(0)+\ddot{\varphi}(0))\\
&+(A'H-AH')(\varepsilon_1\dot{\psi}(0)+\ddot{\psi}(0))]
\end{aligned} \tag{25}$$

where $\psi(t)$ stands for the Kussner function and can be stated approximately as following [16]

$$\psi(s) = 1 - 0.5e^{-0.130s} - 0.5e^{-s}, \ s = \frac{Ut}{b} \qquad (26)$$

Also the coefficients H and H' are introduced in equation (23) as

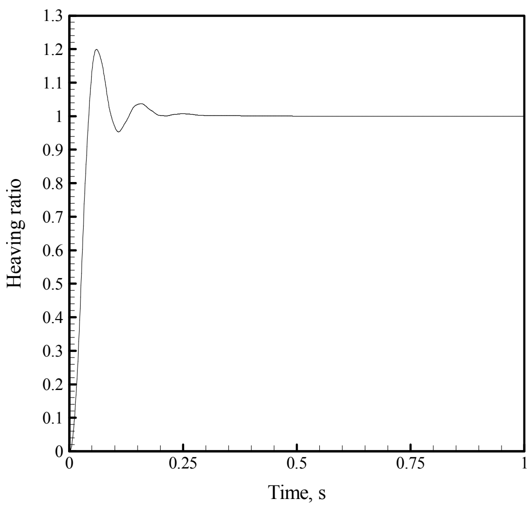
$$\begin{aligned} H &= 2\pi\rho U b w_0 \\ H' &= 2\pi\rho U b^2 (1/2 + a) w_0 \end{aligned} \qquad (27)$$

Where w_0 is the gust speed.
For an airfoil with following specifications the dynamic responses in pre-flutter regimes are shown. The dynamic responses include the plunging and pitching amplitude variation versus time relative to steady state responses.

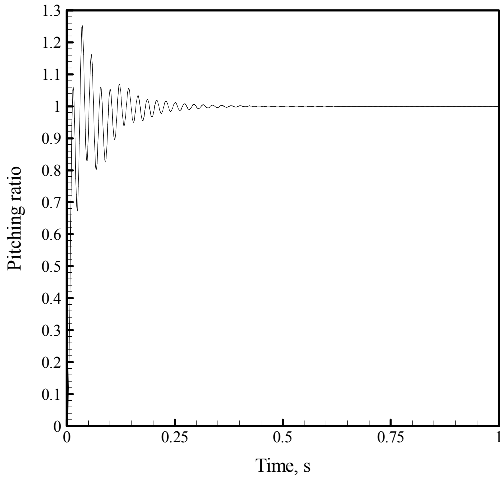
Table (3) : Given airfoil specifications

ρ	0.002378 Sluges/ft^3	k_w	100 lb/ft
V	1(ft/s)	x_θ	0.0
r_θ	0.5	a	0.0
c	1 ft		

The results found for sharp-edged gust with unit ($w_0 = 1\,ft/s$) value in four different conditions and zero initial conditions.

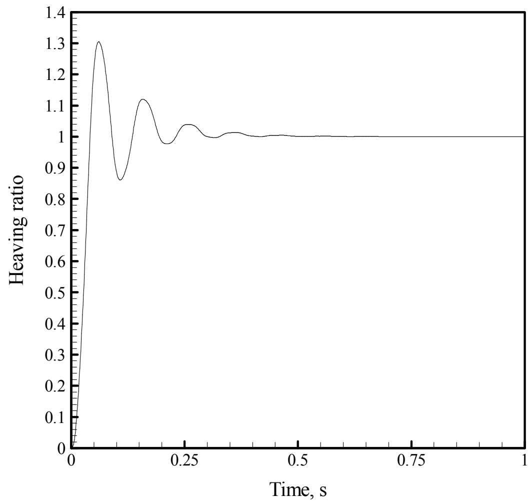


(a)

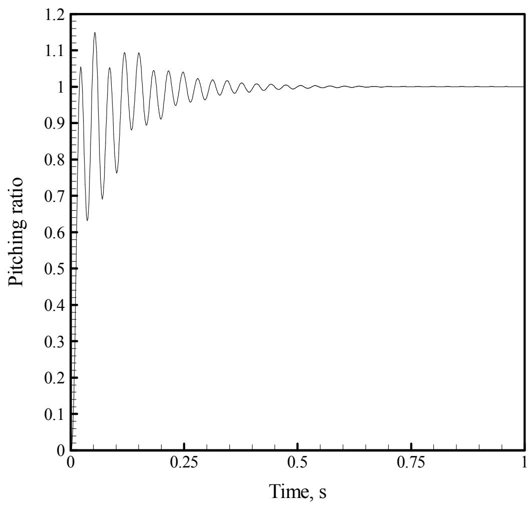


(b)

Figure 3.Dynamic response time histories of the flexible airfoil for $\mu = 14$ and $A = 0.0375$; (a) heaving direction; (b) pitching direction.

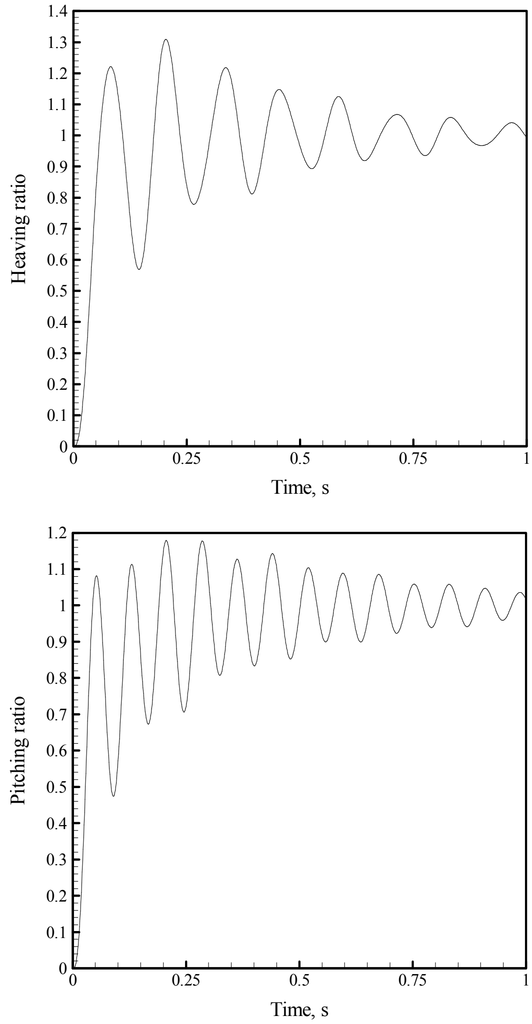


(a)



(b)

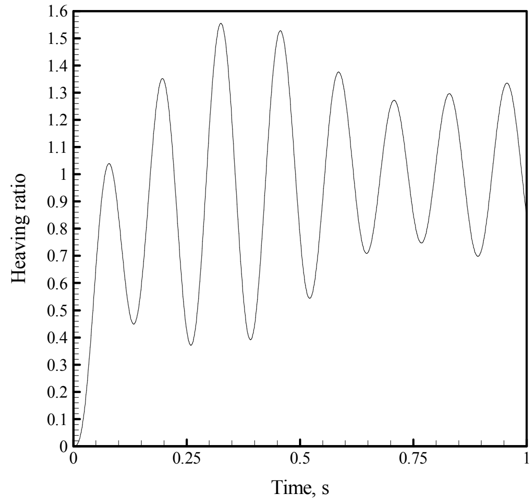
Figure 4.Dynamic response time histories of the flexible airfoil for $\mu = 14$ and $A = 0.0845$; (a) heaving direction; (b) pitching direction.



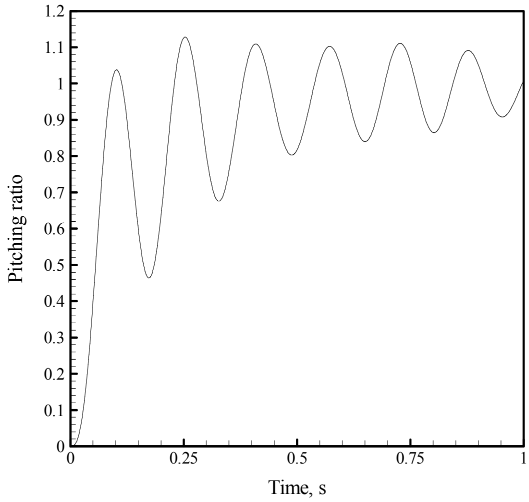
(a)

(b)

Figure 5.Dynamic response time histories of the flexible airfoil for $\mu = 21$ and $A = 0.338$; (a) heaving direction; (b) pitching direction.



(a)



(b)

Figure 6.Dynamic response time histories of the flexible airfoil for $\mu = 21$ and $A = 1.335$; (a) heaving direction; (b) pitching direction.

5 Conclusion

It was shown that modeling of the unsteady aerodynamics by a two-state representation of the Wagner function and simplifying the integro-differential aeroelastic equations of a two-dimensional airfoil (typical section) with the use of some mathematical methods can reduce these equations to a set of fourth order ordinary differential equations. Good simplicity of these equations makes them as a unique tool to obtain dynamic responses to different inputs in time domain solutions. These equations were used to predict flutter speed in comparison with the Theodorsen and Peters' aerodynamics of an airfoil and its dynamic responses due to sharp-edged gust with the given specifications. The results show that the present method is a powerful simplified analytical method for aeroelastic calculations with the least number of states in comparison with other methods.

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