ON THE CORRELATION OF ACOUSTIC MODES IN A NOZZLE AND IN THE FAR-FIELD

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Abstract

Acoustic liners are widely used in jet engine inlet and exhaust ducts as a passive means of noise reduction, and the problem of liner optimization has received some attention recently. The use of circumferentially non-uniform liners has been shown to lead to an attenuation in the acoustic amplitude of some of the modes present in the duct. However, the sound attenuation perceived by an observer in the far-field is arguably the most important effect to be achieved. In this paper the far-field acoustic pressure field radiated from the exit of a lined cylindrical duct is considered and the case of uniform and circumferentially non-uniform liners is compared.

The radiation of sound from the duct to the free space is specified by a Kirchhoff integral, the sound source distribution being specified by the pressure distribution on the duct exit plane. This pressure distribution depend on the radial, axial and circumferential modes in the cylindrical duct, which are determined by the impedance boundary conditions. When the impedance distribution varies circumferentially, the evaluation of the wavenumbers involves the determination of the roots of an infinite determinant.

The evaluation of the radiation integrals show that (i) the total acoustic field consists of a spherical wave multiplying a sum of directivity factors which depend on the radial $n = 1,\ldots,\infty$ and azimuthal $m = 1,\ldots,\infty$ modes; (ii) each mode consists of a monopole term and a dipole term and depend on the frequency and the radial and axial wavenumbers that had been determined by the boundary condition at the duct wall. In the case of external noise of an aircraft, the observer is on the ground, at a distance much greater than the duct diameter, and the radiation integrals for an observer in the far-field can be simplified, since the dipole term is weak.

The evaluation of the acoustic pressure for an observer in the far-field, shows that the directivity factors depend on the radial wavenumbers in the nozzle, which are specified by the wall boundary conditions, and thus depend on the acoustic impedance distribution. This allows comparison of hard-walled nozzles, with liners with constant impedance and non-uniform liners, the latter with impedance distribution varying circumferentially.

1 Introduction

In a cylindrical nozzle noise can be absorbed in its interior by vortical flow [1–6] and at the walls by acoustic liners, which may have uniform [7–9] or non-uniform [10–17] impedances. The sound field received by an observer in the far-field is determined by radiation out of the open end of the nozzle [18–22] or inlet of a fan [23, 24]. In the case of nozzle there is a refraction effect which can be represented either by a vortex sheet [25] or an irregular shear layer [1, 2] issuing from the lip.

The aim of the present paper is to relate the acoustic field radiated through the nozzle exit to the far-field to the acoustic modes in a nozzle with non-uniform wall impedance. The radiation from the disk in the nozzle exit plane, to an observer in the far-field, consists, to leading order, of monopole and dipole terms. The acoustic pressure distribution in the nozzle exit plane is expressed in terms of duct modes, allowing the evaluation of radiation integrals. The latter involve the acoustic eigenfunctions, corresponding to the eigenvalues for propagating or cut-on modes, in the case of rigid walls or walls with uniform impedance, and also walls with impedance varying circumferentially. As an example, the radial wavenumbers and the directivity factors were calculated for uniform liner duct in comparison with weakly and strongly circumferentially non-uniform liners; the benefits of the non-uniform lining are assessed, not including the effects of transmission across the irregular and turbulent shear layer issuing from the jet nozzle lip.

2 Sound radiation from an open pipe

The radiation of sound from an open pipe (Fig. 1) is represented by a pressure distribution on a disk (2.1), *viz.* the exit plane; the radiation integrals are simplified (2.3) for an observer in the far-field (2.2).

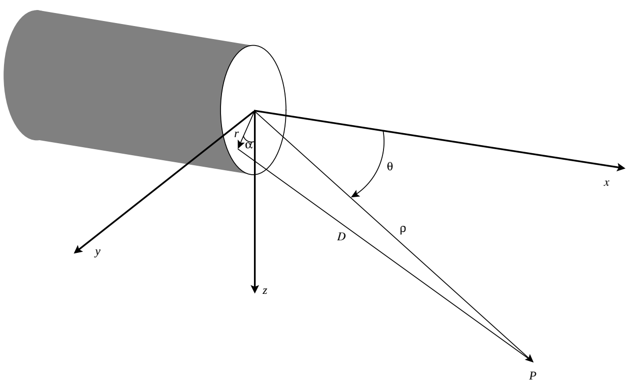

Fig. 1 Sound radiation from the disk on the exit plane of a cylindrical pipe to an observer in the far-field

2.1 Source distribution on a circular disk

The radiation of sound in free space, *i.e.* without obstacles, is specified by the Kirchhof integral:

$$p(\mathbf{x}) = \frac{1}{4\pi}\int_{\mathcal{D}} \frac{1}{|\mathbf{x}-\mathbf{y}|} e^{-i\omega(t-|\mathbf{x}-\mathbf{y}|/c)} q(\mathbf{y})\, d^3\mathbf{y}, \quad (1)$$

for a spatial source distribution of strength q at position $\mathbf{y}$ in the domain $\mathcal{D}$, with frequency ω, and radiation to the observer at $\mathbf{x}$, in a homogeneous medium at rest, for which the sound speed is c. In the case of a disk of radius a on the XOY-plane with centre at the origin, the position of the source is written in polar coordinates (R,α):

$$0 \le \alpha \le 2\pi, \quad 0 \le R \le a; \qquad \mathbf{y} = R(\mathbf{e}_x \cos\alpha + \mathbf{e}_y \sin\alpha), \quad (2)$$

and the position of the observer in spherical coordinates (r,θ,φ):

$$\mathbf{x} = r(\mathbf{e}_x \cos\theta + \mathbf{e}_z \sin\theta), \quad (3)$$

where $\varphi = 0$ because the XOZ-plane can be taken through the observer. The radiation integral (1) becomes in this case

$$p(r,\theta,t) = \frac{e^{-i\omega t}}{4\pi} \times \int_0^{2\pi} d\alpha \int_0^a dR \frac{R}{D(R,\alpha)} e^{i(\omega/c)D(R,\alpha)} q(R,\alpha) \quad (4)$$

where $q(R,\alpha)$ is the source distribution, and

$$D(R,\alpha) = |\mathbf{x}-\mathbf{y}| = |R^2 + r^2 - 2Rr\sin\theta\cos\alpha|^{1/2}, \quad (5)$$

is the distance between observer and source.

2.2 Reception by an observer in the far-field

In the case of external noise of an aircraft, the observer is on the ground, at a distance $D(R,\alpha)$ much greater than the nozzle diameter. Then $R^2 \le a^2 << r^2$ and the distance by inverse distance may be simplified by:

$$D(R,\alpha) = r - R\sin\theta\cos\alpha + O(R^2/r^2), \quad (6a)$$

$$\frac{1}{D(R,\alpha)} = \frac{1}{r} + \frac{R}{r^2}\sin\theta\cos\alpha + O(R^2/r^2). \quad (6b)$$

Substitution of (6a, 6b) in the radiation integral (4) specifies

$$p(r,\theta,t) = \frac{e^{-i\omega t}}{4\pi}\int_0^{2\pi} d\alpha \int_0^a dR\, q(R,\alpha)\times \frac{R}{r}\left[1+\frac{R}{r}\sin\theta\cos\alpha\right] e^{i(\omega/c)(r-R\sin\theta\cos\alpha)} \quad (7)$$

the acoustic field received by the observer in the far-field.

2.3 Decomposition into monopole and dipole terms

The acoustic pressure received in the far-field (7) may be written

$$p(r,\theta,t) = \frac{e^{i\omega(r/c-t)}}{4\pi r}(I_1+I_2), \quad (8)$$

as a spherical wave radiated from the origin (or disk or nozzle centre) to the observer, multiplied by monopole (9a) and dipole (9b) terms

$$I_1 = \int_0^{2\pi} d\alpha \int_0^a dR\, q(R,\alpha)\, R e^{-i(\omega R/c)\sin\theta\cos\alpha} \quad (9a)$$

$$I_2 = \int_0^{2\pi} d\alpha \int_0^a dR\, q(R,\alpha)\times R e^{-i(\omega R/c)\sin\theta\cos\alpha}(R/r)\sin\theta\cos\alpha \quad (9b)$$

where the latter is related to the former by

$$I_2 = -\frac{1}{r}\frac{d}{d(i\omega/c)}I_1, \quad (10)$$

since differentiation with regard to $i\omega/c$ is equivalent to multiplication by $R\sin\theta\cos\alpha$. Substitution of (9a) and (10) in (8) yields:

$$p(r,\theta,t) = \frac{e^{i\omega(r/c-t)}}{4\pi r}\left\{1+i\frac{d}{d(\omega r/c)}\right\} \int_0^{2\pi} d\alpha \int_0^a dR\, q(R,\alpha)\, R e^{-i(\omega R/c)\sin\theta\cos\alpha}, \quad (11)$$

as the acoustic pressure received by an observer in the far-field, from a source distribution $q(R,\alpha)$ on a disk. The latter is specified next in terms of the acoustic modes of a nozzle.

3 Acoustic modes in a cylindrical nozzle

The source distribution on the nozzle exit plane (3.2), which allows the evaluation of radiation integrals, is specified by the radial, axial, and circumferential modes, in the cylindrical nozzle.

3.1 Radial, axial and circumferential modes

In the absence of mean flow, the modes in a cylindrical nozzle are specified by the solution of the classical wave equation in cylindrical coordinates:

$$\left\{\frac{1}{R}\frac{\partial}{\partial R}R\frac{\partial}{\partial R}+\frac{1}{R^2}\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right\}Q=0. \quad (12)$$

Since the coefficients of the wave equation depend only on radius R, it is convenient to use a Fourier decomposition in (z,α,t), *viz.*:

$$Q(R,\alpha,z,t) = e^{-i\omega t}\sum_{m=-\infty}^{+\infty} e^{im\alpha}\int_{-\infty}^{+\infty} dk\, e^{ikz}P_m(R,k), \quad (13)$$

a wave of frequency ω, with longitudinal wavenumber k and circumferential wavenumber m. Substitution of (13) in (12) shows that the radial dependence is specified, for a cylindrical nozzle, by a Bessel function of order m:

$$P_m(R,k) = J_m(\kappa R), \quad (14a)$$

with radial wavenumber

$$\kappa = \sqrt{(\omega/c)^2-k^2}, \quad (14b)$$

specified by a boundary condition at the nozzle wall $R=a$, which specifies the radial modes κ_{nm} with $n=1,\ldots,\infty$, which may be distinct for each circumferential order m.

3.2 Amplitudes of sound generation in a nozzle

Thus the acoustic field in the nozzle consists of a superposition (13, 14a) of:

$$Q(R,\alpha,z,t) = e^{-i\omega t}\sum_{m=-\infty}^{+\infty} e^{im\alpha}\times \sum_{n=1}^{\infty} J_m(\kappa_{mn}R)e^{ik_{mn}z}A_{mn}, \quad (15)$$

where m is the azimuthal and n the radial order, the radial wavenumbers κ_{mn} are specified by the boundary condition at the nozzle wall $R = a$, the axial wavenumbers are related by (14b), *viz.*:

$$k_{mn} = \sqrt{(\omega/c)^2 - (\kappa_{mn})^2}, \tag{16}$$

and the amplitudes A_{mn} of each mode are specified by the source distribution in the nozzle. The pressure distribution $q(R,\alpha)e^{-i\omega t}$ in the nozzle exit plane $z = 0$ is thus specified by:

$$q(R,\alpha) = Q(R,\alpha,0,0) = \sum_{m=-\infty}^{+\infty} e^{im\alpha} \sum_{n=1}^{\infty} J_m(\kappa_{mn}R)A_{mn}, \tag{17}$$

which may be substituted in the radiation integral (11) to specify the sound field received by the observer in the far-field:

$$p(r,\theta,t) = \frac{e^{i\omega(r/c-t)}}{4\pi r} \sum_{m=-\infty}^{+\infty} \sum_{n=1}^{\infty} p_{mn}(r,\theta)A_{mn}, \tag{18a}$$

which is specified by a superposition of radiation integrals for each mode:

$$p_{mn}(r,\theta) = \left\{1 - i\frac{d}{d(\omega r/c)}\right\} \int_0^{2\pi} d\alpha \int_0^a dR\, R e^{-i(\omega R/c)\sin\theta\cos\alpha} e^{im\alpha} J_m(\kappa_{mn}R). \tag{18b}$$

It has been shown [27] that (18b) can be expressed in terms of Bessel functions as:

$$p_{mn}(r,\theta) = a^2 \int_0^1 J_m(\kappa_{mn}as)\left\{J_m(s\Omega\sin\theta) - i\frac{a}{r}\left[\frac{m}{\Omega}J_m(s\Omega\sin\theta) - \sin\theta J_{m-1}(s\Omega\sin\theta)\right]\right\} s\,ds, \tag{19}$$

where a dimensionless radial distance (20a) and a dimensionless frequency (20b) were introduced:

$$s = R/a \tag{20a}$$
$$\Omega = \omega a/c. \tag{20b}$$

Thus: (i) the total acoustic field (18a) consists of a spherical wave multiplying a sum of radial modes $n = 1,\ldots,\infty$ and azimuthal modes of order $m = 0,1,\ldots,\infty$; (ii) each mode (19) consists of a monopole term (first in the braces) and a dipole term (in square brackets); (iii) the parameters are the dimensionless frequency (20b), the ratio of the the nozzle radius to the distance of the observer a/r in the dipole term (which is weak because $a^2 << r^2$ for observers in the far-field), and the radial wavenumbers $\kappa_{mn}a$, determined by the boundary condition at the duct wall.

4 Acoustic effects of non-uniform impedance

The acoustic pressure for an observer in the far-field depend on the radial wavenumbers in the nozzle, which are specified by the wall boundary conditions (4.1). This allows comparison of hard-walled nozzles, with liners with constant impedance and non-uniform liners with impedance distribution varying circumferentially.

4.1 Rigid, impedance and non-uniform walls

The simplest case (I) is a nozzle with rigid walls, for which the normal velocity at the wall is zero, implying from the momentum equation that the normal derivative of the pressure is zero:

$$0 = i\omega v_n(R = a) = \rho^{-1} \left.\frac{\partial p}{\partial r}\right|_{r=a}, \tag{21}$$

where ρ is the mass density. Thus: $J'_m(\kappa_{mn}a) = 0$, so that the radial wavenumbers κ_{mn} are determined by the zeros j_{mn} of the derivative of the Bessel function J_m. Since these zeros are real, the radial wavenumbers κ_{mn} are real, and the corresponding (16) axial wavenumbers:

$$k_{mn}a = \sqrt{\Omega^2 - (j_{mn})^2}, \tag{22}$$

are: (i) either real, for propagating or cut-on modes, if $|j_{mn}| \leq \Omega$; (ii) or imaginary, for evanescent or cut-off modes, if $|j_{mn}| > \Omega$. Since the zeros of the derivative of the Bessel function J'_m form an unbounded sequence $j_{m1}, j_{m2},\ldots$, there is a finite number of cut-on modes, larger for larger dimensionless frequency. The cut-off modes make a negligible contribution to radiation to the far-field, so the sum in the total acoustic pressure (18a) is restricted to cut-on modes. The amplitude of the latter cut-on modes has been found to be weakly dependent on mode order for turbomachinery noise, and thus A_{mn} may be taken as a constant factor, and omitted together with the spherical wave term $e^{i\omega(r/c-t)}/2\pi r$ and a^2, which are common factors, regardless of the wall condition, and thus do not affect the comparison between rigid and lined walls. For a rigid wall, the far-field acoustic pressure is thus specified by a directivity factor:

$$P(\theta) = \sum_{m=0}^{\infty} \sum_{n=1}^{j_{mn}<\Omega} \int_0^1 J_m(j_{mn}s) J_m(s\Omega \sin\theta)\, s\, \mathrm{d}s, \tag{23}$$

where only the monopole term was considered, since it dominates the dipole term.

In the case (II) of a wall with uniform impedance $\bar{Z}_0$, the radial wavenumbers are specified by the roots of:

$$\mathrm{i} Z_0 J'_m(\kappa_{mn}a) = J_m(\kappa_{mn}a), \tag{24}$$

where Z_0 is the specific impedance, *i. e.* the impedance divided by that of a plane wave:

$$Z_0 = \bar{Z}_0/\rho c. \tag{25}$$

The radial wavenumbers κ_{mn} are generally complex, and the axial wavenumbers (16) also:

$$k_{mn}a = \sqrt{\Omega^2 - (\kappa_{mn}a)^2}. \tag{26}$$

Although the distinction between cut-on and cut-off modes is not so clear in the case of an impedance wall, the sum for the total acoustic field is taken for the cut-on modes of a rigid wall, as in (23), but for the complex radial wavenumbers, so that (23) remains valid, and can be evaluated (cfr. [26]). The acoustic pressure received in the far-field is:

$$p_{mn}(\theta) = -2\mathrm{i}k_{mn}a\left[(\kappa_{mn}a)^2 - (\Omega\sin\theta)^2\right]^{-1} \left[\Omega\sin\theta J_{mn}(\kappa_{mn}a)J'_m(\Omega\sin\theta) - \kappa_{mn}aJ_m(\Omega\sin\theta)J'(\kappa_{mn}a)\right]. \tag{27}$$

4.2 Circumferentially non-uniform liners

Expression (27) for the directivity of the acoustic pressure in the far-field can also be used in the case of non-uniform wall impedance $Z(\theta, z)$, except that radial wavenumbers are no longer determined as the roots of (24). In the case of a circumferentially non-uniform distribution, which is represented by the first two terms of a Fourier series:

$$Z_A(\theta) = Z_0(1 + 2\varepsilon\cos\theta), \tag{28}$$

the radial wavenumbers are specified exactly by the roots of an infinite determinant,

$$\det\left[\mathrm{i}\frac{\Omega}{\kappa a}J_m(\kappa a)\delta_{mm'} - Z_{m'-m}J'_m(\kappa a)\right] = 0, \tag{29}$$

where all the impedance Fourier coefficients $Z_m = 0$ except Z_0 and $Z_{\pm 1} = \varepsilon Z_0$.

5 Noise reduction for far-field observer

The eigenvalues and eigenfunctions are calculated for circumferentially non-uniform liners, to assess the noise reduction benefit relative to uniform liners.

5.1 Eigenvalues for the dimensionless radial wavenumbers

Arguably the best measure of the effectiveness of an acoustic liner is the effect on the reduction of far-field noise. As a numerical example consider a nozzle of radius $a = 1$m, and a wave frequency $f = 1$kHz or $\omega = 2\pi f = 6.28 \times 10^3 \mathrm{s}^{-1}$, corresponding, for a sound speed $c = 340\mathrm{ms}^{-}1$, to a dimensionless frequency or Helmholtz number $\Omega = \omega R/c = 18.5$.

The condition $j_{mn} < \Omega$ for the roots j_{mn} of the derivatives of the Bessel functions $J'(j_{mn}) = 0$, specifies the propagating or cut-on modes.

The radial wavenumbers can be determined using (24) for a uniform specific impedance impedance:

$$Z_0 = 2.5 - \mathrm{i}0.4, \tag{30}$$

and (29), for circumferentially non-uniform impedances (28), with relative amplitude of the harmonic

$$\varepsilon = 0.1 + 0.1\mathrm{i}, \quad (31a)$$
$$\varepsilon = 0.2 - 0.3\mathrm{i}. \quad (31b)$$

The former is designated comparatively the 'weakly' (31a) and the latter the 'strongly' (31b) non uniform liner, although in both cases the impedance is the same (30) on the mean, and the variation from the mean is small in absolute terms in both cases.

5.2 Radiation integrals

The non-dimensional radial wavenumbers determined can now be used in the specification of the radiation integrals (27) for an angle θ between the duct axis and the observer in the far field, where $0 \le \theta < \pi/2$.

In figure 2 the modulus and the phase of p_{mn} as a function of θ are represented for radial orders $n = 1-4$, for uniform impedance $Z_0 = 2.5 - 0.4\mathrm{i}$ and for 'weak' (31a) and 'strong' (31b) impedance non-uniformity, for $m = 0$. The radiation integral is almost unchanged for 'weak' non-uniform impedance, but for 'strong' non-uniformity the changes both in modulus and in phase are significant, and are greater for higher radial orders n.

In figure 3 the modulus and the phase of p_{mn} as a function of θ are represented for the radial wavenumbers corresponding to the first radial order n and to circumferential orders $m = 0, 2, 4, 6, 8, 10, 12$, and 14, for uniform impedance (30) and 'weak' and 'strong' non-uniformities. As in the previous cases, the radiation integral is almost unchanged for 'weak' non-uniform impedance, but for 'strong' non-uniformity the changes both in modulus and in phase are significant and depend strongly both on the angle θ and on the circumferential order m.

Note that the maximum of the radiation integrals in the interval $0 \le \theta < \pi/2$ decreases as the circumferential order increases, and is obtained for greater values of θ. As a result, the effects of impedance non-uniformity, which are greater for higher circumferential orders, are stronger for directions far from the duct axis.

6 Discussion

The effect of non-uniform impedance on sound radiation to the far-field has been assessed on the basis of the acoustic pressure on the disk corresponding to the nozzle exit plane, without accounting for edge diffraction effects which can be quite important [25], in particular in the presence of a mean flow, when a turbulent and irregular shear layer [1, 2] is issued from the nozzle lip. The latter causes spectral and directional broadening of sound, which further reduces the noise levels. Although the latter effects have not been modelled here, it is clear that an attenuation of the 'input' sound field incident on the shear layer from the interior of the jet will result in a further attenuated sound field transmitted to an observer in the far-field outside the shear layer. The attenuations shown for the basic in-duct sound field are clear even though: (i) the non-uniform liner (28) has the same average impedance as the uniform liner; (ii) only one liner impedance 'harmonic' was used, involving a single parameter ε, which was not optimized. By considering multi-harmonic impedance distribution

$$Z(\theta) = Z_0 \left[1 + \sum_{l=1}^{L} \varepsilon_l \cos(l\theta) \right], \quad (32)$$

and optimizing the parameter ε_l, greater noise reduction could be obtained. In the present example a relative impedance variation $|\varepsilon|$ of $20-30\%$ was shown to be sufficient to affect significantly the sound fields.

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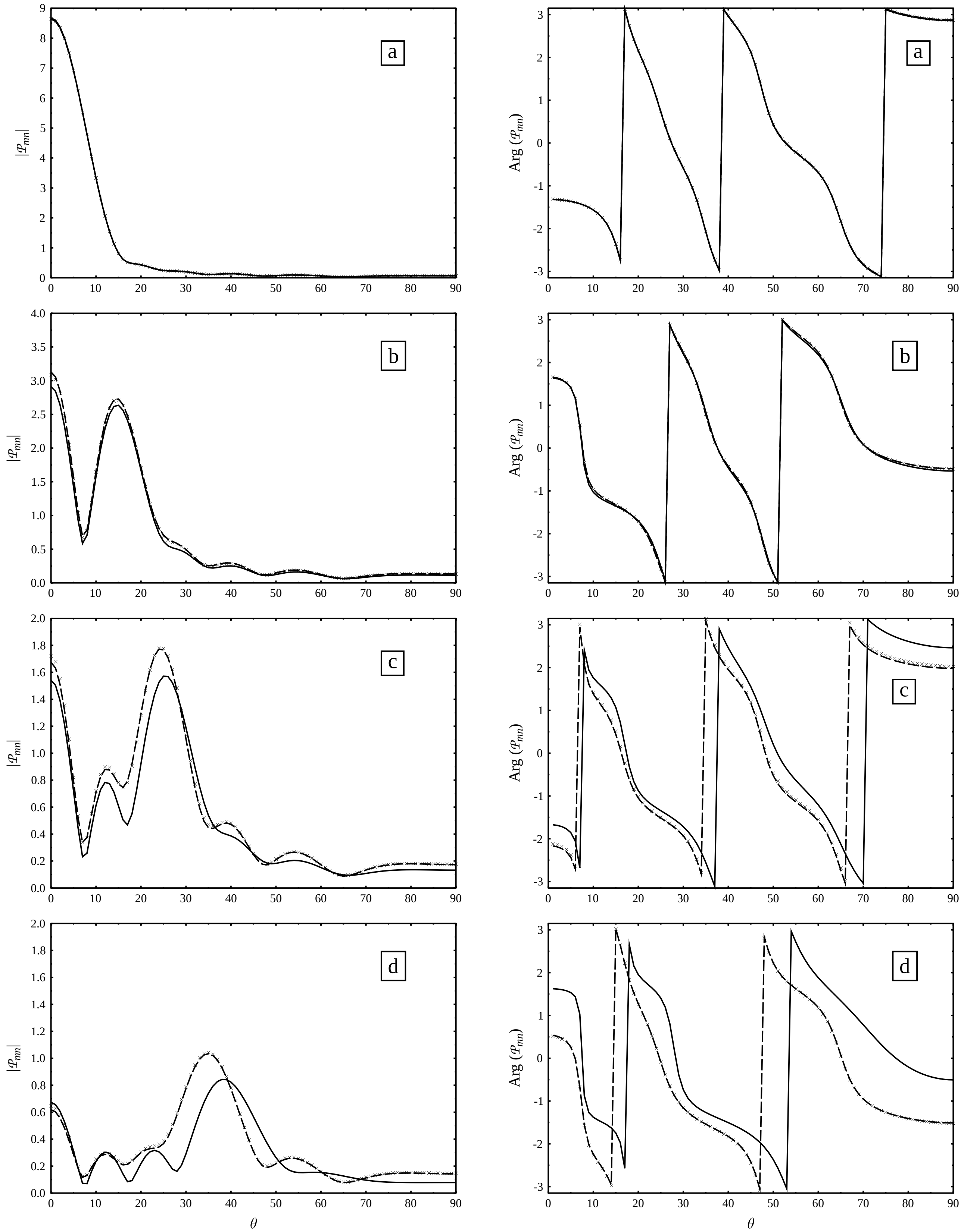

Fig. 2 Modulus and phase of the radiation integral p_{mn} as a function of θ in the cases of uniform impedance $Z_0 = 2.5 - 0.4i$ ($\times$) and non-uniform impedance with $\varepsilon = 0.1 + 0.1i$ (– –) and $\varepsilon = 0.2 - 0.3i$ (—) for $m = 0$ and radial orders: (a) $n = 1$; (b) $n = 2$; (c) $n = 3$; (d) $n = 4$.

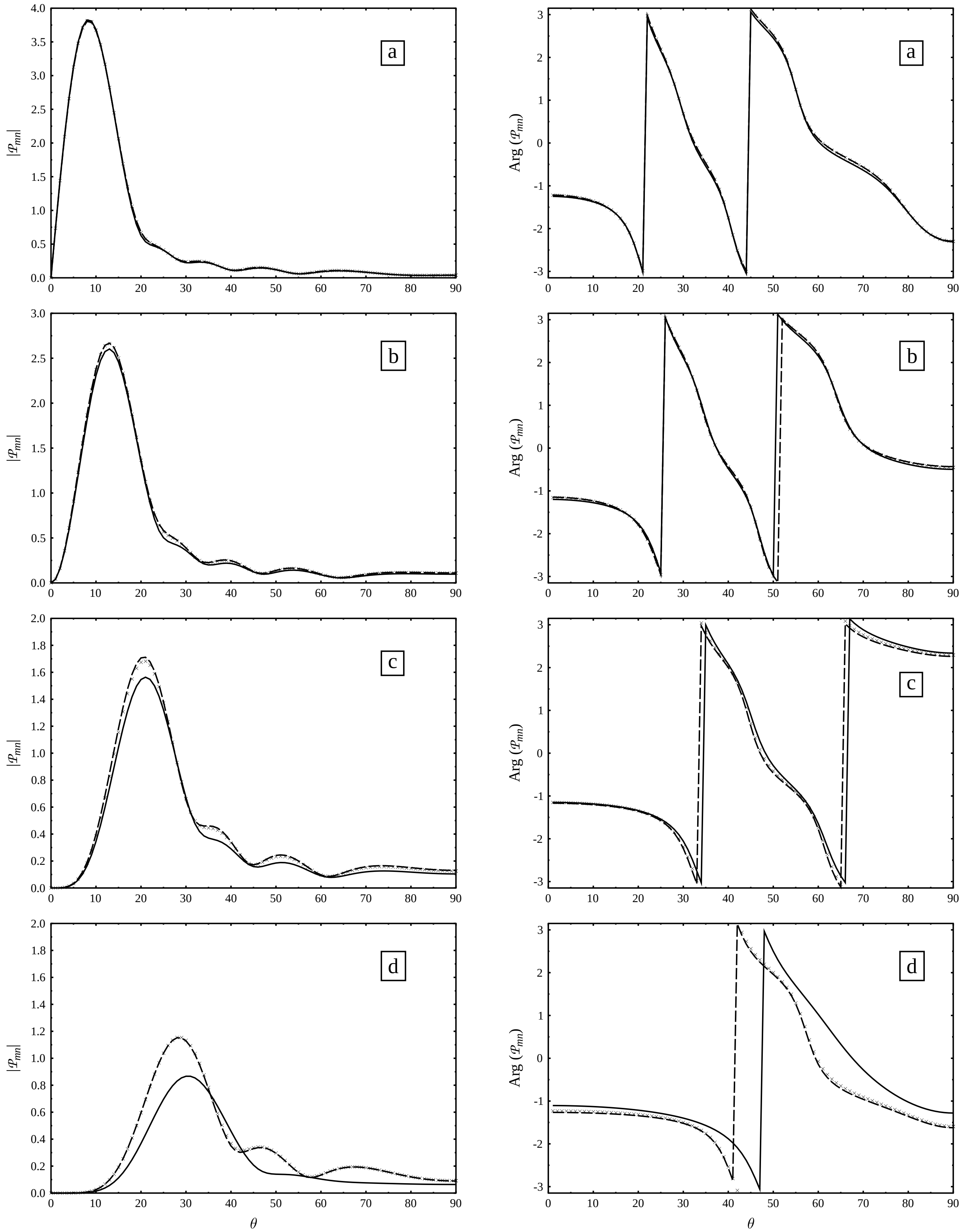

Fig. 3 Modulus and phase of the radiation integral p_{mn} as a function of θ in the cases of uniform impedance $Z_0 = 2.5 - 0.4\mathrm{i}$ ($\times$) and non-uniform impedance with $\varepsilon = 0.1 + 0.1\mathrm{i}$ (– –) and $\varepsilon = 0.2 - 0.3\mathrm{i}$ (—) for radial order $n = 1$ and circumferential orders: (a) $m = 1$; (b) $m = 2$; (c) $m = 4$; (d) $m = 6$.

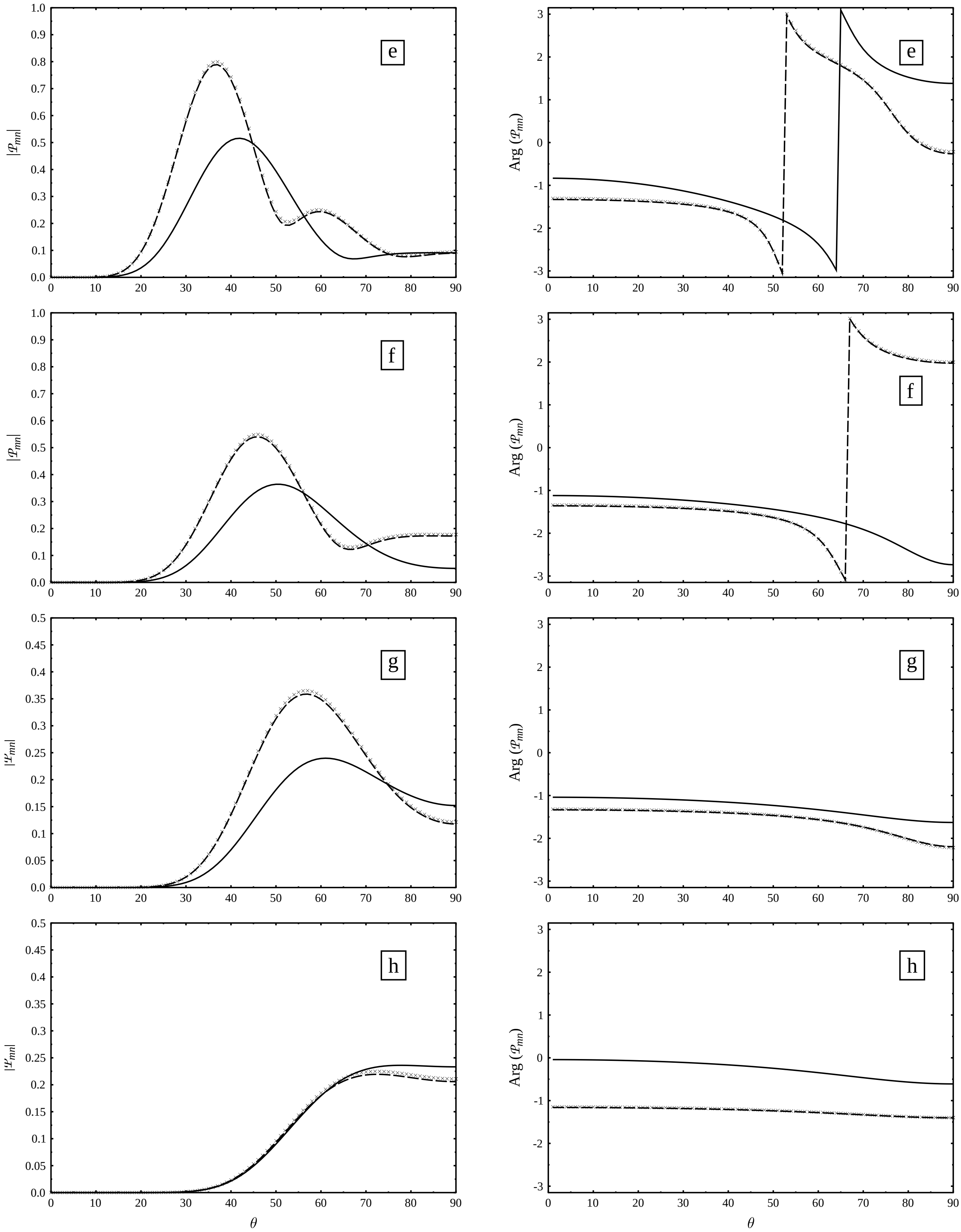

Fig. 3 (Cont.) Modulus and phase of the radiation integral p_{mn} as a function of θ in the cases of uniform impedance $Z_0 = 2.5 - 0.4\text{i}$ ($\times$) and non-uniform impedance with $\varepsilon = 0.1 + 0.1\text{i}$ (– –) and $\varepsilon = 0.2 - 0.3\text{i}$ (—) for radial order $n = 1$ and circumferential orders: (e) $m = 8$; (f) $m = 10$; (g) $m = 12$; (h) $m = 14$.

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