ON AN ABSOLUTE SAFETY CRITERION FOR THE PROBABILITY OF THE COLLISION BETWEEN AIRCRAFT

L.M.B.C. Campos, J.M.G. Marques
Secção de Mecânica Aeroespacial, Instituto Superior Técnico, 1049-001 Lisboa, Portugal

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Abstract

The safety of separation of aircraft can be assessed by calculating the probability of collision is the worst possible case, which for certain scenarios can be related to the probability of coincidence, as an alternative safety metric. Examples of safety metrics in the case of two aircraft flying always at minimum separation distance, are: (i) the maximum probability of coincidence P_0*, at the one point; (ii) the one-dimensional cumulative probability of coincidence* P_1 *along the line joining the two aircraft; (iii) the three-dimensional probability of coincidence* P_3 *over all space; (iv) a two dimensional probability of coincidence* P_2*. The latter two are defined in the present paper. It is shown that the general formula for* P_n *which holds for* $n = 0,1,3$ *also holds for* $n = 2$*. The ICAO Target Level of Safety of probability of collision is comparable to* P_1*. Since* P_2 *has no dimensions, it can lead to an absolute safety standard, which is independent of time or distance flown. A value of this absolute safety standard is proposed, which is consistent with the ICAO TLS standard for the usual flight separation rules.*

1 Introduction

The safety of air traffic is based on separation rules [1], and when they fail to be observed, on conflict resolution measures [2] to avoid a collision. The chosen separation can be reduced to increase air traffic capacity [3] if the risk of collision remains below the threshold set by ICAO Target Level of Safety of 5×10^{-9} per hour, which is an example of safety metric [4]. The pioneering work on the calculation of collision probabilities [5] is based on the penetration of a safety volume around an aircraft, by another aircraft. It can be shown that in the particular but important case of air corridors, i.e. aircraft flying at the same speed on parallel tracks, the probability of coincidence is an upper bound for the probability of collision [6], and thus is a safety metric which is easier to calculate.

The value of the probabilities of collision depends on the statistical distribution of aircraft position errors [7]. It can be argued that the Gaussian [8] is suited to frequent events [9] like small flight path deviations; collisions are due to large deviations [5,7], which are rare events [10] modeled by Laplacian or generalized error distributions [11]. A combined gamma and generalized error distribution can be used [12] to model the probability of flight path deviations over the whole range from small to large. The distributions of probabilities of flight path deviations may be different for dissimilar aircraft, e.g. in the Gaussian case the r.m.s. position errors could be different [13].

In the present paper two simple but important cases are considered, both with aircraft flying at the same speed, on: (case I) the same flight path at a given distance (Figure 1); (case II) on parallel flight paths at given distance (Figure 2). Using six assumptions (Sec. 2), including that Gaussian statistics, with dissimilar r.m.s. position errors for each aircraft, are calculated (Sec. 3) the: (Sec. 3.1) probability of coincidence (zero-dimensional) and the cumulative probability of overlap in one- or three-dimensions; (Sec. 3.2) these are compared with a two-dimensional probability of overlap, leading to a general formula (Sec. 3.3) applicable to all four safety metrics (Sec. 4). Comparison with the ICAO TLS (Sec. 4.1) shows that one of the metrics is dimensionless, and hence universal (Sec. 4.2). A correction factor between the Gaussian and the generalized error distribution is introduced (Sec. 4.3), before the discussion (Sec. 5) on the merits of the alternative safety metrics.

2 Assumptions

The simplified model of collision probability used, makes six assumptions. The first assumption is that the three dimensional position error is decomposed into horizontal along track and across track errors and vertical error: these three are considered statistically independent, and thus treated separately. For example, applications can be made to rather different separation distances e.g., lateral separation in transoceanic ($L = 60nm$) or controlled ($L = 5nm$) air space, and normal ($L = 2000\,ft$) or reduced ($L = 1000\,ft$) vertical separation minima; in this way only a one-dimensional collision problem needs to be considered at a time.

The second assumption is that the aircraft are treated as a mass points located at centers of mass. One approach to account for finite aircraft sizes L_1 and L_2, is to associate a volume to each aircraft. Since a "collision" occurs when the distance between the centers of mass is less than $L_1 + L_2$, an alternative is to reduce the separation distance from L to $L - L_1 - L_2$. Another alternative is to incorporate aircraft sizes L_1 and L_2 into the r.m.s. position errors σ_1 and σ_2. It can be shown [6] that the aircraft size affects collision probability if it is comparable to the r.m.s. position error. Generally the aircraft size multiplies the probability of coincidence.

The third assumption is that the aircraft are assumed to move in unbounded space, which is valid only if far from altitude limitations, e.g. the ground or the service ceiling. The space is also assumed to be unbounded horizontally, which will not be true close the boundaries of restricted airspace or geographical obstacles.

The fourth assumption is that the position errors are specified by a Gaussian probability distribution. This appears justified on the basis of the central limit theorem of the theory of statistics [8], using the Lindeberg [9] necessary and sufficient condition, viz.: (i) consider a long sequence of N deviations, spaced in time by at least a time τ, so that they are statistically independent; (ii) assume that the large deviations are improbable, in the sense that they make a small contribution to the total variance. Then a Gaussian probability distribution is satisfied with an error $O\left(1/\sqrt{N}\right)$; this reasoning can be flawed, in that collisions are very rare events, and thus a law of large numbers is not applicable. The Lindeberg condition assumes that events with large deviations from the mean make a small contribution to the variance; since collisions are rare events corresponding to large deviations from the mean, the Lindeberg condition is not met either. It is known from flight data records and radar tracks that the Gaussian underestimates the probability of large flight path deviations and collision [3], and whereas the Laplace distribution is an improvement [5]; a more accurate representation is provided by the generalized error distribution [11], in unimodal [6] or bi-modal [7] forms. These probability distributions can be extended to a combined Gamma and generalized error distribution [12] to model the whole range of flight path deviations from small to large. These distributions can be introduced [15] as a correction [13] to the Gaussian (§4.3).

The fifth assumption, is the geometry considered, of two aircraft flying at constant distance, along the same or parallel tracks, is one of the cases in which the probabilities of collision can be related to probabilities of overlap [6], and the latter can serve as alternative safety metrics.

The sixth assumption, allows the calculation of the probabilities of collision as a function of position if aircraft dynamics do not appear explicitly; since aircraft dynamics would limit the possible displacements, the probabilities of collision calculated in this way are upper bounds.

It should be noted that the calculation of probabilities of collision between aircraft involves more than one kind of probability density function, e.g. Gaussian with one or two variables, conditional or joint probabilities, and various possible integrations over space, leading to different dimensions. The conversion between probabilities per unit distance flown and per unit time can be made using the constant flight speed, in the present case. The comparison with ICAO Target Level of Safety of 5×10^{-9} per flight hour depends on how the probabilities are defined, and thus are affected by the assumptions of the model.

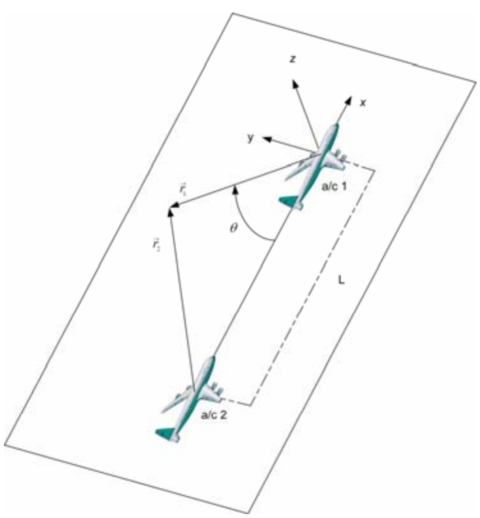

Fig. 1- For two aircraft flying along the same flight path at a minimum separation distance, the two-dimensional cumulative probability of collision, which is absolute, i.e. has no dimensions, is calculated by integrating over any plane passing through the flight path.

3 Compararison of probabilities of coincidence

3.1 One and three-dimensional probabilities of overlap

The calculation of maximum coincidence probabilities is made for the case of two aircraft flying on the same straight flight path at a distance L. A coincidence occurs if the first aircraft deviates by $\mathbf{r}_1$ and the second by $\mathbf{r}_2$ such that $\mathbf{r}_2 = \mathbf{r}_1 + L\mathbf{e}_x$ in Figure 1. The coincidence occurs on a plane through the flight path, and thus it is possible to introduce polar coordinates in this plane, with origin at aircraft one, and axis along the flight path:

$$\mathbf{r}_1 = \left(-r\cos\theta, r\sin\theta\right), \mathbf{r}_2 = \left(L - r\cos\theta, r\sin\theta\right). \tag{1a,b}$$

The probability distributions are initially assumed to be Gaussian for both aircraft:

$$P(\mathbf{r}_1) = \left[1/\left(\sigma_1\sqrt{2\pi}\right)\right]\exp\left[-\left(\|\mathbf{r}_1\|/\sigma_1\right)^2/2\right] = \left[1/\left(\sigma_1\sqrt{2\pi}\right)\right]\exp\left[-\left(r/\sigma_1\right)^2/2\right],$$

$$P(\mathbf{r}_2) = \left[1/\left(\sigma_2\sqrt{2\pi}\right)\right]\exp\left[-\left(\|\mathbf{r}_2\|/\sigma_2\right)^2/2\right] = \left[1/\left(\sigma_2\sqrt{2\pi}\right)\right]\exp\left[-\left(L/\sigma_2\right)^2/2\right] \times\exp\left\{\left(-r^2+2rL\cos\theta\right)/\left[2\left(\sigma_2\right)^2\right]\right\}, \tag{2a,b}$$

with r.m.s. position errors respectively σ_1 and σ_2, which may or may not be equal (other probability distributions will be considered in Sec. 4.3). Exactly the same formulas (1a,b; 2a,b) will apply (case II) for two aircraft on parallel flight paths at a distance L, using polar coordinates with origin on the first aircraft and axis perpendicular to the flight paths as shown in Figure 2. Assuming that the position errors are statically independent for the two aircraft, the probability of coincidence is the product of (2a,b);

$$P(r,\theta) = P(\mathbf{r}_1)P(\mathbf{r}_2)$$
$$= \left[1/(2\pi\sigma_1\sigma_2)\right]\exp\left[-(L/\sigma_2)^2/2\right] \quad , \quad (3)$$
$$\exp\left[-(r^2/2)(\sigma_1^{-2}+\sigma_2^{-2})+rL\sigma_2^{-2}\cos\theta\right]$$

and depends only on (r,θ) in both cases I and II.

From (3) can be defined a one-dimensional cumulative probability of coincidence, by integrating over $-\infty < r < +\infty$ along the polar axis $\theta = 0$:

$$\bar{P} \equiv \int_{-\infty}^{+\infty} P(r,0)\,dr \equiv P_1 \ , \qquad (4a)$$

viz. : (case I) the integration is along the flight path; (case II) the integration is along a line perpendicular to the flight paths passing through the aircraft. Substitution of (3) into (4a) leads [13] to the one-dimensional probability of collision:

$$P_1 = \left[1/\left(2\bar{\sigma}\sqrt{\pi}\right)\right]\exp\left\{-\left[L/(2\bar{\sigma})\right]^2\right\}, \qquad (4b)$$

which involves the r.m.s. position error $\bar{\sigma}$ corresponding to the arithmetic mean of the variances of the position errors of the two aircraft:

$$\bar{\sigma} \equiv \sqrt{\left[(\sigma_1)^2+(\sigma_2)^2\right]/2} \quad . \qquad (4c)$$

The three-dimensional cumulative probability of coincidence involves an integration over all space in spherical coordinates:

$$\bar{\bar{P}} \equiv \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\theta \sin\theta \int_0^{+\infty} \mathrm{d}r r^2 P(r,\theta) \equiv P_3 \ , \qquad (5a)$$

and leads via a broadly similar integration to:

$$P_3 = \left(\sqrt{\pi}/2\right)\left(\bar{\sigma}/f^2\right)\exp\left\{-\left[L/(2\bar{\sigma})\right]^2\right\} \ , \qquad (5b)$$

involving $\bar{\sigma}$ in (4c) and the aircraft dissimilarity function:

$$f \equiv (\sigma_1/\sigma_2+\sigma_2/\sigma_1)/2 . \qquad (5c)$$

The extremum of the cumulative probabilities of coincidence (3) occurs for:

$$\partial P/\partial\theta = 0 = \partial P/\partial r : \theta_0 = 0,$$
$$r_0 = L/\left(1+(\sigma_2/\sigma_1)^2\right) \quad , \qquad (6a)$$

and it corresponds to a maximum because:

$$\theta = \theta_0, r = r_0 :$$
$$\partial^2 P/\partial\theta^2 > 0 > \left(\partial^2 P/\partial\theta^2\right)\left(\partial^2 P/\partial r^2\right) - \left(\partial^2 P/\partial r\partial\theta\right)^2 \qquad (6b)$$

The value of the probability of collision at the maximum is:

$$P(r_0,\theta_0) = \left[1/(2\pi)\right]\left(f/\bar{\sigma}^2\right)\exp\left\{-\left[L/(2\bar{\sigma})\right]^2\right\} \equiv P_0 \qquad (6c)$$

and this may be considered as a zero-dimensional probability, since it applies at a point (i.e. the maximum), which is a domain of dimension zero.

3.2 General formula and two-dimensional case

The collision probabilities P_n of dimensions $n=0$ or zero in (6c), $n=1$ or one in (4b) and $n=3$ or three in (5b), all satisfy a common formula:

$$n = 0,1,3 :$$
$$P_n = (1/2)\pi^{n/2-1} f^{1-n}\bar{\sigma}^{n-2}\exp\left\{-\left[L/(2\bar{\sigma})\right]^2\right\} \ ,(7)$$

the simplest formula is for dimension *n=1*, and in terms of this the others can be expressed:

$$P_n = P_1 \pi^{(n-1)/2} f^{1-n}\bar{\sigma}^{n-1} \quad . \qquad (8)$$

This formula is an identity for $n=1$, and is non-trivial for $n=0,3$. The one-dimensional probability of coincidence P_1 per unit distance $P_1 \sim \bar{\sigma}^{-1}$ has the dimensions of inverse distance can be compared to the ICAO TLS standard 5×10^{-9} per hour after multiplication by the velocity. The others have different dimensions, e.g. $P_n \sim \bar{\sigma}^{n-2}$ has the dimensions of distance to the power $n-2$ and can be compared to the ICAO TLS standard in a less straightforward way (Sec. 4). Thus the only case of dimensionless probability of coincidence is the two-dimensional case $n=2$. In order to find out whether the formulas (7,8) apply to $n=2$ as well as to $n=0,1,3$, it is necessary to evaluate the two-dimensional cumulative probability of coincidence:

$$P_2 \equiv \int_0^{2\pi} \mathrm{d}\theta \int_0^{+\infty} \mathrm{d}r rP(r,\theta) \quad (9)$$

which may be interpreted as the collision probability integrated over: (case I) a plane passing through the trajectory (figure 1); (case II) a plane perpendicular to the trajectories passing through both aircraft (figure 2).

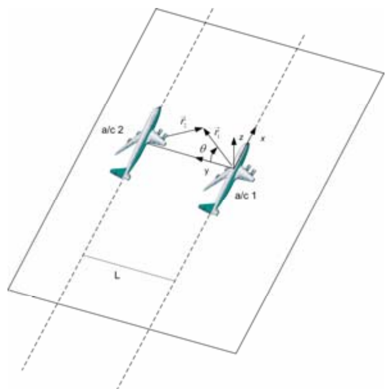

Figure 2 - In the case of two-aircraft flying along parallel flight paths at the minimum separation distance the integration is over a plane perpendicular to the two flight paths and passing through both aircraft.

It may be expected that the two-dimensional cumulative probability of coincidence can be put into the form (7) for $n=2$:

$$P_2 = (1/2)(g/f)\exp\left\{-\left[L/(2\bar{\sigma})\right]^2\right\}, \quad (10)$$

or (8) in terms of P_1 viz.:

$$P_2 = P_1\sqrt{\pi}\, g(\bar{\sigma}/f), \quad (11)$$

by inserting a correction function g, which would be unity $g=1$ if (7,8) hold for $n=2$. In order to find out whether this is or not the case, the value of g will be calculated by comparing (10) with the result of evaluating the two-dimensional probability integral (9,3), viz.:

$$P_2 = \left[1/(2\pi\sigma_1\sigma_2)\right]\exp\left[-(L/\sigma_2)^2/2\right] \int_0^\infty dr \exp\left[-(r^2/2)(\sigma_1^{-2}+\sigma_2^{-2}) I(r) r\right], \quad (12a)$$

where the dr integration is performed after

$$I(r) = \int_0^{2\pi} \exp(rL\sigma_2^{-2}\cos\theta)\, \mathrm{d}\theta, \quad (12b)$$

the $\mathrm{d}\theta$ integration in (12b).

3.3 Calculation of the two-dimensional

Starting with the $\mathrm{d}\theta$ integration (12b), use of the power series for exponential yields:

$$a \equiv rL\sigma_2^{-2} : I(r) = \sum_{n=0}^{\infty}(a^n/n!)\int_0^{2\pi}\cos^n\theta\, d\theta. \quad (13a)$$

It is clear that the integrals in (13a) vanish unless n is an even integer

$$n=2p: \quad I(r) = \sum_{n=0}^{\infty}\left[(a^{2p})/(2p)!\right]\int_0^{2\pi}\cos^{2p}\theta\, d\theta. \quad (13b)$$

The definition of cosine is used in the last integral:

$$\int_0^{2\pi}\cos^{2p}\theta\, d\theta = 2^{-2p}\int_0^{2\pi}\left(\mathrm{e}^{i\theta}+\mathrm{e}^{-i\theta}\right)^{2p}\, d\theta = 2^{-2p}\sum_{q=0}^{\infty}\binom{2p}{q}\int_0^{2\pi}\mathrm{e}^{i2(p-q)\theta}\, d\theta, \quad (14a)$$

together with the binomial expansion; it is clear that the last integral in (14a) vanishes unless $q=p$, in which case it equals 2π . Thus the whole sum reduces to one term:

$$\int_0^{2\pi}\cos^{2p}\theta\, d\theta = 2^{-2p}\binom{2p}{q}2\pi = \pi 2^{1-2p}\left[(2p)!/(p!)^2\right] \quad (14b)$$

Substitution of (14b) into (13a,b) yields:

$$I(r) = 2\pi\sum_{p=0}^{\infty}\left(rL\sigma_2^{-2}/2\right)^{2p}(p!)^{-2}, \quad (15a)$$

which completes the $\mathrm{d}\theta$-integration in the *2-D* probability (12a):

$$P_2 = \left[1/(\sigma_1\sigma_2)\right]\exp\left[-(L/\sigma_2)^2/2\right] \times\sum_{p=0}^{\infty}(L/2)^{2p}\sigma_2^{-4p}(p!)^{-2}\int_0^\infty r^{2p+1}\exp\left[-(r^2/2)(\sigma_1^{-2}+\sigma_2^{-2})\right]dr, \quad (15b)$$

leaving only the dr-integration.

Concerning the dr-integration, it is convenient to make the change of variable:

$$\zeta \equiv (r/\sqrt{2})(\sigma_1^{-2}+\sigma_2^{-2})^{1/2}, \mathrm{d}r \equiv \sqrt{2}(\sigma_1^{-2}+\sigma_2^{-2})^{-1/2}\mathrm{d}\zeta \quad (16a,b)$$

which simplifies (15b) to:

$$P_2 = \left[2(\sigma_1^{-2}+\sigma_2^{-2})^{-1}/(\sigma_1\sigma_2)\right]\exp\left[-(L/\sigma_2)^2/2\right] \sum_{p=0}^{\infty}(L^2/2)^p\sigma_2^{-4p}(p!)^{-2}(\sigma_1^{-2}+\sigma_2^{-2})^{-p}I_p \quad (17a)$$

where the integrals

$$\xi \equiv \zeta^2: \qquad I_p \equiv \int_0^{+\infty} \zeta^{2p+1} \exp\left(-\zeta^2\right) \mathrm{d}\zeta = \left(1/2\right) \int_0^{+\infty} \xi^p \mathrm{e}^{-\xi} \mathrm{d}\xi = \left(p!\right)/2 \tag{17b}$$

are evaluated in terms of the Gamma function:

$$\int_0^{+\infty} \xi^p \mathrm{e}^{-\xi} \mathrm{d}\xi = \Gamma\left(p+1\right) = p! \tag{18}$$

This completes the evaluation of the *2-D* probability of collision. Substitution of (17b) into (17a) specifies the two-dimensional cumulative probability of coincidence:

$$P_2 = \left[1/\left(\sigma_1/\sigma_2 + \sigma_2/\sigma_1\right)\right] \exp\left[-\left(L/\sigma_2\right)^2/2\right] \sum_{p=0}^{\infty} \left\{\left[\left(L^2/2\right)\left(\sigma_1/\sigma_2\right)^2\right] / \left[\left(\sigma_1\right)^2 + \left(\sigma_2\right)^2\right]\right\}^p / p! \tag{19}$$

where the last factor is the series for the exponential

$$\sum_{p=0}^{\infty} \left\{\left[\left(L/2\right)\left(\sigma_1/\sigma_2\right)\right]^2 / \left[\left(\sigma_1\right)^2 + \left(\sigma_2\right)^2\right]\right\}^p / p! = \exp\left\{\left[\left(L/2\right)\left(\sigma_1/\sigma_2\right)\right]^2 / \left[\left(\sigma_1\right)^2 + \left(\sigma_2\right)^2\right]\right\}. \tag{20}$$

Substitution of (20) into (19) yields:

$$P_2 = \left[1/\left(\sigma_1/\sigma_2 + \sigma_2/\sigma_1\right)\right] \exp\left[-\left(L^2/2\right)/\left(\left(\sigma_1\right)^2 + \left(\sigma_2\right)^2\right)\right] \tag{21}$$

Introducing $\bar{\sigma}$ from (4c) and the dissimilarity function *f* from (5c) yields:

$$P_2 = \left(2/f\right) \exp\left\{\left[-L/\left(2\bar{\sigma}\right)\right]^2\right\} \tag{22}$$

which coincides with (10) without need for a correction factor $g = 1$. It follows that (7) holds not only for $n = 0,1,3$ but also for $n = 2$; also, (8) applies as well for $n = 0,1,2,3$.

4 Compararison with the ICAO TLS standard

4.1 The original and alternative ICAO TLS standards

The coincidence probabilities have been considered, in the two cases of aircraft flying with minimum separation L: (I) along the same trajectory (Figure 1); (II) along parallel trajectories (Figure 2).The coincidence probabilities have been considered in four 'dimensions': ($n = 0$) the dimension zero is the maximum probability of coincidence P_0 at a point between the two aircraft (on the flight path in case I, and between the flight paths in case II), and has the dimensions of inverse of distance squared; ($n = 1$) the dimension one corresponds to the cumulative probability of coincidence P_1 along a line (the trajectory in case I and a line perpendicular to the trajectories and passing through the aircraft in case II), and has the dimensions of inverse of distance; ($n = 2$) the two-dimensional case corresponds to the cumulative probability of coincidence on a plane P_2 (any plane passing through the trajectory in case I, and the plane perpendicular to the trajectories and passing through both aircraft in case II) and has no dimensions; ($n = 3$) the three-dimensional case corresponds to the cumulative probability of coincidence in all space (both in cases I and II), and has the dimensions of distance.

The ICAO target level of safety (TLS) specifies a probability of collision 5×10^{-9} per hour flown, which can be converted in probability of collision per nautical mile S_1/V by dividing by the speed V in knots. Thus the ICAO TLS standard is directly comparable to the one-dimensional cumulative probability of coincidence $P_1 V \le S_1$. In order to use the maximum probability of coincidence , it would be necessary to use as safety metric a modified ICAO TLS standard $P_0 V^2 \le S_0$, with the dimensions of inverse of square of hour flown. To apply the three-dimensional cumulative probability of coincidence $P_3/V \le S_3$ would need the introduction of another safety metric or modified ICAO TLS standard with the dimensions of hour flown. The general formula

$$n = 0,1,2,3: \qquad P_n V^{2-n} \le S_n, \tag{23}$$

shows that the modified ICAO TLS standard would be absolute i.e. independent of velocity, only in the case $n = 2$, because the two-dimensional cumulative probability of collision is independent of distance, i.e. the 'safety relation'

$$n = 2: \qquad P_2 \le S_2 \tag{24}$$

is dimensionless. The possibility of introducing an absolute, dimensionless ICAO TLS standard is discussed next.

4.2 The original and alternative ICAO TLS standards

Taking as reference case aircraft with identical r.m.s. position errors, the ratio to the minimum separation distance which typically meets [13] the current ICAO TLS standard:

Table I- Two-dimensional probability of coincidence for similar and dissimilar aircraft, and three ratios of separation L to r.m.s. position error, using Gaussian statistics.

$L/\bar{\sigma}$	10	11	12
P_2	6.95×10^{-12}	3.64×10^{-14}	1.16×10^{-16}
P_2/f	1.26×10^{-12}	6.62×10^{-15}	2.11×10^{-17}

(25)

is indicated in the first line, together with: (i) the two-dimensional probability of coincidence calculated from (22) for identical aircraft, i.e. with the same r.m.s. position errors:

$$\sigma_1 = \sigma_2 = \bar{\sigma} \equiv \sigma : P_2 = (1/2)\exp\left[-(L/\sigma)^2/4\right]; \tag{26}$$

(ii) the aircraft dissimilarity factor (5c) has the upper bound

$$\sigma_2 \le \sigma_1 \le 10\sigma_2 : f \le (10+1)/2 = 5.5, \tag{27}$$

for dissimilar aircraft with r.m.s. position errors in a ratio of not more than one order of magnitude. Taking in (25) the geometric mean of: (i) the least strict condition for dissimilar aircraft (1.26×10^{-12}) and the intermediate condition for similar aircraft (3.64×10^{-14}) leads to $\sqrt{1.26\times10^{-12}\times3.64\times10^{-14}} = 2.14\times10^{-13}$, which suggests:

$$P_2 \le S_2 = 2\times10^{-13}, \tag{28}$$

as the alternative absolute ICAO TLS standard, which will be checked next.

In order to asses the implications of this choice of absolute safety standard, it is applied to the following four typical ATM cases: (i) lateral separation in transoceanic airspace $L_1 = 60nm$; (ii) lateral separation in controlled airspace $L_2 = 5nm$; (iii) reduced vertical separation minima (RVSM) $L_3 = 1000\,ft$ in controlled airspace at lower flight levels (above FL 290); (iv) vertical separation $L_4 = 2000\,ft$ elsewhere. For these four values L_m with $m = 1, 2, 3, 4$, the proposed absolute target level of safety (28) corresponds by (26) to $L_m/\sigma_m = 10.7$ and thus to a r.m.s. position error σ_m indicated in

Table II- One-dimensional probability of coincidence based in Gaussian statistics, and maximum velocity for which the ICAO TLS standard is met.

m	1	2	3	4
Lm	60 nm	5 nm	1000 ft	2000 ft
σ_m	5.61 nm	0.468 nm	93.6 ft	187 ft
P1m (per nm)	2.01×10^{-14}	2.41×10^{-12}	7.33×10^{-12}	3.66×10^{-12}
$V_m \le S_1/P_{1n}$	$2.48\times10^{5}kt$	$2.07\times10^{4}kt$	6.82×10^{2}	$1.36\times10^{3}kt$

(29)

together with the one-dimensional probability of coincidence (4b) per nautical mile, which satisfies the ICAO TLS standard:

$$S_1 = 5\times10^{-9} \text{ per hour}, \tag{30}$$

for airspeeds up to V_m. Since V_m exceeds the speed capability of all current subsonic airliners, the absolute alternative ICAO TLS standard (28) is safe in all these conditions. The values of

V_m suggest that the absolute ICAO TLS standard (28) is stricter than the original ICAO TLS standard (30) in the four cases considered. This is the price to be paid for having an ICAO TLS standard which is absolute, i.e. applies to all separation conditions, not just the four examples given.

4.3 Correction factor

The suggested alternative TLS standard is dimensionless for all probability distribution of aircraft deviations. The preceding examples using the Gaussian distribution can be extended to other distributions, e.g. using a correction factor. This is illustrated by the generalized error distribution, which is appropriate to describe large flight path deviations [7,12] relevant to potential collisions. The preceding calculations were based on the Gaussian probability distribution

$$P(x;2)=1/\left(\sigma\sqrt{2\pi}\right)\exp\left[-x^2/\left(2\sigma^2\right)\right] \quad (31)$$

which is the particular case *k=2* of [11] the generalized error distribution

$$P(x;k)=A\exp\left[-a\left(|x|/\sigma\right)^k\right] \quad (32)$$

where [6]:

$$A\equiv 1/\left(2\sigma\Gamma\left(1+1/k\right)\right)\sqrt{\Gamma\left(3/k\right)/\Gamma\left(1/k\right)}, \quad (33a)$$

$$a\equiv\left[\Gamma\left(3/k\right)/\Gamma\left(1/k\right)\right]^{k/2} \quad (33b)$$

where Γ denotes the Gamma function. For $k=2$, it follows from (33b) that $a\equiv\left[\Gamma\left(3/2\right)/\Gamma\left(1/2\right)\right]=1/2$ and $1/A=2\sigma\sqrt{2}\Gamma\left(3/2\right)=\sigma\sqrt{2}\Gamma\left(1/2\right)=\sigma\sqrt{2\pi}$, in agreement with (31). The Gaussian is well-know to underestimate probabilities of collision [5,15], and a better fit to flight records and radar tracks of aircraft deviations in flight is provided by the generalized error distribution [7,12].

A simple, unimodal approximation is provided by the generalized error distribution for $k=1/2$, for which (33b) specifies $a=\left[\Gamma\left(6\right)/\Gamma\left(2\right)\right]^{1/4}=\sqrt[4]{5!}=\sqrt[4]{120}$ and (33a) gives $A=\sqrt[4]{5!}/\left[2\sigma\Gamma\left(3\right)\right]$; leading to:

$$P\left(x;1/2\right)=\left[\left(\sqrt{15/2}\right)/\sigma\right]\exp\left[-\sqrt[4]{120}\sqrt{|x|/\sigma}\right] \quad (34)$$

Substituting the generalized error distribution (34) for the Gaussian (31) implies a correction factor

$$C\geq\left|P\left(x;1/2\right)/P\left(x;2\right)\right|^2 =15\pi\exp\left\{\left(x/\sigma\right)^2-2\sqrt[4]{120}\sqrt{|x|/\sigma}\right\}\equiv G\left(x\right), \quad (35)$$

for the probabilities of coincidence, which are all quadratic in the probabilities of deviation. As a simple estimate, the correction factor (35) will be calculated at the mid-position between the aircraft:

$$C\left(L/\sigma\right)=G\left(x=L/2\right)= 15\pi\exp\left\{\left[L/\left(2\sigma\right)\right]^2-\sqrt[4]{120}\sqrt{L/\left(2\sigma\right)}\right\}. \quad (36)$$

which is the position of highest probability of coincidence for aircraft with identical r.m.s. position errors.

Applying the correction factor (36) leads to much higher *n*-dimensional probabilities of coincidence (7), viz..

$$CP_m=\left(15/2\right)\pi^{n/2}f^{1-n}\bar{\sigma}^{n-2} \times\exp\left\{-2\sqrt[4]{120}\sqrt{|x|/\sigma}\right\}. \quad (37)$$

This can be illustrated by re-calculating the two-dimensional probabilities of coincidence in Table I:

Table III- Correction factor for generalized error distribution, relative to the Gaussian used in Table I to calculate two-dimensional coincidence probabilities for similar and dissimilar aircraft.

$L/\bar{\sigma}$	10	11	12
C	$1.27\text{x}10^{6}$	$1.17\text{x}10^{8}$	$1.84\text{x}10^{10}$
CP_2	$8.80\text{x}10^{-6}$	$4.26\text{x}10^{-6}$	$3.98\text{x}10^{-6}$
CP_2/f	$1.60\text{x}10^{-6}$	$7.75\text{x}10^{-7}$	$3.89\text{x}10^{-7}$

(38)

using the correction factor (36); the generalized error distribution gives higher coincidence probabilities (38) than the Gaussian (25). A similar conclusion can be reached re-calculating Table II with a constant correction factor $C = 2.85\times10^{7}$ for $L/\sigma = 10.7$, viz.:

Table IV- Maximum probability of coincidence using the generalized error distribution, for the same ATM scenarios as for the Gaussian distribution in Table II.

n	1	2	3	4
CP_m	$5.72\text{x}10^{-7}$	$6.86\text{x}10^{-6}$	$2.09\text{x}10^{-4}$	$1.04\text{x}10^{-4}$

(39)

The suggested absolute ICAO TLS standard (28) need not be changed, because in (39) appears a different value. Rather (39) shows that since the generalized error distribution gives a higher probability of coincidence than the Gaussian, a smaller r.m.s. position error is needed to achieve the same low probability of collision.

4 Discussion

In conclusion the ICAO TLS of safety $S_1 = 5\times10^{-9}$ per hour is comparable to the one-dimensional probability coincidence $P_1V \le S_1$. For the maximum probability of coincidence $P_0V^2 \le S_0$ a modified TLS standard $S_0 = 5\times10^{-9}$ per hour squared is needed, if the same value is chosen. For the three-dimensional probability of coincidence P_3/V another modified TLS standard $S_3 = 5\times10^{-9}$ times hour would be needed. Of course, the value of S_0 and S_3 need not be numerically equal to S_1. Since the two-dimensional probability of coincidence $P_2 \le S_2$ is dimensionless, the modified TLS standard $S_2 = 5\times10^{-9}$ would also be dimensionless. The numerical value of S_2 need not equal S_0 or S_1 or S_3. The procedure indicated in (Sec. 4) has lead to a value (28) of S_2 consistent with S_1, justifying the following reasoning: (i) the original ICAO TLS standard (30) has been applied to three of the most common ATM traffic situations; (ii) for these situations it is comparable to the absolute level of safety (28). The latter is preferable to the former, because it is dimensionless, and thus independent of flight time or speed. Thus the absolute level of safety (TLS) in (28) can be proposed as a more general dimensionless substitute to the original ICAO TLS in (30

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