MODELLING RESPONSE OF FLEXIBLE HIGH ASPECT RATIO WINGS TO WIND TURBULENCE

A.V. Balakrishnan
Flight System Research Center
University of California, Los Angeles

Abstract

Response of flexible high aspect ratio wings to wind turbulence is analyzed using the continuum cantilever beam model of Goland for the structure and typical-section compressible attached flow aerodynamics. The wind turbulence model is derived from the Kolmogorov power-law spectrum random field, invoking the Taylor hypothesis. The gust loading and the wing bending/torsion spectral response are studied in detail for the extreme cases M=0 and M=1, as typical.

1 Introduction

In this paper we model the aeroelastic response of a flexible wing structure of high aspect ratio due to air turbulence. Unlike classical treatments limited to "sinusoidal" gust models (see [1] and [2]) we consider the full space-time random turbulence field as well as a continuum model of the flexible wing. Specifically, the cantilever beam model of Goland [3] is used for the wing with two degrees of freedom — bending and torsion. The aerodynamic model is the linearized subsonic/transonic ($0 \leq M \leq 1$) inviscid — attached flow with the Kutta-Joukowski boundary conditions typical section theory valid for assumed high aspect ratio [6]. The turbulence model is the classical Kolmogorov isotropic power-law spectrum model [4] combined with the Taylor hypothesis.

We begin in Section 2 with the wind gust model leading to the spectral density of the vertical component of the velocity field. In Section 3 we calculate the gust loads in inviscid compressible air flow. The bending-torsion structure response to the gust loading is calculated in Section 4 using the continuum cantilever model of Goland [3]. In particular the temporal spectral density of the plunge/pitch gust response at any point on the wing is evaluated. Some numerical results are presented in Section 5.

2 Turbulence Field

Let $v(x, y, z)$ denote the 3×1 vector turbulence wind field. Let $\vec{k}$ denote the unit vector normal to the wing, assumed plane. By vertical wind gust $w_g(\cdot)$ we mean the component:

$$w_g(x, y, z) = \left[v(x, y, z), \, \vec{k} \right].$$

We assume that this is isotropic and is Gaussian with the 3D Kolmogorov spectral density (cf. [8]):

$$Q(\nu_1 \, , \, \nu_2 \, , \, \nu_3) = \frac{c_n^2}{(k^2 + \nu_1^2 + \nu_2^2 + \nu_3^2)^{11/6}} \, , \qquad -\infty < \nu_i < \infty \qquad (1)$$

where c_n^2 is a constant that we assume is known. This constant characterizes the turbulence strength and is considered arbitrary for our purposes here.

With X-axis as the pitch axis and x denoting the chord-wise coordinate:

$$-b \leq x \leq b$$

and s the spanwise coordinate:

$$0 \le s \le \ell.$$

(We consider only one side of the wing. The other side is taken care of by making the functions symmetric in s.) We are interested in the 2D random field:

$$w_g(x, s, z_0), \quad -\infty < x, s < \infty$$

where z_0 is the fixed altitude (level flight). The spectral density of

$$w_g(x, s, z_0), \quad -\infty < x < \infty, \quad -\infty < s < \infty$$

is given by (2D-Kolmogorov)

$$\begin{aligned} Q_2(\nu_1\,,\,\nu_2) &= \int_{-\infty}^{\infty} Q(\nu_1\,,\,\nu_2\,,\,\nu_3)\; d\nu_3 \\ &= \frac{\text{constant}}{(k^2 + 4\pi^2(\nu_1^2 + \nu_2^2))^{4/3}}\;. \end{aligned} \qquad (2)$$

The constants in (1), (2) are not the same. In fact from now on we will ignore the multiplicative constant, since it is irrelevant for our purposes.

2.1 Wind Gust Model: Temporal

For an aircraft in motion, the wind gust at a fixed point on the wing becomes a function of time. We invoke the Taylor "frozen field" hypothesis to characterize the temporal gust field. Thus we assume that the aircraft velocity at the fixed altitude z_0, relative to the wind (to be distinguished from turbulence) is along the negative X-axis, the magnitude (speed) being U (level flight, constant speed), the turbulence component assumed small enough in comparison with U that it may be neglected. Then at any point (x, s, z_0) on the moving wing the vertical wind turbulence is a function of time t given by:

$$w_g(t, x, s) \;=\; w_g(x - Ut,\; s,\; z_0). \qquad (3)$$

For each x, s we have a stationary Gaussian process. The covariance function:

$$\mathbf{E}[\, w_g(t, x, s)\; w_g(t + \tau,\; x,\; s)\,] \;=\; R_g(U\tau, 0, 0) \qquad (4)$$

where $R_g(x, y, z)$ denotes the covariance function of the 3D field. We may simplify it as

$$R_g(U\tau, 0)$$

where

$$R_g(x, s)$$

is the covariance function of the 2D field with spectral density given by (2), since z_0 is fixed. In either case we have for the temporal covariance function:

$$\begin{aligned} &R_1(\tau) \\ &\quad= \mathbf{E}[\, w_g(t, x, s)\; w_g(t + \tau,\; x,\; s)\,] \\ &\quad= \text{constant} \int_{-\infty}^{\infty} e^{2\pi i \nu U\tau}\, \frac{d\nu}{(k^2 + 4\pi^2\nu^2)^{5/6}}\;, \end{aligned} \qquad (5)$$

using the fact that

$$\int_{-\infty}^{\infty} \frac{d\nu_2}{(k^2 \,+\, 4\pi^2(\nu_1^2 + \nu_2^2))^{4/3}} = \frac{1}{(k^2 \,+\, 4\pi^2\nu_1^2)^{5/6}}\;.$$

From (5) we see that the spectral density of the temporal process

$$w_g(t, x, s)$$

at the point (x, s) on the aircraft is given by

$$Q_1(\nu) \;=\; \frac{\text{constant} \cdot U^{2/3}}{(k^2U^2 + 4\pi^2\nu^2)^{5/6}}\;, \quad -\infty < \nu < \infty \qquad (6)$$

and of course the same at all points x, s. We note that (5) is the spectral density of any straight-line (1D) scan of of the 3D field $w_g(x, y, z)$ at speed U.

We also need the covariance function of the space-time field (again stationary)

$$w_g(x - Ut,\; s,\; z_0), \quad -\infty < t, s < \infty.$$

This covariance function is defined by:

$$\begin{aligned} &R_2(\tau, x, s) \\ &= \mathbf{E}[w_g(x_1 - Ut_1, s_1, x_0)\, w_g(x_2 - Ut_2, s_2, x_0)] \end{aligned}$$

where

$$x_2 - x_1 = x, \quad t_2 - t_1 = \tau, \quad s_2 - s_1 = s$$

is given by (using (2)),

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2\pi i\nu_1(x-U\tau)+2\pi i\nu_2 s} \cdot \frac{d\nu_1\, d\nu_2}{(k^2+4\pi^2(\nu_1^2+\nu_2^2))^{4/3}} \tag{7}$$

generalizing (6).

3 Aerodynamic Loads Due to Wind Turbulence

To calculate the aerodynamic loads due to wind turbulence, we begin by noting that the vertical component of the turbulence velocity is the downwash function. In the small-disturbance linearized theory the corresponding force and moment (per unit span) can be expressed:

$$F(t,s) = \int_0^t \int_{-b}^{b} H_F(t-\sigma,\, \xi)\, w_a(\sigma,\xi,s)\, d\xi\, d\sigma, \quad 0 < t, \quad 0 < s < \ell$$

for the force, and

$$M(t,s) = \int_0^t \int_{-b}^{b} H_M(t-\sigma,\, \xi)\, w_a(\sigma,\xi,s)\, d\xi\, d\sigma, \quad 0 < t, \quad 0 < s < \ell.$$

The kernels $H_F(\cdot,\cdot)$ and $H_M(\cdot,\cdot)$ can be derived by solving the Possio Equation [4], valid for the inviscid small-disturbance (linearized) subsonic ($0 \le M \le 1$) flow [5, 6]. Note that the kernels do not involve the span variable because of the assumed high aspect ratio.

Following [1] we may take

$$w_a(\cdot) = -w_g(\cdot)$$

and since for the random field the sign does not matter, the gust induced lift $F_g(t,s)$ can be expressed conveniently as the chordwise integral

$$F_g(t,s) = \int_{-b}^{b} f_g(t,\xi,s)\, d\xi \tag{8}$$

where

$$f_g(t,\xi,s) = \int_0^t H_F(t-\sigma,\, \xi)\, w_g(\xi - U\sigma,\, s)\, d\sigma. \tag{9}$$

We are interested only in the steady state (temporal) response. Hence (see [4]) we may define

$$f_g(t,\xi,s) = \int_0^\infty H_F(\sigma,\xi)\, w_g(\xi - U(t-\sigma),\, s)\, d\sigma. \tag{10}$$

Similarly the steady-state moment is given by:

$$M_g(t,s) = \int_{-b}^{b} m_g(t,\xi,s)\, d\xi \tag{11}$$

$$m_g(t,\xi,s) = \int_0^\infty H_M(\sigma,\xi)\, w_g(\xi - U(t-\sigma),\, s)\, d\sigma. \tag{12}$$

The covariance function of the gust-induced force can then be expressed as the double integral:

$$\mathbf{E}[\, F_g(t_1\,,\, s_1)\, F_g(t_2\,,\, s_2)\,] = \int_{-b}^{b}\int_{-b}^{b} \mathbf{E}[\, f_g(t_1,\xi_1,s_1)\, f_g(t_2,\xi_2,s_2)]\, d\xi_1\, d\xi_2$$

where we can calculate the integrand:

$$\begin{aligned}
\mathbf{E}[\, f_g(t_1\,,\,\xi_1\,,\, s_1)\, f_g(t_2\,,\,\xi_2\,,\, s_2)\,]
&= \int_0^\infty \int_0^\infty H_F(\sigma_1\,,\,\xi_1) \\
&\quad \cdot \mathbf{E}[\, w_g(\xi_1 - U(t_1 - \sigma_1),\, s_1) \\
&\quad \cdot w_g(\xi_2 - U(t_2 - \sigma_2),\, s_2)\,] \\
&\quad \cdot H_F(\sigma_2\,,\,\xi_2)\, d\sigma_1\, d\sigma_2, \\
&= \int_0^\infty \int_0^\infty H_F(\sigma_1\,,\,\xi_1) \\
&\quad \cdot R_2(\xi_2 - \xi_1 - Ut + U(\sigma_2 - \sigma_1),\ s) \\
&\quad \cdot H_F(\sigma_2,\xi_2)\, d\sigma_1\, d\sigma_2\,, \\
&\qquad t = t_2 - t_1\,; \quad s = s_2 - s_1 \qquad (13) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\int_0^\infty H_F(\sigma_1,\xi_1) e^{-2\pi i\nu_1 U\sigma_1}\, d\sigma_1\right) \\
&\quad \cdot \left(\int_0^\infty H_F(\sigma_2\,,\,\xi_2) e^{2\pi i\nu_1 U\sigma_2}\, d\sigma_2\right) \\
&\quad \cdot e^{-2\pi i\nu_1 Ut}\, e^{2\pi i\nu_1(\xi_2-\xi_1)}\, e^{2\pi i\nu_2 s} \\
&\quad \cdot \frac{d\nu_1\, d\nu_2}{(k^2 + 4\pi^2(\nu_1^2+\nu_2^2))^{4/3}}\,.
\end{aligned}$$

Let $\hat{H}_F(i\omega,\xi)$ denote the Fourier transform

$$\int_0^\infty e^{-i\omega\sigma}\, H_F(\sigma,\xi)\; d\sigma.$$

Then (13)

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2\pi i\nu_1(\xi_2-\xi_1)} \cdot \hat{H}_F(i\nu,U,\xi_1)\;\overline{\hat{H}_F(i\nu,U,\xi_2)} \cdot e^{2\pi i\nu_2 s-2\pi i\nu_1 Ut}\;\frac{d\nu_1\; d\nu_2}{(k^2\;+\;4\pi^2(\nu_1^2+\nu_2^2))^{4/3}}\;.$$

Hence

$$\mathbf{E}[\,F_g(t_1\,,\,s_1)\;F_g(t_2\,,\,s_2)\,] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi i\nu_1 Ut}\,e^{+2\pi i\nu_2 s} \cdot \frac{\left|\hat{\hat{H}}_F(i\nu_1 U,\,i\nu_1)\right|^2}{(k^2\;+\;4\pi^2(\nu_1^2+\nu_2^2))^{4/3}}\;d\nu_1\;d\nu_2$$

where

$$\hat{\hat{H}}_F(i\nu_1 U,\,i\nu_1)$$

is the double Fourier transform

$$= \int_{-b}^{b}\int_0^\infty e^{-2\pi i\nu_1 Ut+2\pi i\nu_1\xi}\,H_F(t,\xi)\;dt\;d\xi \quad (14)$$

This shows that $F_g(t,s)$ is steady-state stationary in both variables, with stationary covariance function $R_F(t,s)$,

$$R_F(t,s) \;=\; \frac{1}{U}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi i\nu_1 t+2\pi i\nu_2 s} \cdot \frac{\left|\hat{\hat{H}}_F(i\nu_1,i\nu_1)\right|^2}{\left(k^2+4\pi^2\left(\frac{\nu_1^2}{U^2}+\nu_2^2\right)\right)^{4/3}}\;d\nu_1\;d\nu_2$$

and spectral density $P_F(\nu_1\,,\,\nu_2)$

$$P_F(\nu_1\,,\,\nu_2) \;=\; \frac{1}{U}\;\frac{\left|\hat{\hat{H}}_F(i\nu_1\,,\,i\nu_1)\right|^2}{\left(k^2\;+\;4\pi^2\left(\frac{\nu_1^2}{U^2}+\nu_2^2\right)\right)^{4/3}}\;,$$

$$\infty < \nu_1\,,\,\nu_2 < \infty. \quad (15)$$

Similarly, for the stationary covariance function of the moment:

$$\begin{aligned} R_M(t,s) \;&=\; E[\,M_g(t_1\,,s_1)\,M_g(t_2\,,s_2)\,], \\ &\qquad t_2-t_1=t;\quad s_2-s_1=s \qquad (16) \\ &=\; \frac{1}{U}\int_{-\infty}^{\infty}\int_0^\infty \left|\hat{H}_M(i\nu_1\,;\;i\nu_1)\right|^2 \cdot \frac{e^{2\pi i\nu_2 s+2\pi i\nu_1 t}}{\left(k^2+4\pi^2\left(\frac{\nu_1^2}{U^2}+\nu_2^2\right)\right)^{4/3}}\;d\nu_1\;d\nu_2 \end{aligned}$$

where (17)

$$\hat{H}_M(i\nu_1\,;\;i\nu_1) = \int_0^\infty e^{-2\pi i\nu_1\sigma}\int_{-b}^{b} e^{2\pi i\nu_1\xi}\,H_M(\sigma,\xi)\;d\sigma\;d\xi. \quad (18)$$

Next the cross covariance:

$$\mathbf{E}[\,F_g(t_1\,,s_1)\,M_g(t_2\,,s_2)\,] = \frac{1}{U}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi i\nu_1 t}\,e^{-2\pi i\nu_2 s} \cdot \frac{\hat{H}_F(i\nu_1\,,\,i\nu_1)\;\overline{\hat{H}_M(i\nu_1\,,\,i\nu_1)}}{\left(k^2+4\pi^2\left(\frac{\nu_1^2}{U^2}+\nu_2^2\right)\right)^{4/3}}\;d\nu_1\;d\nu_2\,,$$

$$t_2-t_1\;=\;t,\;\;s_2-s_1\;=\;s, \qquad (19)$$

which we denote by

$$R_{FM}(t,s)$$

and

$$\mathbf{E}[\,M_g(t_1\,,s_1)\,F_g(t_2\,,s_2)\,] = \frac{1}{U}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi i\nu_1 t}\,e^{-2\pi i\nu_2 s} \cdot \frac{\hat{H}_M(i\nu_1\,,\,i\nu_1)\;\overline{\hat{H}_F(i\nu_1\,,\,i\nu_1)}}{\left(k^2+4\pi^2\left(\frac{\nu_1^2}{U^2}+\nu_2^2\right)\right)^{4/3}}\;d\nu_1\;d\nu_2\,,$$

$$t_2-t_1\;=\;t,\;\;s_2-s_1\;=\;s, \qquad (20)$$

which we denote by

$$R_{MF}(t,s).$$

For calculating the aeroelastic response we need to consider the 2×1 process

$$N_g(t,s) \;=\; \begin{vmatrix} F_g(t,s) \\ M_g(t,s) \end{vmatrix} \qquad (21)$$

and the corresponding (stationary) covariance matrix functions:

$$R_g(t,s) = \begin{vmatrix} R_F(t,s) & R_{FM}(t,s) \\ R_{MF}(t,s) & R_{MM}(t,s) \end{vmatrix} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2\pi i\nu_1 t+2\pi i\nu_2 s}\, P_g(\nu_1\,,\nu_2)\; d\nu_1\; d\nu_2 \quad (22)$$

where

$$P_g(\nu_1\,,\,\nu_2) = \frac{1}{U}\,\frac{1}{\left(k^2+4\pi^2\left(\frac{\nu_1^2}{U^2}+\nu_2^2\right)\right)^{4/3}}\;P_1(\nu_1) \quad (23)$$

where

$$P_1(\nu_1) = \begin{vmatrix} \left|\hat{H}_F(i\nu_1\,,i\nu_1)\right|^2 & \hat{H}_F(i\nu_1\,,i\nu_1)\,\overline{\hat{H}_M(i\nu_1\,,i\nu_1)} \\ \hat{H}_M(i\nu_1\,,i\nu_1)\,\overline{\hat{H}_F(i\nu_1\,,i\nu_1)} & \left|\hat{H}_M(i\nu_1\,,i\nu_1)\right|^2 \end{vmatrix}. \quad (24)$$

We should note in particular that if we fix s, say $s = s_0$, then the temporal process

$$N_g(t,s_0)$$

is such that the statistical properties in the steady state do not depend upon s_0, and we have only to set $s = 0$ in (22) for the corresponding covariance function, and obtaining for the spectral density (omitting multiplicative constants)

$$P_g(\nu) = \frac{1}{U}\,\frac{1}{\left(k^2+4\pi^2\,\frac{\nu^2}{U^2}\right)^{5/6}}\;P_1(\nu), \qquad -\infty<\nu<\infty. \quad (25)$$

4 Wing Structure Response

We can now formulate the structure response to the aerodynamic gust loads. We use the uniform cantilever beam model of Goland [3], endowed with two degrees of freedom: bending and torsion — or in aeroelastic terms, plunging (displacement in Z-axis) and pitching, the pitching axis as shown in Figure 1. Referring to [5] for more details on this, let

$$x(t,s) = \begin{vmatrix} h(t,s) \\ \theta(t,s) \end{vmatrix}, \quad 0<t, \quad 0<s<\ell$$

where $h(\cdot,\cdot)$ is the bending displacement (plunge) and $\theta(\cdot,\cdot)$ the pitch angle. Then the structural dynamics equations are given by:

$$M_S\ddot{x}(t,s) + D_S\dot{x}(t,s) + \begin{vmatrix} EI\,\frac{\partial^4}{\partial s^4} & 0 \\ 0 & -GJ\,\frac{\partial^2}{\partial s^2} \end{vmatrix} x(t,s) = \int_0^t\int_{-b}^{b} H(t-\sigma,x)\,w_a(\sigma,x,s)\;d\sigma \;+\; N_g(t,s) \quad (26)$$

where the structure normal velocity

$$w_a(t,x,s) = -\dot{h}(t,s) \;-\; (x-a)\,\dot{\theta}(t,s) \;-\; U\cos\alpha\,\theta(t,s)$$

$$H(t,x) = \begin{vmatrix} H_F(t,x) \\ H_M(t,x) \end{vmatrix}$$

$$N_g(t,s) = \begin{vmatrix} F_g(t,s) \\ M_g(t,s) \end{vmatrix}$$

where M_S, D_S are the structure mass/moment of inertial and damping (if known) matrices. With the cantilever end conditions:

$$h(0,t) = h'(0,t) = 0 = \theta(0,t)$$

$$\theta'(\ell,t) = 0 = h''(\ell,t) = h'''(\ell,t)$$

(see [7] for modification if self-straining controller action is to be included). Since we are only dealing with linear dynamics, we begin by taking Laplace transforms treating $N_g(t,s)$ as a deterministic input. Our primary interest is in developing the system "Transfer Function." Then using the notation

$$\hat{x}(\lambda,s) = \int_0^\infty e^{-\lambda t}\,x(t,s)\;dt, \quad \operatorname{Re}\lambda>0$$

$$\hat{N}_g(\lambda,s) = \int_0^\infty e^{-\lambda t}\,N_g(t,s)\;dt$$

and, taking transforms in (26), assuming zero initial conditions, we obtain:

$$\left.\begin{aligned} \hat{h}''''(\lambda,s) &= w_1(\lambda)\,\hat{h}(\lambda,s) \\ &\quad + w_2(\lambda)\,\hat{\theta}(\lambda,s) + \tfrac{1}{EI}\,\hat{F}_g(\lambda,s) \\ \hat{\theta}''(\lambda,s) &= w_3(\lambda)\,\hat{h}(\lambda,s) \\ &\quad + w_4(\lambda)\,\hat{\theta}(\lambda,s) - \tfrac{1}{GJ}\,\hat{M}_g(\lambda,s) \end{aligned}\right\} \qquad (27)$$

where $w_1(\cdot)$, $w_2(\cdot)$, $w_3(\cdot)$, $w_4(\cdot)$ are given in [5]. We note that (26) with the boundary conditions is a two-point boundary-value problem, to solve which we define (keeping λ fixed)

$$Y(s) = \begin{vmatrix} \hat{h}(\lambda,s) \\ \hat{h}'(\lambda,s) \\ \hat{h}''(\lambda,s) \\ \hat{h}'''(\lambda,s) \\ \hat{\theta}(\lambda,s) \\ \hat{\theta}'(\lambda,s) \end{vmatrix}.$$

Then (27) becomes:

$$Y'(s) = A(\lambda)\,Y(s) + B_{62}N_g(s), \quad 0 < s < \ell \qquad (28)$$

where

$$A(\lambda) = \begin{vmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ w_1 & 0 & 0 & 0 & w_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ w_3 & 0 & 0 & 0 & w_4 & 0 \end{vmatrix}$$

$$B_{62} = \begin{vmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{1}{EI} & 0 \\ 0 & 0 \\ 0 & \frac{-1}{GJ} \end{vmatrix}$$

$$N_g(s) = \begin{vmatrix} \hat{F}_g(\lambda,s) \\ \hat{M}_g(\lambda,s) \end{vmatrix}.$$

And (27) has the solution:

$$\begin{vmatrix} \hat{h}(\lambda,s) \\ \hat{\alpha}(\lambda,s) \end{vmatrix} = C_{26}Y(s)$$

$$C_{26} = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{vmatrix}$$

$$Y(s) = e^{A(\lambda)s}\,Y(0) + \int_0^s e^{A(\lambda)(s-\sigma)}\,B_{62}\,N_g(\sigma)\,d\sigma. \qquad (29)$$

The next step is to satisfy the boundary conditions. Let

$$P_{36} = \begin{vmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}.$$

Then

$$Y(\ell) = e^{A(\lambda)\ell}\,P^*_{36}\begin{vmatrix} \hat{h}''(\lambda,0) \\ \hat{h}'''(\lambda,0) \\ \hat{\alpha}'(\lambda,0) \end{vmatrix} + \int_0^\ell e^{A(\lambda)(\ell-\sigma)}\,B_{62}\,N_g(\sigma)\,d\sigma \qquad (30)$$

and we require that

$$P_{36}Y(\ell) = 0$$

or

$$D_3(\lambda)\begin{vmatrix} \hat{h}''(\lambda,0) \\ \hat{h}'''(\lambda,0) \\ \hat{\alpha}'(\lambda,0) \end{vmatrix} + P_{36}\int_0^\ell e^{A(\lambda)(\ell-\sigma)}\,B_{62}\,N_g(\sigma)\,d\sigma = 0$$

where

$$D_3(\lambda) = P_{36}\,e^{A(\lambda)\ell}\,P^*_{36} \qquad (31)$$

which is an entire function with a countable number of zeros which are the aeroelastic modes. Hence, omitting this sequence, we have

$$\begin{aligned} Y(s) &= e^{A(\lambda)s}\,(-1)P_{36}^*\,D_3(\lambda)^{-1} \\ &\quad \cdot \int_0^\ell P_{36}\,e^{A(\lambda)(\ell-\sigma)}\,B_{62}\,N_g(\sigma)\;d\sigma \\ &\quad + \int_0^s e^{A(\lambda)(s-\sigma)}\,B_{62}\,N_g(\sigma)\;d\sigma. \end{aligned}$$

Hence the solution of (26) satisfying the end conditions is given by (the inverse transforms of):

$$\begin{aligned} \hat{x}(\lambda, s) &= \int_0^s C_{26}\,e^{A(\lambda)(s-\sigma)}\,B_{62}\,N_g(\sigma)\;d\sigma \\ &\quad - C_{26}\,e^{A(\alpha)s}\,P_{36}^*\,D_3(\lambda)^{-1} \\ &\quad \cdot \int_0^\ell P_{36}\,e^{A(\lambda)(\ell-\sigma)}\,B_{62}\,N_g(\sigma)\;d\sigma \end{aligned}$$

which can be expressed as:

$$= \int_0^\ell G(\lambda, s, \sigma)\,N_g(\sigma)\;d\sigma, \quad 0 \le s \le \ell \quad (32)$$

where $G(\lambda, \cdot, \cdot)$ is the Green's Function

$$\begin{aligned} G(\lambda, s, \sigma) &= C_{26}\,e^{A(\lambda)(s-\sigma)}\,B_{62} \\ &\quad - C_{26}\,e^{A(\lambda)s}\,P_{36}^*\,D_3(\lambda)^{-1} \\ &\quad \cdot P_{36}\,e^{A(\lambda)(\ell-\sigma)}\,B_{62}\,, \\ &\quad 0 < \sigma < s \\ &= -C_{26}\,e^{A(\lambda)s}\,P_{36}^*\,D_3(\lambda)^{-1} \\ &\quad \cdot P_{36}\,e^{A(\lambda)(\ell-\sigma)}\,B_{62}\,, \\ &\quad s < \sigma < \ell. \end{aligned}$$

Note that

$$G(\lambda, 0, \sigma) = 0, \quad 0 < \sigma < \ell.$$

Denoting now the inverse Laplace transform of $G(\lambda, s, \sigma)$ by

$$W(t, s, \sigma)$$

we have that the *steady-state* time-domain gust response satisfying (4.1) is given by:

$$\begin{aligned} x(t, s) &= \int_0^\ell d\sigma \int_0^\infty W(\tau, s, \sigma) \\ &\quad \cdot N_g(t-\tau,\, \sigma)\;d\tau. \end{aligned} \quad (33)$$

Hence the temporal covariance

$$\begin{aligned} &\mathbf{E}[\,x(t_1\,,\,s)\,x(t_2\,,\,s)^*\,] \\ &\quad = \int_0^\ell \int_0^\ell \int_0^\infty \int_0^\infty W(\tau_1\,,\,s,\,\sigma_1) \\ &\quad\quad \cdot \mathbf{E}[\,N_g(t_1-\tau_1\,,\,\sigma_1)\;N_g(t_2-\tau,\,\sigma_2)^*\,] \\ &\quad\quad \cdot W(\tau_2\,,\,s,\,\sigma_2)^*\;d\tau_1\;d\tau_1\;d\sigma_1\,d\sigma_2 \end{aligned}$$

which using

$$\begin{aligned} &\mathbf{E}[\,N_g(t_1-\tau_1\,,\,\sigma_1)\,N_g(t_2-\tau,\,\sigma_2)^*\,] \\ &\quad = \int_{-\infty}^\infty \int_{-\infty}^\infty e^{2\pi i\nu_1(t_2-t_1)+2\pi i\nu_2(\sigma_2-\sigma_1)} \\ &\quad\quad \cdot P_2(\nu_1\,,\,\nu_2)\;d\nu_1\;d\nu_2 \end{aligned}$$

becomes

$$\begin{aligned} &= \int_{-\infty}^\infty e^{-2\pi i\nu_1 t}\;d\nu_1 \\ &\quad \cdot \int_{-\infty}^\infty \hat{G}(2\pi i\nu_1\,,\,s,\,2\pi i\nu_2) \\ &\quad \cdot P_2(\nu_1\,,\,\nu_2)\,\hat{G}(2\pi i\nu_1\,,\,s,\,2\pi i\nu_2)^*\;d\nu_2 \end{aligned}$$

where

$$\hat{G}(2\pi i\nu, s, 2\pi i\nu_2) = \int_0^\ell e^{-2\pi i\nu_2\sigma}\,G(\nu, s, \sigma)\;d\sigma.$$

Hence the temporal spectral density of the process $x(t, s)$ at s is given by

$$\begin{aligned} P_s(\nu) &= \int_{-\infty}^\infty \hat{G}(2\pi i\nu, s, 2\pi i\nu_2) \\ &\quad \cdot P_2(\nu, \nu_2)\,\hat{G}(2\pi i\nu, s, 2\pi i\nu_2)^*\;d\nu_2\,, \\ &\quad -\infty < \nu < \infty. \end{aligned}$$

We can take advantage of the fact that (cf. (23)):

$$P_2(\nu_1\,,\,\nu_2) = \frac{1}{U}\,\frac{P_1(\nu_1)}{\left(k^2 + 4\pi^2\left(\frac{\nu_1^2}{U^2} + \nu_2^2\right)\right)^{4/3}}\,.$$

Hence we can express $P_s(\cdot)$ as:

$$\begin{aligned} P_s(\nu) &= \int_{-\infty}^\infty g(2\pi i\nu, s, 2\pi i\nu_2) \\ &\quad \cdot P_1(\nu)\,g(2\pi i\nu, s, 2\pi i\nu_2)^*\;d\nu_2 \end{aligned} \quad (34)$$

where

$$g(2\pi i\nu, s, 2\pi i\nu_2) = \frac{1}{U}\,\frac{\hat{G}(2\pi i\nu, s, 2\pi i\nu_2)}{\left(k + 2\pi i\sqrt{\frac{\nu_1^2}{U^2} + \nu_2^2}\right)^{4/3}}.$$

Also

$$\begin{aligned}
&\hat{\hat{G}}(2\pi i\nu, s, 2\pi i\nu_2) \\
&= \int_0^\ell \hat{G}(2\pi i\nu, s, \sigma)\, e^{-2\pi i\nu_2\sigma}\, d\sigma \\
&= C_{26}\Big[e^{A(2\pi i\nu)s}\left(I - e^{-(A(2\pi i\nu)+2\pi i\nu_2)s}\right) \\
&\qquad \cdot\, (A(2\pi i\nu) + 2\pi i\nu_2 I)^{-1} \\
&\quad - e^{A(2\pi i\nu)s}\left(P_{36}^* D_3(2\pi i\nu)^{-1} P_{36}\right) \\
&\quad \cdot\, e^{A(2\pi i\nu)\ell}\left(I - e^{-(A(2\pi i\nu)+2\pi i\nu_2)\ell}\right) \\
&\quad \cdot\, (A(2\pi i\nu) + 2\pi i\nu_2 I)^{-1}\Big]\, B_{62}.
\end{aligned}$$

4.1 Accelerometer Output

The displacement at (x, s), where x is the chordwise coordinate and s the spanwise coordinate is given by

$$z(t, x, s) = h(t, s) + x\,\theta(t, s)$$

and the acceleration at (x, s) is therefore

$$\ddot{z}(t, x, s) = \ddot{h}(t, s) + x\,\ddot{\theta}(t, s).$$

The corresponding spectral density is

$$\nu^2 [\,1\,,\, x\,]\, P_s(\nu) \begin{vmatrix} 1 \\ x \end{vmatrix}, \quad \infty < \nu < \infty$$

This is zero at $s = 0$, and we expect the intensity to increase with s at $x = 0$.

5 Numerical Results

We begin with the spectra of the aerodynamic loads since they do not require the structure parameters. The aerodynamic Transfer Functions $\hat{\hat{H}}_F(i\nu, i\nu)$ and $\hat{\hat{H}}_M(i\nu, i\nu)$ depend on M and we select $M = 0$ and $M = 1$ as typical of subsonic (small M) and transonic (large M).

Using the closed form solutions of the Possio Equation given in [4], we can calculate:

For $M = 0$

$$\begin{aligned}
&\hat{\hat{H}}_F(i\nu, i\nu) \\
&= \rho\, U 2\pi b\left[C\left(\frac{2\pi i\nu b}{U}\right)(J_0(2\pi b\nu) + iJ_1(2\pi b\nu)) + \frac{i}{U}\, J_1(2\pi b\nu)\right] \qquad (35)
\end{aligned}$$

(named as the Küssner function in [1]) where $C(\cdot)$ is the Theodorsen function:

$$C(k) = \frac{K_1(k)}{K_0(k) + K_1(k)}$$

$$k = \frac{2\pi i\nu b}{U}$$

$$\begin{aligned}
&\hat{\hat{H}}_M(i\nu, i\nu) \\
&= \rho\, U b\left[(1 - (1{+}2a)\, C\left(\frac{2\pi i\nu b}{U}\right)\pi(J_0(2\pi\nu b) + iJ_1(2\pi\nu b)) + \left(\frac{4\pi i a\nu}{U} - 2\right)\frac{J_1(2\pi\nu b)}{2\nu} - \frac{\pi}{U}\, J_2(2\pi\nu b)\right]. \qquad (36)
\end{aligned}$$

For $M = 1$

$$\begin{aligned}
&\hat{\hat{H}}_F(i\nu, i\nu) \\
&= \frac{\rho\, U}{\pi b(1 + 2U)\nu^2}\Bigg[e^{-2\pi i b\nu}(ib(1 + 2U)\nu) \cdot \mathrm{Erf}\left[\sqrt{\frac{2\pi i b\nu}{U}}\,\right] \\
&\quad - i\, e^{4\pi i b\nu}\, U(1 + U)\sqrt{\frac{b^2}{U^2}\,(1 + 2U)\nu^2} \cdot \mathrm{Erf}\sqrt{4\pi i b\pi\nu + \frac{2\pi i b\nu}{U}}\,\Bigg]. \qquad (37)
\end{aligned}$$

The moment loading transfer function

$$H_M(i\nu, i\nu)$$

turns out to be rather complex. We have for $a = 0$:

$$H_M(i\nu, i\nu)$$

$$= \rho U \left(\frac{2b^2 e^{-2ib\pi\nu} \left(-\frac{2e^{-k}}{k\sqrt{\pi}} + \frac{(1-k)\,\mathrm{Erf}[\sqrt{k}]}{2k^3} \right)}{\sqrt{k}} \right.$$

$$- \left(4b^2 e^{-k-2ib\pi\nu} \left(-2(k + 4ib\pi\nu) + e^{k+4ib\pi\nu} \sqrt{\pi}\sqrt{k + 4ib\pi\nu}\,\mathrm{Erf}[\sqrt{k + 4ib\pi\nu}]\right)\right) \Big/ \left(\sqrt{k^3}\sqrt{\pi}\,(k + 4ib\pi\nu)^2\right)$$

$$- \left(e^{-k-2ib\pi\nu} \left(e^k (k + 4ib\pi\nu)^2\,\mathrm{Erf}[\sqrt{k}] + \sqrt{k}\left(4b\sqrt{\pi}\,\nu(ik - 4b\pi\nu) - e^{k+4ib\pi\nu}\sqrt{k + 4ib\pi\nu} \cdot (k - 4ibk\pi\nu + 2b\pi\nu(3i+8b\pi\nu)) \cdot \mathrm{Erf}[\sqrt{k + 4ib\pi\nu}]\right)\right)\right) \Big/$$

$$\left. \left(2k\pi^2\nu^2(k + 4ib\pi\nu)^2\right) \right). \qquad (38)$$

And for $a \neq 0$,

$$= (38) - a(37)$$

Plots of the power density spectra for lift and moment

$$10 \log |H_F(i\nu, i\nu)|^2 P(\nu)$$

$$10 \log |H_M(i\nu, i\nu)|^2 P(\nu)$$

where

$$P(\nu) = \frac{1}{U} \frac{1}{\left(\kappa^2 + \frac{4\pi^2\nu^2}{U^2}\right)^{5/6}}$$

with

$$\kappa = \frac{U}{1000}$$

are given in Figures 2 and 3 for $M = 0$ and $M = 1$ as well as various values of U with $b = 1$.

Finally as may be expected the bending/torsion wing response spectra depend heavily on the specific parameters of the wing and are not included here; a representative case will be presented at the Conference.

References

[1] Bisplinghoff, R L, Ashley, H, and Halfman, R L. *Aeroelasticity.* Addison-Wesley Publishing Company, Inc., 1955.

[2] Fung, Y C. *An Introduction to the Theory of Aeroelasticity.* Dover Publications, 1969.

[3] Goland, M. The Flutter of a Uniform Cantilever Wing. *Journal of Applied Mechanics.* Vol. 12, No. 4, pp. A-197–A-208, 1945.

[4] Balakrishnan, A V. Possio Integral Equations of Aeroelasticity Theory. *Journal of Aerospace Engineering*, Vol. 16, No. 4, pp. 139–154, 2003.

[5] Balakrishnan, A V and Iliff, K W. A Continuum Aeroelastic Model for Inviscid Subsonic Bending-Torsion Wing Flutter. *Proceedings of the International Forum on Aeroelasticity and Structural Dynamics*, Amsterdam, June 4-6, 2003, published on CD, file us-03.pdf, 2003.

[6] Balakrishnan,A V. On the Transonic Small Disturbance Potential Equation. *AIAA Journal*, Vol. 42, No. 6, pp. 1081–1088, 2004.

[7] Balakrishnan, A V. Subsonic Flutter Suppression Using Self-Straining Actuators. *Journal of the Franklin Institute*, Vol. 338, pp. 149–170, 2001.

[8] Balakrishnan, A V. *Introduction to Random Processes in Engineering.* John Wiley & Sons, Inc., 1995.

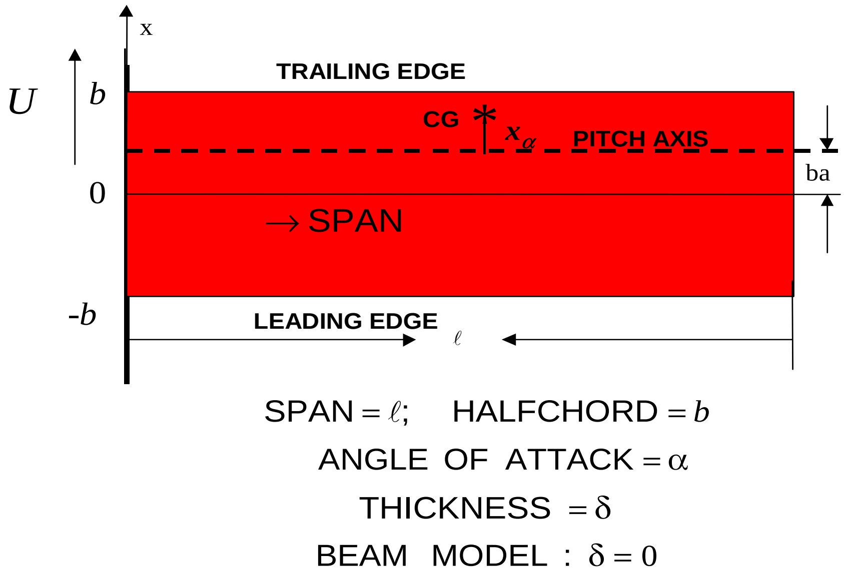

Fig. 1. Wing Structure Beam Model

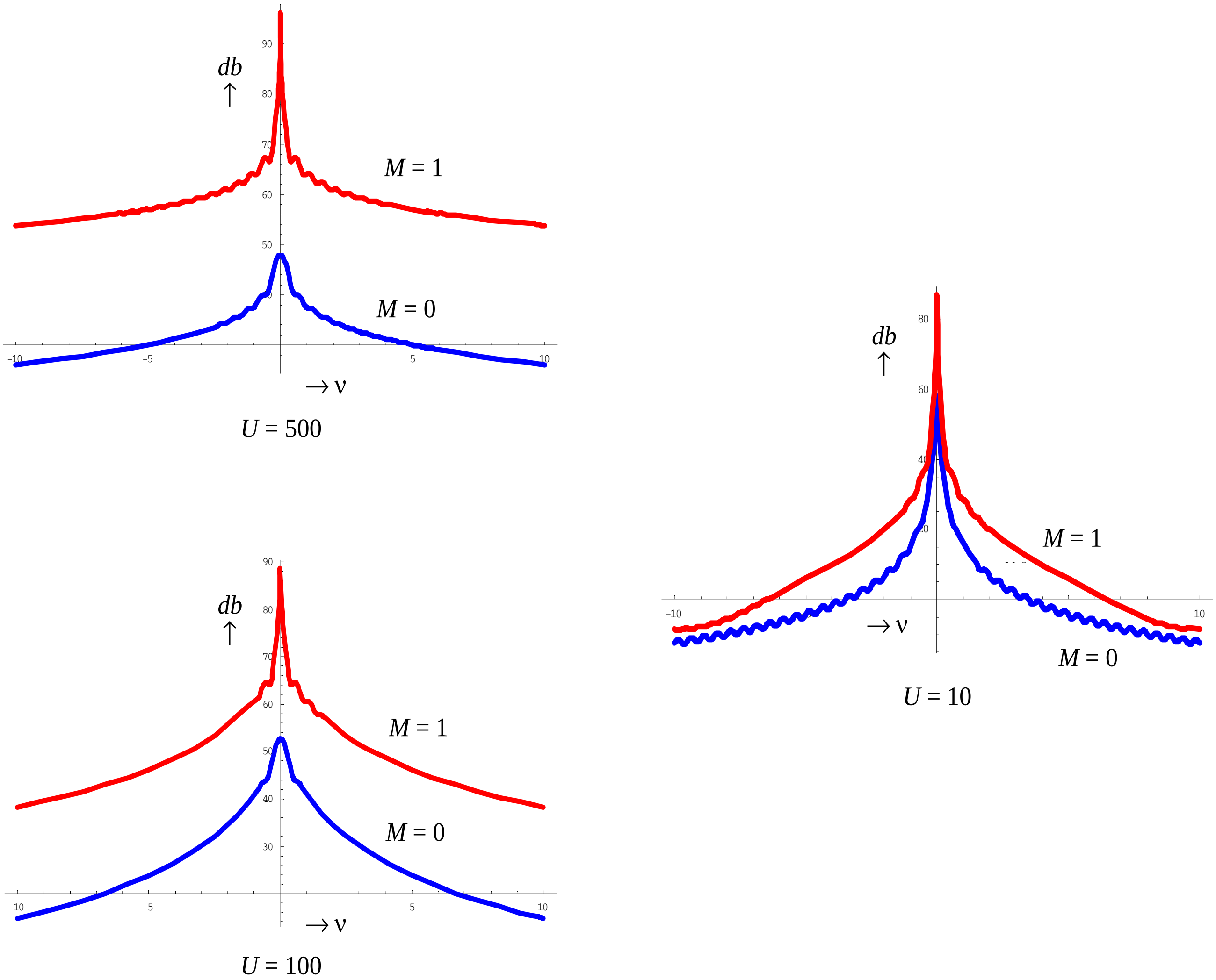

Fig. 2. Aerodynamic Loading: Lift Spectra

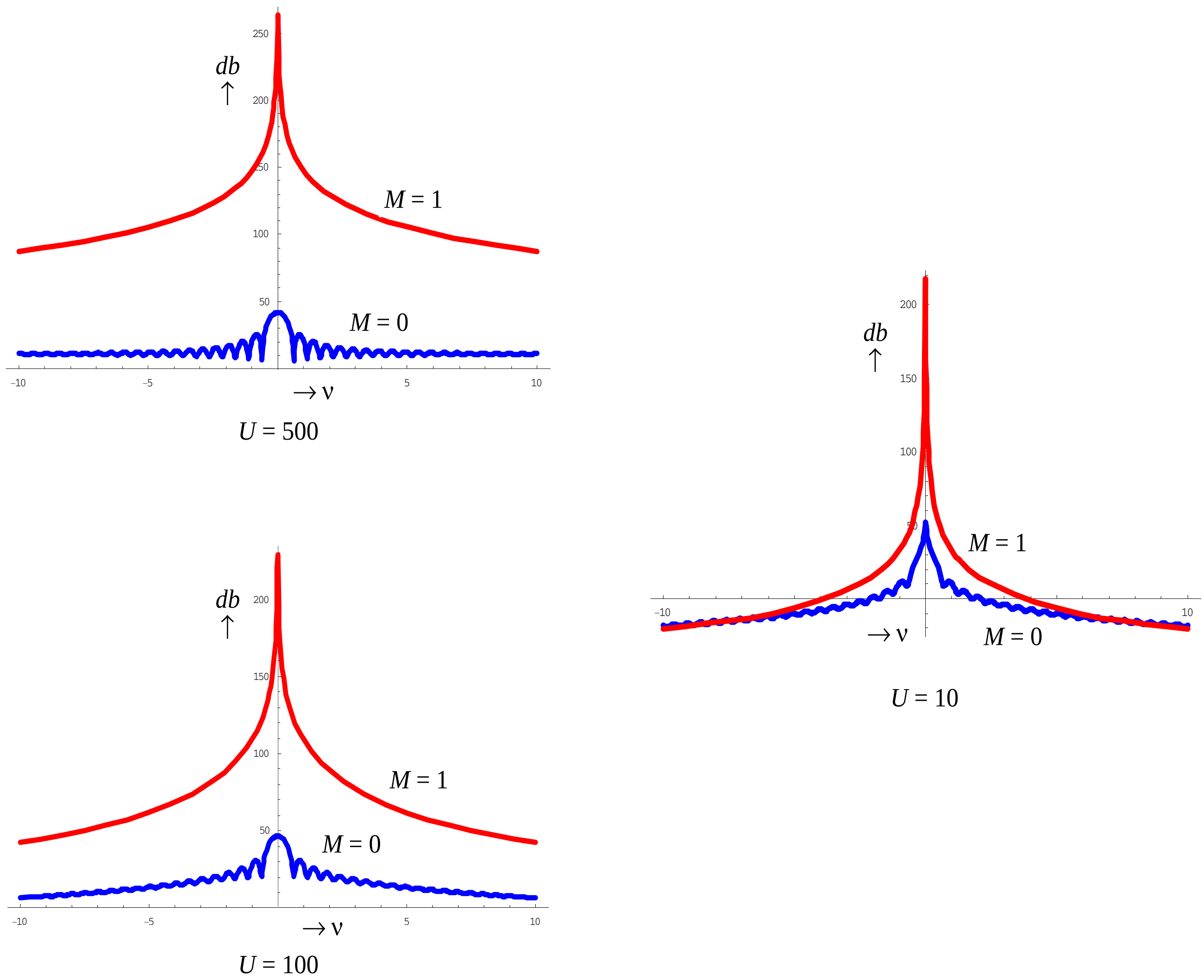

Fig. 3. Aerodynamic Loading: Moment Spectra