NON-HOMOGENEOUS COMPOSITE BEAMS: ANALYTIC FORMULATION AND SOLUTION

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Abstract

The paper presents a derivation for the analysis of monoclinic beams of non-homogeneous cross-section that undergo axially distributed surface loads and body forces. The derivation has been proved to be exact by symbolic computational tools, and includes illustrative numerical examples.

1 Introduction

This paper deals with anisotropic beams of non-homogeneous cross-section, namely, that consist of various *Z-monoclinic* materials, where the (x-y) planes of the cross-sections coincide with the planes of elastic symmetry.

The origin of the method developed seems to be [1], where the problem solution for homogeneous *isotropic* beam under surface loads that do not vary along the generators is expressed by three harmonic and one biharmonic functions. In an independent way, [2], presented an analytical (*level based*) solution for the same problem when the surface loads are polynomials of the beam axis variable. Hence, many of the related works are generally referred to as 'Michell-Almansi (*recursive*) method'. An analogous methodology, founded on prescribed stress (and not deformation) distributions, has been presented much later by [3]. Similar problems were also discussed by [4]-[6] and others. The above solutions were further evolved by [7] for homogeneous isotropic beams undergoing both surface and body loads. The formulation was first valid for loads that may be longitudinally expressed by third order polynomials only. Later on, [8], [9] extended the method for the case of Z-monoclinic beams and generic polynomial loading.

This paper presents an improved derivation of the above approach. We use notations from [10] for non-homogeneous domain, interlaminar and boundary conditions, generic Neumann-type problems and the auxiliary problems of plane deformation. To prove the symbolic exactness of the expressions, the entire methodology is documented and verified symbolically by the Maple programs. The illustrative examples are also produced by the Maple programs.

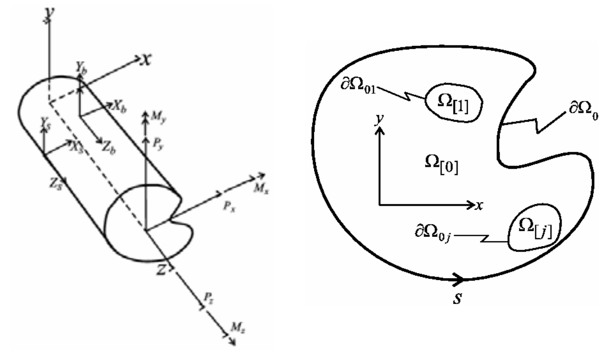
2 The solution hypothesis and procedure

The constitutive relations for *Z-monoclinic* material are given by the linear Hook's low $[\varepsilon_x,\varepsilon_y,\varepsilon_z,\gamma_{yz},\gamma_{xz},\gamma_{xy}] = a\,[\sigma_x,\sigma_y,\sigma_z,\gamma_{yz},\gamma_{xz},\gamma_{xy}]$ with a positive definite matrix $\boldsymbol{a}$ (with 13 independent coefficients)

$$a = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & a_{16} \\ & a_{22} & a_{23} & 0 & 0 & a_{26} \\ & & a_{23} & 0 & 0 & a_{36} \\ & & & a_{44} & a_{45} & 0 \\ & Sym. & & & a_{55} & 0 \\ & & & & & a_{66} \end{bmatrix}. \quad (1)$$

Such typical materials may be obtained by rotating *orthotropic* material about the beam (z) axis. The *reduced elastic constants* are defined as $b_{ij} = a_{ij} - \dfrac{a_{i3}a_{3j}}{a_{33}}$. We assume the most general *surface loading form* (per unit area), $\mathbf{F}_s=\{X_s, Y_s, Z_s\}$, and *distributed body forces* (per unit volume), $\mathbf{F}_b=\{X_b, Y_b, Z_b\}$, see Figure 1, which are expressed as vector polynomials of degree $K \geq 0$,

$$\begin{aligned} \mathbf{F}_s &= \sum\nolimits_{k=0}^{K} \{X_s^{(k)}, Y_s^{(k)}, Z_s^{(k)}\}(x,y)z^k, \\ \mathbf{F}_b &= \sum\nolimits_{k=0}^{K} \{X_b^{(k)}, Y_b^{(k)}, Z_b^{(k)}\}(x,y)z^k. \end{aligned} \tag{2}$$



a) General View *b*) A Cross-Section

Fig. 1. Beam.

Hence, the case of K=0 stands for *uniform* distributed loads in the z-direction, K=1 stands for *linear* distribution, etc. In general, we let each of the two stress components τ_{xz}, τ_{yz} to be polynomials of degree K+1 in z, while $\sigma_x, \sigma_y, \tau_{xy}$ and σ_z are polynomials of degree K+2 in z as

$$\begin{aligned} \{\sigma_x, \sigma_y, \tau_{xy}, \sigma_z\} &= \sum\nolimits_{k=0}^{K+2} \{\sigma_x^{(k)}, \sigma_y^{(k)}, \tau_{xy}^{(k)}, \sigma_z^{(k)}\} z^k, \\ \{\tau_{xz}, \tau_{yz}\} &= \sum_{k=0}^{K+1} \{\tau_{xz}^{(k)}, \tau_{yz}^{(k)}\} z^k, \end{aligned} \tag{3}$$

where $\sigma_i^{(k)}, \tau_{ij}^{(k)}$ are functions of x, y. In homogeneous case, we let each of the three stress components $\sigma_x, \sigma_y, \tau_{xy}$ to be expressed as polynomials of degree K in z. A scheme of the level-based solution methodology is presented in Table 1. The process is initiated for k=K and continues for lower levels down to k=0. For each $K \geq k \geq 0$ level, a set of the biharmonic and the Neumann problems in non-homogeneous domain Ω should be solved. Each solution level is driven by its level loading and the quantities obtained in previous (higher) levels.

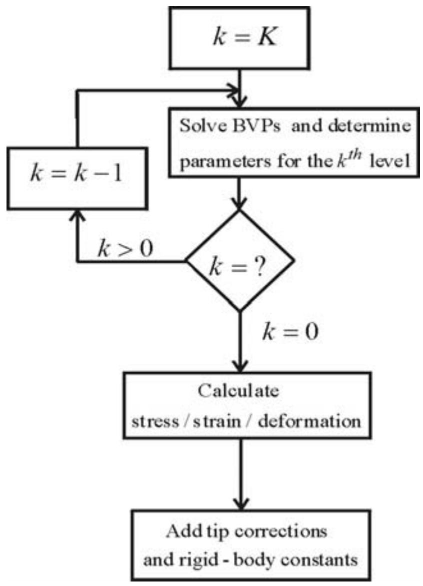


Table 1. Solution Procedure for Axially Non-Uniform Loading of a Z-Monoclinic Beam.

As shown by Table 2, the solution ingredients are gradually introduced according to their level of appearance.

Component	Level
$\sigma_x \leftrightarrow \sigma_y$, $\tau_{xy} \leftrightarrow \tau_{xy}$	K+2
$\tau_{xz} \leftrightarrow \tau_{yz}$, $\gamma_{xz} \leftrightarrow \gamma_{yz}$	K+1
$\sigma_z \leftrightarrow \sigma_z$, $\varepsilon_z \leftrightarrow \varepsilon_z$, $\varepsilon_x \leftrightarrow \varepsilon_y$, $\gamma_{xy} \leftrightarrow \gamma_{xy}$	K+2
$u \leftrightarrow v$	K+4
$w \leftrightarrow w$	K+3
$\omega_x \leftrightarrow -\omega_y$	K+3
$\omega_z \leftrightarrow -\omega_z$	K+2
H, $S_x \leftrightarrow S_y$	K+1
$p \leftrightarrow q$, $\tau \leftrightarrow -\tau$, $d \leftrightarrow d$	K
Φ, ω, $L \leftrightarrow M$	K
X_b, Y_b, Z_b, X_s, Y_s, Z_s, $\bar{U}_1 \leftrightarrow \bar{U}_2$	K
$Q^\omega \leftrightarrow P^\omega$, $Q_2 \leftrightarrow P_1$, $Q_1 \leftrightarrow P_2$	K

Table 2. Maximal Level of Appearance and Symmetry of Various Solution Components.

It should be noted that the solution is 'symmetric' under the following parameter interchange: $x \leftrightarrow y$, $1 \leftrightarrow 2$, $4 \leftrightarrow 5$, $X_b \leftrightarrow Y_b$, $X_s \leftrightarrow Y_s$, etc. One may obtain symmetric expressions using operation '*Sym*'. For example, $u \leftrightarrow v$ means $u = Sym(v)$ and $v = Sym(u)$. Also, $Q^{\chi_1} \leftrightarrow P^{\chi_2}$, $Q^{\chi_2} \leftrightarrow P^{\chi_1}$, $Q^{\varphi} \leftrightarrow -P^{\varphi}$, $D_2 = -Sym(D_1)$ and $\tau = -Sym(\tau)$, $\sigma_z = Sym(\sigma_z)$.

This solution does not ensure that the three forces and three moments at the beam tip vanish, see stress integral

$$\iint_{\Omega} \{\sigma_z, \sigma_z y, \sigma_z x, \tau_{xz}, \tau_{xz}, \tau_{yz}x - \tau_{xz}x\} = \{P_z, M_x, -M_y, P_x, P_y, M_z\}. \tag{4}$$

Hence, one needs to superimpose a series of solutions for tip loads (see [10]) in order to cancel out these resultants.

2.1 Stress components

The detailed stress expressions are

$$\sigma_x = \sum_{k=0}^{K} z^k \{\Phi^{(k)}_{,yy} + \overline{U}^{(k)}_1 - \frac{k+1}{a_0}[a_{44}H^{(k+1)} - \frac{d_{k+1}a_0}{4a_{33}}(\frac{3a_{23}+a_{44}}{a_{13}}y^2 - x^2) + S^{(k+1)}_x] - \frac{z}{k+1}d_k\overline{\sigma}^{(3)}_y + \frac{z^2}{(k+1)(k+2)}(p_k\overline{\sigma}^{(1)}_x + q_k\overline{\sigma}^{(2)}_x)\},$$

$$\sigma_z = \sum_{k=0}^{K} \frac{z^k}{a_{33}}\{-a_{13}(\Phi^{(k)}_{,yy} + \overline{U}^{(k)}_1) - a_{23}(\Phi^{(k)}_{,xx} + \overline{U}^{(k)}_2) + a_{36}\Phi^{(k)}_{,xy} + (p_k y + q_k x)xy + \frac{k+1}{a_0}[(a_{13}a_{44} + a_{23}a_{55} - a_{36}a_{45} + a_0)H^{(k+1)} - \frac{d_{k+1}a_0}{2a_{33}}(a_{45}+a_{36})xy + a_{13}S^{(k+1)}_x + a_{23}S^{(k+1)}_y] - \frac{zd_k}{k+1}(1 + a_{33}\overline{\sigma}^{(3)}_z) - \frac{z^2}{(k+1)(k+2)}[p_k(x - a_{33}\overline{\sigma}^{(1)}_z) + q_k(y - a_{33}\overline{\sigma}^{(2)}_z)]\},$$

$$\tau_{yz} = \sum_{k=0}^{K+1} \frac{z^k}{a_0}(a_{55}H^{(k)}_{,y} - a_{45}H^{(k)}_{,x} + \frac{d_k a_0}{2a_{33}}y + S^{(k)}_{y,y}),$$

$$\tau_{xy} = \sum_{k=0}^{K} z^k[-\Phi^{(k)}_{,xy} + (k+1)\frac{a_{45}}{a_0}H^{(k+1)} - \frac{zd_k}{k+1}\overline{\tau}^{(3)}_{xy} + \frac{z^2}{(k+1)(k+2)}(p_k\overline{\tau}^{(1)}_{xy} + q_k\overline{\tau}^{(2)}_{xy})],$$

$$\sigma_y = Sym(\sigma_y), \quad \tau_{xz} = Sym(\tau_{yz}). \tag{5}$$

The body force potentials $\overline{U}^{(k)}_j$ (k=0,…,K), that appear in the above terms are

$$\overline{U}_1 = -\int_0^x X_b dx = \sum_{k=0}^{K} \overline{U}^{(k)}_1(x,y)z^k, \; \overline{U}_2 = Sym(\overline{U}_1). \tag{6}$$

The expressions for $S^{(k)}_x, S^{(k)}_y$ (k=0,…,K+1) are

$$S^{(k)}_x = \int_0^x [(k+1)(a_{44}L^{(k+1)} - a_{45}M^{(k+1)}) - \tilde{\delta}_{k0}(p_{k-1}P^{\chi_1} + q_{k-1}P^{\chi_2} - \tau_{k-1}P^{\varphi}) - d_k(a_{44}\overline{u}^{(3)} - a_{45}\overline{v}^{(3)})]dx, \quad S^{(k)}_y = Sym(S^{(k)}_x) \tag{7}$$

where $\tilde{\delta}_{k0} = 1/k$ for $k \neq 0$ and $\tilde{\delta}_{00} = 0$, the polynomials $P^{\chi_k}, Q^{\chi_k}, P^{\varphi}, Q^{\varphi}$ are defined in [10]. In (5) and in all expressions below, we replace $H^{(k)}$ $(k \le K+1)$ by the RHS of

$$H^{(k)} = \omega^{(k)} + \tilde{\delta}_{k0}(p_{k-1}\chi_1 + q_{k-1}\chi_2 - \tau_{k-1}\varphi) \tag{8}$$

where $\omega^{(k)}$ $(k \le K)$ is an additional series of *longitudinal stress functions* as required by the single-value conditions for the present problem, see item (**e**) in Section 2.

Remark 1. The problem without single-valued requirements for the biharmonic stress function may be considered. In this case one may assume $p_k = q_k = \tau_k = 0$ and use the harmonic stress functions $H^{(k)}$ only, i.e., without definition (8).

2.2 Displacements

The strain components are derived from (5), using Hook's low, see (1). Displacements are determined via integration of strains. The rigid body displacements are not included in these expressions and are introduced by the tip loads correction, see Section 1:

$$u=\sum_{k=0}^{K} z^k\{L^{(k)}-\frac{zd_k}{k+1}(\frac{2a_{13}x+a_{36}y}{2a_{33}}+\overline{u}^{(3)})$$
$$-\frac{z^4 p_k}{(k+1)(k+2)(k+3)(k+4)}$$
$$-\frac{z^2}{(k+1)(k+2)}[p_k(\frac{a_{23}y^2-a_{13}x^2}{2a_{33}}-\overline{u}^{(1)})$$
$$-q_k(\frac{a_{36}y^2+2a_{13}xy}{2a_{33}}+\overline{u}^{(2)})-y\tau_k]\},$$
$$v=Sym(u), \qquad (9)$$

$$w=\sum_{k=0}^{K+1} z^k\{H^{(k)}+\frac{d_k}{4a_{33}}[(2a_{13}+a_{55})x^2$$
$$+2(a_{36}+a_{45})xy+(2a_{23}+a_{44})y^2]-\frac{z^2 d_k}{(k+1)(k+2)}$$
$$-\frac{z}{k+1}(q_k x+p_k y)xy+\frac{z^3(p_k x+q_k y)}{(k+1)(k+2)(k+3)}\}.$$

2.3 Biharmonic stress functions

To enable further handling of the boundary conditions, we shall also use the following general identities for C^2-differentiable functions:

$$\frac{d}{ds}\Phi^{(k)}_{,y}=\Phi^{(k)}_{,yy}\cos(\vec{n},x)-\Phi^{(k)}_{,xy}\cos(\vec{n},y),$$
$$\frac{d}{ds}\Phi^{(k)}_{,x}=\Phi^{(k)}_{,xy}\cos(\vec{n},x)-\Phi^{(k)}_{,xx}\cos(\vec{n},y).$$

The *biharmonic stress functions* $\Phi^{(k)}$ (k=0,…,K) are governed by

$$\nabla_1^{(4)}\Phi^{(k)}=F_0^{(k)} \quad \text{over} \quad \Omega, \qquad (10a)$$
$$\frac{d}{ds}\{\Phi^{(k)}_{,x},\Phi^{(k)}_{,y}\}=\{-F_1^{(k)},F_2^{(k)}\} \quad \text{in } \partial\Omega, \quad (10b)$$
$$\frac{d}{ds}\{\Phi^{(k)}_{,x},\Phi^{(k)}_{,y}\}^{[j]}_{[i]}=\{-F_1^{(k)},F_2^{(k)}\}^{[j]}_{[i]} \text{ in } \partial\Omega_{ij}, \quad (10c)$$
$$\{L^{(k)},M^{(k)}\}^{[j]}_{[i]}=\{0,0\} \quad \text{in } \partial\Omega_{ij} \qquad (10d)$$

where $F_i^k=P_i^k\cos(\vec{n},x)+Q_i^k\cos(\vec{n},y)$ and

$$P_1^{(k)}=Y_1^{(k)}-(k+1)\frac{a_{45}}{a_0}H^{(k+1)},$$

$$Q_1^{(k)}=Y_2^{(k)}-\overline{U}_2^{(k)}+\frac{k+1}{a_0}[a_{55}H^{(k+1)}$$
$$-\frac{d_{k+1}a_0}{4a_{33}}(\frac{3a_{13}+a_{55}}{a_{23}}x^2-y^2)+S_y^{(k+1)}],$$

$$Q_2^{(k)}=Sym(P_1^{(k)}),\ P_2^{(k)}=Sym(Q_1^{(k)}), \quad (11)$$

$$F_0^{(k)}=-(b_{11}\overline{U}_1^{(k)}+b_{12}\overline{U}_2^{(k)})_{,yy}$$
$$+(b_{16}\overline{U}_1^{(k)}+b_{26}\overline{U}_2^{(k)})_{,xy}-(b_{12}\overline{U}_1^{(k)}+b_{22}\overline{U}_2^{(k)})_{,xx}$$
$$-\frac{2p_k}{a_{33}}(a_{13}x-a_{36}y)+\frac{2q_k}{a_{33}}(a_{36}x-a_{23}y)$$
$$+(k+1)\{(\frac{b_{11}a_{44}+b_{12}a_{55}-b_{16}a_{45}}{a_0}-\frac{a_{13}}{a_{33}})H^{(k+1)}_{,yy}$$
$$+(\frac{b_{12}a_{44}+b_{22}a_{55}-b_{26}a_{45}}{a_0}-\frac{a_{23}}{a_{33}})H^{(k+1)}_{,xx}$$
$$-(\frac{b_{16}a_{44}+b_{26}a_{55}-b_{66}a_{45}}{a_0}-\frac{a_{36}}{a_{33}})H^{(k+1)}_{,xy}$$
$$+\frac{d_{k+1}}{2a_{33}}[2a_{12}+\frac{a_{36}}{a_{33}}(a_{36}+a_{45})-\frac{a_{11}}{a_{13}}(3a_{23}+a_{44})$$
$$-\frac{a_{22}}{a_{23}}(3a_{13}+a_{55})]+\frac{1}{a_0}[(b_{11}S_x^{(k+1)}+b_{12}S_y^{(k+1)})_{,yy}$$
$$+(b_{12}S_x^{(k+1)}+b_{22}S_y^{(k+1)})_{,xx}+(b_{16}S_x^{(k+1)}+b_{26}S_y^{(k+1)})_{,xy}]\}.$$

The *biharmonic operator* of (10a) is defined as

$$\nabla_1^{(4)}=b_{22}\frac{\partial^4}{\partial x^4}-2b_{26}\frac{\partial^4}{\partial x^3\partial y}$$
$$+(2b_{12}+b_{66})\frac{\partial^4}{\partial x^2\partial y^2}-2b_{16}\frac{\partial^4}{\partial x\partial y^3}+b_{11}\frac{\partial^4}{\partial y^4}. \qquad (12)$$

The ellipticity of $\nabla_1^{(4)}$ follows from positive definite Hook's low, (1), see [12]. The $\partial/\partial s$ – type boundary conditions (10b,c) leave the values of the functions $\Phi^{(k)}$ and the derivatives $\Phi^{(k)}_{,x},\Phi^{(k)}_{,y}$ undetermined by a constant for each homogeneous domain component. As a general rule we select a point over each dividing curve, say, $(x_1^{ij},y_1^{ij})\in\partial\Omega_{ij}$, where we force the functions to be equal for both neighbor domains, namely,

$$\{\Phi^{(k)},\Phi^{(k)}_{,x},\Phi^{(k)}_{,y}\}(x_1^{ij},y_1^{ij})^{[j]}_{[i]}=\{0,0,0\}.$$

For the sake of simplicity we also assume

$$\{\Phi^{(k)},\Phi^{(k)}_{,x},\Phi^{(k)}_{,y}\}(0,0)=\{0,0,0\}.$$

The single-valued conditions for biharmonic function $\Phi^{(k)}$ on a simply connected (homogeneous) domain are, see [11],

$$\oint_{\partial\Omega} \{F_1^{(k)}, F_2^{(k)}, F_2^{(k)} y - F_1^{(k)} x\} = \{0,0,0\}. \quad (13)$$

Subsequently, the single-valued type conditions, necessary for solution of (10) existence on a non-homogeneous domain are

$$\begin{aligned} &\oint_{\partial\Omega} \{F_1^{(k)}, F_2^{(k)}, F_2^{(k)} y - F_1^{(k)} x\} \\ &+ \sum_{ij} \oint_{\Omega_{ij}} \{F_1^{(k)}, F_2^{(k)}, F_2^{(k)} y - F_1^{(k)} x\}_{[i]}^{[j]} = \{0,0,0\}. \end{aligned} \quad (14)$$

These equalities become clearer if (13) are written first for each domain component $\Omega_{[j]}$ and then summed up.

2.4 Longitudinal stress functions

The Laplace-type operator and Neumann-type boundary operator are, see [10],

$$\nabla_3^{(2)} = a_{44} \frac{\partial^2}{\partial x^2} - 2a_{45} \frac{\partial^2}{\partial x \partial y} + a_{55} \frac{\partial^2}{\partial y^2}, \quad (15a)$$

$$\begin{aligned} D_1^n &= (a_{44} \frac{\partial}{\partial x} - a_{45} \frac{\partial}{\partial y}) \cos(\vec{n}, x) \\ &+ (-a_{45} \frac{\partial}{\partial x} + a_{55} \frac{\partial}{\partial y}) \cos(\vec{n}, y). \end{aligned} \quad (15b)$$

The *longitudinal stress functions* $\omega^{(k)}$ (k=0,…,K) are governed by the Neumann problem

$$\nabla_3^{(2)} \omega^{(k)} = F_0^{\omega^{(k)}} \text{ over } \Omega, \quad (16a)$$

$$D_1^n \omega^{(k)} = F_3^{(k)} \text{ on } \partial\Omega, \quad (16b)$$

$$[\frac{1}{a_0}(D_1^n \omega^{(k)} - F_3^{(k)})]_{[i]}^{[j]} = 0 \text{ on } \partial\Omega_{ij}, \quad (16c)$$

$$[\omega^{(k)}]_{[i]}^{[j]} = F^{\omega^{(k)}, ij} \text{ on } \partial\Omega_{ij} \quad (16d)$$

where $F_3^{(k)} = P^{\omega^{(k)}} \cos(\vec{n}, x) + Q^{\omega^{(k)}} \cos(\vec{n}, y)$ and

$$\begin{aligned} P^{\omega^{(k)}} &= a_0 Z_1^{(k)} + d_k (-\frac{a_0}{2a_{33}} x + a_{44} \overline{u}^{(3)} - a_{45} \overline{v}^{(3)}) \\ &- (k+1)(a_{44} L^{(k+1)} - a_{45} M^{(k+1)}), \end{aligned}$$

$$Q^{\omega^{(k)}} = Sym(P^{\omega^{(k)}}), \quad (17)$$

$$\begin{aligned} F_0^{\omega^{(k)}} &= -a_0 Z_b^{(k)} - \frac{a_0}{a_{33}}(k+1)\{-a_{13}(\Phi_{,yy}^{(k+1)} + \overline{U}_1^{(k+1)}) \\ &- a_{23}(\Phi_{,xx}^{(k+1)} + \overline{U}_2^{(k+1)}) + a_{36} \Phi_{,xy}^{(k+1)} + (p_{k+1} y + q_{k+1} x) xy \\ &- (k+1)(a_{44} L_{,x}^{(k+1)} + a_{55} M_{,y}^{(k+1)} - a_{45} L_{,y}^{(k+1)} - a_{45} M_{,x}^{(k+1)}) \\ &+ (k+2)[\frac{a_{13}a_{44} + a_{23}a_{55} - a_{36}a_{45} + a_0}{a_0} H^{(k+2)} \\ &+ \frac{d_{k+2}}{2a_{33}^2}(a_{45} + a_{36}) xy + \frac{a_{13}}{a_0} S_x^{(k+2)} + \frac{a_{23}}{a_0} S_y^{(k+2)}]\} \\ &+ d_k (a_{45} \overline{\varepsilon}_x^{(3)} + a_{55} \overline{\varepsilon}_y^{(3)} - a_{45} \overline{\gamma}_{xy}^{(3)} + a_0 \overline{\sigma}_z^{(3)}), \end{aligned}$$

$$\begin{aligned} F^{\omega^{(k)}, ij} &= -d_k \{\frac{1}{4a_{33}}[(2a_{13} + a_{55}) x^2 + 2(a_{36} + a_{45}) xy \\ &+ (2a_{23} + a_{44}) y^2]\}_{[i]}^{[j]}. \end{aligned}$$

or convenience we assume $\omega^{(k)}(0,0) = 0$. The necessary condition for boundary value problem (16) solution existence, see [10], is

$$\oint_{\partial\Omega} \frac{1}{a_0} F_3^{(k)} + \sum_{ij} \int_{\Omega_{ij}} [\frac{1}{a_0} F_3^{(k)}]_{[i]}^{[j]} = \iint_{\Omega} \frac{1}{a_0} F_0^{\omega^{(k)}}. \quad (18)$$

In contrast to torsion and bending stress functions, $\omega^{(k)}$ may be discontinuous along dividing contours Ω_{ij}, see (16d).

2.5 Auxiliary functions

The functions $L^{(k)}(x,y)$ (k=0,…,K) are

$$\begin{aligned} L^{(k)} &= b_{12} \Phi_{,x}^{(k)} - b_{16} \Phi_{,y}^{(k)} + \int_0^x \{b_{11}(\Phi_{,yy}^{(k)} + \overline{U}_1^{(k)}) + b_{12} \overline{U}_2^{(k)} \\ &- (k+1)[(\frac{b_{11}a_{44} + b_{12}a_{55} - b_{16}a_{45}}{a_0} - \frac{a_{13}}{a_{33}}) H^{(k+1)} + \frac{b_{11}}{a_0} S_x^{(k+1)} \\ &+ \frac{b_{12}}{a_0} S_y^{(k+1)}]\} dx + \frac{p_k}{a_{33}} \{[\frac{1}{a_0}((b_{26}a_{45} + b_{12}a_{44})(a_{23} + 2a_{33}) \\ &+ a_{55} b_{22}(2a_{33} - a_{23})) - 2a_{23}] \frac{y^4}{24} + a_{13} \frac{x^2 y^2}{2}\} + \frac{q_k}{a_{33}} \{a_{13} \frac{x^3 y}{3} \\ &- \frac{1}{a_0}[a_{45} b_{22}(2a_{33} - a_{13}) + b_{12}(a_{36}a_{44} - a_{23}a_{45}) - b_{26}(a_{23}a_{55} \\ &- a_{36}a_{45})] \frac{y^4}{24}\} + \frac{\tau_k}{a_0}(b_{12}a_{44} - b_{22}a_{55} + b_{26}a_{45}) \frac{y^3}{6} \end{aligned}$$

$$+(k+1)\frac{d_{k+1}}{4a_{33}}\{[\frac{a_{12}(3a_{13}+a_{55})}{a_{23}}-a_{11}]\frac{x^3}{3}$$
$$+\frac{a_{13}(a_{45}+a_{36})}{a_{33}}x^2y+[\frac{a_{11}(3a_{23}+a_{44})}{a_{13}}-a_{12}]xy^2$$
$$-[a_{26}+\frac{a_{23}(a_{45}+a_{36})}{a_{33}}-\frac{a_{16}(3a_{23}+a_{44})}{a_{13}}]\frac{y^3}{3}\}-\int_0^y l^{(k)}(y)dy,$$

$$l^{(k)}(y)=\{\int_0^y[b_{22}(\Phi^{(k)}_{,xxx}+\bar U^{(k)}_{2,x})+b_{12}\bar U^{(k)}_{1,x}]dy$$
$$-(k+1)[(\frac{b_{12}a_{44}+b_{22}a_{55}-b_{26}a_{45}}{a_0}-\frac{a_{23}}{a_{33}})\int_0^y H^{(k+1)}_{,x}dy$$
$$-(\frac{b_{16}a_{44}+b_{26}a_{55}-b_{66}a_{45}}{a_0}-\frac{a_{36}}{a_{33}})H^{(k+1)}$$
$$+b_{12}[2\Phi^{(k)}_{,xy}-\Phi^{(k)}_{,xy}(0,0)]-b_{16}[2\Phi^{(k)}_{,yy}-\Phi^{(k)}_{,yy}(0,0)+\bar U^{(k)}_1]$$
$$-b_{26}[2\Phi^{(k)}_{,xx}-\Phi^{(k)}_{,xx}(0,0)+\bar U^{(k)}_2]+b_{66}[\Phi^{(k)}_{,xy}-\frac{1}{2}\Phi^{(k)}_{,xy}(0,0)]$$
$$+(k+1)(k+2)[\frac{b_{12}}{a_0}\int_0^y(a_{44}L^{(k+2)}-a_{45}M^{(k+2)})dy$$
$$-\frac{b_{22}}{a_0}\int_0^y(\int_0^y(a_{55}M^{(k+2)}-a_{45}L^{(k+2)})_{,x}dy)dy$$
$$+\frac{b_{26}}{a_0}\int_0^y(a_{55}M^{(k+2)}-a_{45}L^{(k+2)})dy]$$
$$-\frac{b_{12}}{a_0}\int_0^y(a_{44}\bar u_k-a_{45}\bar v_k)dy+\frac{b_{26}}{a_0}\int_0^y(a_{55}\bar v_k-a_{45}\bar u_k)dy$$
$$-\frac{b_{22}}{a_0}\int_0^y(\int_0^y(a_{55}\bar v_k-a_{45}\bar u_k)_{,x}dy)dy\}_{x=0},$$

and $M^{(k)}=Sym(L^{(k)})$, $m^{(k)}=Sym(l^{(k)})$.

In the above, for efficient writing, we have introduced the notation

$$\bar u_k=p_k\bar u^{(1)}+q_k\bar u^{(2)}-(k+1)d_k\bar u^{(3)},$$
$$\bar v_k=p_k\bar v^{(1)}+q_k\bar v^{(2)}-(k+1)d_k\bar v^{(3)}.$$

2.6 Loading constants

The loading constants p_k, q_k, τ_k, d_k are defined for k=0,…,K by

$$p_k=\frac{\bar I_{12}}{\bar I_{11}}q_k+\frac{1}{\bar I_{11}}\oint_{\partial\Omega}X^{(k)}_s+\frac{1}{\bar I_{11}}\iint_\Omega\{X^{(k)}_b$$
$$+\frac{k+1}{a_0}[a_{44}\omega^{(k+1)}_{,x}-a_{45}\omega^{(k+1)}_{,y}-(k+2)(a_{44}L^{(k+2)} \qquad (19)$$
$$-a_{45}M^{(k+2)})+d_{k+1}(\frac{a_0}{2a_{33}}x-a_{44}\bar u^{(3)}+a_{45}\bar v^{(3)})]\},$$

$$q_k=Sym(p_k),$$
$$\tau_k=p_k\frac{D_1}{D}+q_k\frac{D_2}{D}+\frac{1}{D}\oint_{\partial\Omega}(xY^{(k)}_s-yX^{(k)}_s)$$
$$+\frac{1}{D}\iint_\Omega\{xY^{(k)}_b-yX^{(k)}_b+\frac{k+1}{a_0}[(a_{55}x+a_{45}y)(\omega^{(k+1)}_{,y}$$
$$+(k+2)M^{(k+2)})-(a_{45}x+a_{44}y)(\omega^{(k+1)}_{,x}+(k+2)L^{(k+2)})$$
$$+d_{k+1}((a_{44}\bar u^{(3)}-a_{45}\bar v^{(3)})y+(a_{45}\bar u^{(3)}-a_{55}\bar v^{(3)})x)]\},$$

$$d_k=\frac{1}{\bar I_{33}}\oint_{\partial\Omega}Z^{(k)}_s+\frac{1}{\bar I_{33}}\iint_\Omega\{Z^{(k)}_b+\frac{k+1}{a_{33}}[a_{36}\Phi^{(k+1)}_{,xy}$$
$$-a_{13}(\Phi^{(k+1)}_{,yy}+\bar U^{(k+1)}_1)-a_{23}(\Phi^{(k+1)}_{,xx}+\bar U^{(k+1)}_2)$$
$$+(p_{k+1}y+q_{k+1}x)xy+(k+2)(\frac{a_{13}}{a_0}S^{(k+2)}_x$$
$$+\frac{a_{23}}{a_0}S^{(k+2)}_y+(k+1)d_{k+2}\frac{a_{45}+a_{36}}{2a_{33}}xy$$
$$+\frac{a_{13}a_{44}+a_{23}a_{55}-a_{36}a_{45}+a_0}{a_0}H^{(k+2)})]\}.$$

where $\bar I_{ij}$ are defined in [10] by

$$\bar I_{33}=\iint_\Omega(\frac{1}{a_{33}}+\sigma^{(3)}_z),$$
$$\{\bar I_{12},\bar I_{22}\}=-\iint_\Omega(\frac{y}{a_{33}}-\sigma^{(2)}_z)\{-x,y\}, \qquad (20)$$
$$\{\bar I_{11},\bar I_{21}\}=-\iint_\Omega(\frac{x}{a_{33}}-\sigma^{(1)}_z)\{-x,y\},$$

and D, D_1, D_2 are

$$D=-\iint_\Omega\frac{1}{a_0}[Q^\varphi(\varphi_{,y}+x)+P^\varphi(\varphi_{,x}-y)], \qquad (21)$$
$$D_i=\iint_\Omega\frac{1}{a_0}(yP^{\chi_i}-xQ^{\chi_i}+P^\varphi\chi_{i,x}+Q^\varphi\chi_{i,y}).$$

As shown in [10], the existence conditions for χ_1, χ_2 require a selection of the coordinate system origin so that

$$\iint_\Omega\{\frac{x}{a_{33}}-\bar\sigma^{(1)}_z,\ \frac{y}{a_{33}}-\bar\sigma^{(2)}_z\}=\{0,0\}.$$

3 Verification of solution hypothesis

Equations (5) provide an exact solution that satisfies all requirements of the theory of elasticity. To carry out the above task, we employ the equilibrium equations, the compatibility equations and the outer surface boundary conditions. The process is broken into subsequent steps, where in general, at each one, we use all relevant relations that were found in previous steps.

(a) **Equilibrium equations**. Considering the stress terms of (5), and using the fact that the solutions of the auxiliary problems satisfy the equilibrium equations, one may verify that the equilibrium equations with X_b and Y_b are satisfied identically. From the third equilibrium equation (with Z_b) the terms of $F_0^{\omega^{(k)}}$ in (17), are extracted as the coefficients of the 'free term' z^{k+1}.

(b) **Boundary conditions**. From the outer contour condition of X_s, Y_s one may deduce the terms for the derivatives $\frac{d}{ds}\Phi^{(k)}_{,y}, \frac{d}{ds}\Phi^{(k)}_{,x}$ over the contour $\partial\Omega$, see (11). From the third boundary condition of Z_s the additional terms for the normal derivatives $D_1^n\omega^{(k)}$ presented in (17) are obtained as the coefficients of z^{k+1}.

(c) **Compatibility equations**. The terms of the strain components satisfy the compatibility equations due to the fact that the solutions of the auxiliary problems are consistent and inherently satisfy the compatibility equations.

(d) **Displacement interface continuity**.

The in-plane displacement components: Continuity for levels $k>K+2$, see (9:a,b), follows from the fact that loading constants of Section 1.6 are not domain dependent (i.e., constants over the entire non-homogeneous domain, Ω). Displacement continuity for levels $k=K+2$, $K+1$ is achieved by the z^{K+1}, z^{K+1} terms of (9:a,b), which cancel out the discontinuity of u, v. For levels $k \le K$ displacement continuity is part of the biharmonic problem, since by explicit use of (9:a,b), the interface conditions of (10b) may also be written as

$$\{u^{(k)}, v^{(k)}\}^{[j]}_{[i]} = \{0,0\} \quad on \;\; \partial\Omega_{ij}.$$

The out-of-plane displacement component: Continuity of w may be verified by examining (9c). This equation shows that the coefficients of z^{k+1}, z^{k+2} and z^{k+3} do not contribute any discontinuity due to the fact that the loading constants are not domain dependent. For lower levels this continuity condition is imposed as part (16d) of the Neumann problem for $\omega^{(k)}$.

(e) **Singled-value conditions**. The single-valued-type conditions (14) for $\Phi^{(k)}$ and its first derivatives yield the expressions (19a-c) for the constants q_k, p_k, τ_k, respectively.

(f) **Existence of the longitudinal stress function**. The existence condition (18) of a longitudinal stress function $\omega^{(k)}$ yields the definition (19d) of the constants d_k.

4 Applications

4.1 Homogeneous beam under constant distributed body force

The loading constants of homogeneous beam are

$$\begin{aligned}
p_k &= \frac{a_{33}}{I_y}\oint_{\partial\Omega} X_s^{(k)} + \frac{a_{33}}{I_y}\iint_{\Omega}\{X_b^{(k)} + \frac{k+1}{a_0}[a_{44}\omega^{(k+1)}_{,x} \\
&\quad -a_{45}\omega^{(k+1)}_{,y} - (k+2)(a_{44}L^{(k+2)} - a_{45}M^{(k+2)})]\}, \\
q_k &= Sym(p_k), \\
d_k &= \frac{a_{33}}{S_\Omega}\oint_{\partial\Omega} Z_s^{(k)} + \frac{a_{33}}{S_\Omega}\iint_{\Omega}\{Z_b^{(k)} + \frac{k+1}{a_{33}}[a_{36}\Phi^{(k+1)}_{,xy} \\
&\quad -a_{13}(\Phi^{(k+1)}_{,yy} + \overline{U}_1^{(k+1)}) - a_{23}(\Phi^{(k+1)}_{,xx} + \overline{U}_2^{(k+1)}) \\
&\quad +(k+2)(\frac{a_{13}a_{44} + a_{23}a_{55} - a_{36}a_{45}}{a_0} + 1)H^{(k+2)} \\
&\quad +(p_{k+1}y + q_{k+1}x)xy + \frac{a_{13}}{a_0}S_x^{(k+2)} + \frac{a_{23}}{a_0}S_y^{(k+2)})]\}.
\end{aligned} \tag{22}$$

We derive here the longitudinal and biharmonic stress functions Φ, ω for a homogeneous Z-monoclinic beam that undergoes constant body loads, i.e., K=0. Hence, we set X_b=X_0, Y_b=Y_0, Z_b=Z_0, where X_0, Y_0, Z_0 are constants, while no other surface or tip loads are applied. The loading constants p_0, q_0, τ_0 and d_0 in (22) are

$$p_0 = \frac{S_\Omega}{I_y} a_{33} X_0, \quad q_0 = \frac{S_\Omega}{I_x} a_{33} Y_0, \quad \tau_0 = 0, \quad d_0 = a_{33} Z_0.$$

We employ (17) to find

$$F_0^\omega = -a_0 Z_0, \quad P_3 = -\frac{1}{2} a_0 Z_0 x, \quad Q_3 = -\frac{1}{2} a_0 Z_0 y.$$

The solution ω is independent of the domain shape

$$\omega = -\frac{Z_0}{4}(a_{55}x^2 + 2a_{45}xy + a_{44}y^2). \qquad (23)$$

For the biharmonic stress function Φ (11) show the following expansions:

$$F_0 = 2S_\Omega[\frac{X_0}{I_y}(a_{36}y - a_{13}x) + \frac{Y_0}{I_x}(a_{36}x - a_{23}y)],$$

$$F_1 = [Y_0 y - \frac{a_{55}}{a_0}(p_0\chi_1 + q_0\chi_2)]\cos(\vec{n}, y),$$

$$F_2 = Sym(F_1).$$

Here we used expression (8) for $H^{(1)}$, and (6) for $\bar{U}_1, \bar{U}_2$.

4.2 Homogeneous beam under linear body force distribution

We consider here a Z-monoclinic beam of generic cross section when the body force distribution is given by $X_b^{(0)} = \gamma_x x,\ Y_b^{(0)} = 0,\ Z_b^{(0)} = \gamma_z z,$ while all other surface and tip loads vanish. A physical example for such a loading is the rotating beam shown in Figure 2.

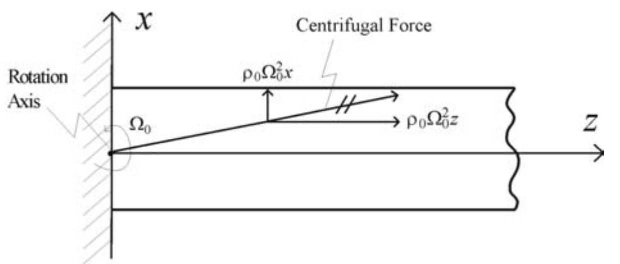


Fig. 2. Notation for Rotating Beam.

In such a case $\gamma_x = \gamma_z = \rho_0 / \Omega_0^2$, where ρ_0 is the specific weight (density) of the material, and Ω_0 is the angular velocity. We shall now discuss the two levels of this problem solution one by one.

Level k=1: Equations (19) show that

$$p_1 = q_1 = \tau_1 = 0, \quad d_1 = \frac{a_{33}}{S_\Omega}\iint_\Omega Z_b^{(1)} = \frac{a_{33}}{\gamma_z}.$$

For the harmonic function $\omega^{(1)}$ we find

$$F_0^{\omega^{(1)}} = -a_0\gamma_z,$$

$$P_3^{(1)} = -\frac{1}{2}a_0\gamma_z x, \quad Q_3^{(1)} = -\frac{1}{2}a_0\gamma_z y.$$

Analogously to the solution presented in (23), $\omega^{(1)}$ is independent of the domain shape

$$\omega^{(1)} = -\frac{\gamma_z}{4}(a_{55}x^2 + 2a_{45}xy + a_{44}y^2). \qquad (24)$$

Level k=0: Equations (19) and Green Theorem show that

$$p_0 = \frac{a_{33}}{I_y}\int_\Omega [X_b^{(0)} + \frac{1}{a_0}(a_{44}\omega_{,x}^{(1)} - a_{45}\omega_{,y}^{(1)})] = 0,$$

$$q_0 = Sym(p_0) = 0$$

and that $d_0 = 0,\ \tau_0 \neq 0.$ Equations (17) show that $\omega^{(0)} = 0$. At this stage one should solve the biharmonic problem (10) for k=0.

4.3 Non-homogeneous beam under constant axial body force

Consider a non-homogeneous Z-monoclinic beam that undergoes a constant body force in the z-direction, namely Z_b=Z_0=const., X_b=Y_b=0, while no other surface or tip loads are applied. The loading constants of (19) are p_0=q_0=0, $d_0 = Z_0 S_\Omega / \bar{I}_{33}$, and

$$\tau_0 = \frac{Z_0 S_\Omega}{D\bar{I}_{33}}\iint_\Omega \frac{1}{a_0}[(a_{44}\bar{u}^{(3)} - a_{45}\bar{v}^{(3)})y - (a_{55}\bar{v}^{(3)} - a_{45}\bar{u}^{(3)})x]$$

where D, $\bar{I}_{33}$ are given by (20:a), (21:a). Since K=0, only the stress functions Φ, ω and the auxiliary functions L, M of level k=0 should be considered, and hence, for the sake of convenience, in what follows we shall omit the index superscript. The Neumann problem (16) for $\omega(x,y)$ should be written with

$$P^{\omega}=d_0(-\frac{a_0}{2a_{33}}x+a_{44}\overline{u}^{(3)}-a_{45}\overline{v}^{(3)}),$$

$$Q^{\omega}=Sym(P^{\omega}),$$

$$F_0^{\omega}=d_0(a_{44}\overline{\varepsilon}_x^{(3)}+a_{55}\overline{\varepsilon}_y^{(3)}-a_{45}\overline{\gamma}_y^{(3)}+a_0\overline{\sigma}_z^{(3)})-a_0Z_0,$$

$$F^{\omega,ij}=-d_0\{\frac{1}{4a_{33}}[(2a_{13}+a_{55})x^2$$

$$+2(a_{36}+a_{45})xy+(2a_{23}+a_{44})y^2]\}_{[i]}^{[j]},$$

while for convenience we assume $\omega(0,0)=0$. The biharmonic problem for $\Phi(x,y)$ should have the type of (10) with F_0=F_1=F_2=0 and

$$L=A+B\tau_0,\quad M=Sym(L)$$

where B depends on φ and A is a linear differential operator of Φ. We assume $\{\Phi^{(k)},\Phi_{,x}^{(k)},\Phi_{,y}^{(k)}\}(0,0)=\{0,0,0\}$.

The stresses become

$$\sigma_x=\Phi_{,yy}-zd_0\overline{\sigma}_x^{(3)},\quad \sigma_y=Sym(\sigma_x),$$

$$\tau_{xy}=-\Phi_{,xy}-zd_0\overline{\tau}_{xy}^{(3)},$$

$$\sigma_z=\frac{1}{a_{33}}(a_{36}\Phi_{,xy}-a_{13}\Phi_{,yy}-a_{23}\Phi_{,xx})-(\frac{1}{a_{33}}+\sigma_z^{(3)})zd_0,$$

$$\tau_{yz}=\frac{1}{a_0}(a_{55}\omega_{,y}-a_{45}\omega_{,x})-\frac{d_0}{a_0}(a_{55}\overline{v}^{(3)}-a_{45}\overline{u}^{(3)})$$

$$-\frac{z\tau_0}{a_0}(a_{55}\varphi_{,y}-a_{45}\varphi_{,x}-Q^{\varphi})+\frac{d_0y}{2a_{33}},$$

$$\tau_{xz}=Sym(\tau_{yz}).$$

Once Φ, ω are determined, one needs superimpose suitable St. Venant's solutions of [10] in order to cancel out the tip resultants which are induced by the above stresses.

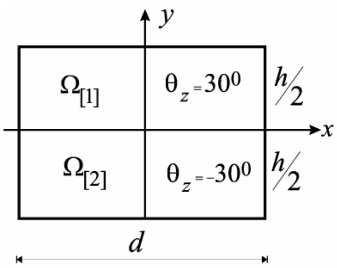


Fig. 3. A y-Laminated Rectangle, N=2.

Let a beam cross-section is geometrically symmetric about the x-axis with anti-symmetric lamination, based on orthotropic material turned about angles $\pm\theta_z$, for example, a non-homogeneous rectangle, see Figure 3. In this case the elastic moduli are identical in two domains $\Omega_{[1]},\Omega_{[2]}$ except for a_{16}, a_{26}, a_{36} and a_{45} that are of identical magnitude but opposite signs. The same is true for reduced elastic constants b_{ij}. By (initially) placing the coordinate system at the cross-section midpoint show that the solution for the third auxiliary problem, see [10], is zero $\overline{\Phi}^{(3)}=0$, and $\overline{\varepsilon}_x^{(3)}=\overline{\varepsilon}_y^{(3)}=\overline{\sigma}_z^{(3)}=0$. Hence $\overline{I}_{33}=S_{\Omega}/a_{33}$ and the displacements are rigid,

$$\overline{u}^{(3)}=\frac{a_{36}}{2a_{33}}y,\quad \overline{v}^{(3)}=-\frac{a_{36}}{2a_{33}}x.$$

The loading constants are p_0=q_0=0, $d_0=Z_0a_{33}$,

$$\tau_0=\frac{Z_0}{2Da_0}\iint_{\Omega}a_{36}(a_{44}y^2+a_{55}x^2)=0.$$

Since in this case the biharmonic problem (10) is homogeneous, $\Phi=0$. We obtain

$$\omega=-\frac{Z_0}{4}(a_{55}x^2+2a_{45}xy+a_{44}y^2)+\frac{Z_0}{2}a_{36}^{[1]}\overline{\varphi}$$

where $\overline{\varphi}$ is known harmonic function with the symmetry $\overline{\varphi}(x,-y)=\overline{\varphi}(x,y)$, see Figure 4.

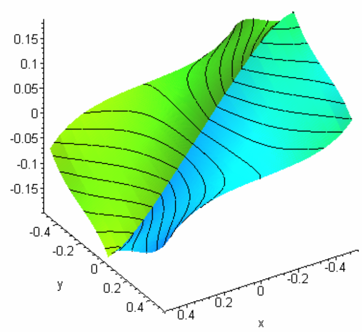


Fig. 4. The Function $\overline{\varphi}$.

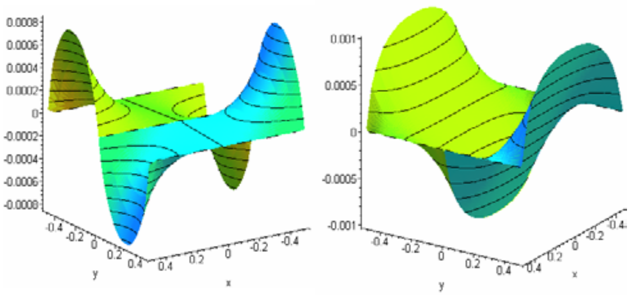
The stress solution that <u>does not produce any tip loads</u> become

$$\sigma_x=\sigma_y=\tau_{xy}=0,\quad \sigma_z=Z_0(z-l),$$

$$\tau_{yz}=\frac{Z_0a_{36}^{[1]}}{2a_0}[a_{55}\overline{\varphi}_{,y}-a_{45}\overline{\varphi}_{,x}+\frac{a_{36}}{a_{36}^{[1]}}(a_{55}x+a_{45}y)],$$

$$\tau_{xz}=Sym(\tau_{yz}).$$

Referring to (9) the axis extension becomes $u(0,0,z) = v(0,0,z) = 0,\ w(0,0,z) = Z_0 a_{33} z(l - z/2)$. The shear stresses τ_{yz} and τ_{xz} for Z_0=1 are presented in Figures 5.



(a) $\tau_{yz}(x,y)$. (b) $\tau_{xz}(x,y)$.

Fig. 5. The Shear Stresses.

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