A POLYTOPIC MODEL OF AIRCRAFT AND GAIN SCHEDULING STATE FEEDBACK CONTROL

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Abstract

This paper proposes an exact polytopic model for the linear parameter varying (LPV) system of aircraft. The LPV system of the linearized equation of aircraft is represented by a descriptor form which reserves physical features of the equation. A polytopic model, called descriptor polytopic model, is derived through a variable transformation to satisfy conditions for the polytope. Using the obtained descriptor polytopic model, a gain scheduling state feedback law is then designed by means of a linear matrix inequality (LMI) formulation. It is shown in a numerical example of a longitudinal flight control that the proposed descriptor polytopic model of aircraft had no model error and exactly represented the original LPV system without any needless flight region.

1 Introduction

Linearized equations of aircraft are regarded as linear time invariant (LTI) systems if the altitude and the flight velocity are constant, but linear parameter varying (LPV) systems if they are varying. Recently, a number of flight control designs in which aircraft are treated as LPV systems have been proposed by gain scheduling techniques [1], [2]. In those gain scheduling designs, the constraints which guarantee the global stability and the global performance over the entire operating region are expressed as linear matrix inequalities (LMIs) [3]. A gain scheduling controller is numerically obtained to satisfy the LMI constraints. As a problem in control designs for LPV systems, the number of the LMI constraints and/or the LMI variables is generally infinite. One of methods for reducing the LMI constraints finitely is to express an LPV system as a polytopic model which is constructed by a linear combination of LTI models at the vertices of the operating region [3]. Then, the LMI constraints are imposed at the vertices of the operating region. Unfortunately, in general it is not always possible to exactly transform LPV systems into polytopic models. It depends on the structure of the LPV systems. There are undesirable cases where polytopic models have model errors [4], [5] and contain needless operating regions to cover the operating region by a convex hull [6]. Consequently, gain scheduling controllers designed may be conservative. The global stability and the performance may be not guaranteed for the LPV systems. It is therefore necessary for construction of polytopic models to make the model errors and the needless operating region small as much as possible.

This paper proposes an exact polytopic model for the LPV system of aircraft. The LPV system of the linearized equation of aircraft is represented by a descriptor form which reserves physical features of the equation. A polytopic model, called descriptor polytopic model, is then derived through a variable transformation to satisfy conditions for the polytope. Using the descriptor polytopic model, a gain scheduling state feedback law is designed by means of an LMI formulation. A numerical example of a longitudinal flight control is shown to demonstrate the proposed method.

2 Statement of Problem

Consider a descriptor LPV system

$$E(\theta)\dot{x}(t) = A(\theta)x(t) + B(\theta)u(t) \tag{1}$$

where $x(t)$ is the n-dimensional state vector, $u(t)$ the m-dimensional input vector and $\theta(t)$ the g-dimensional varying parameter vector

$$\theta(t) \triangleq [\theta_1(t), \cdots, \theta_g(t)]^T \tag{2}$$

The range of each varying parameter is given by

$$\theta_j(t) \in [\underline{\theta}_j, \overline{\theta}_j], \quad \underline{\theta}_j < \overline{\theta}_j \quad (j = 1, \cdots, g) \tag{3}$$

It is assumed that matrix $E(\theta)$ is nonsingular for the range of Eq. (3). Equation (3) represents a convex hull whose number of the vertices is 2^g. It corresponds to the operating region of the LPV system (1) and is also called the parameter box [3]. The vertex set of the parameter box (3) is denoted as

$$\Theta \triangleq \{\theta(t) \in \Re^g : \ \theta_j(t) = \underline{\theta}_j \text{ or } \overline{\theta}_j, \ \ j = 1, \cdots, g\} \tag{4}$$

The purpose of this study is to design a gain scheduling state feedback law for the LPV system (1)

$$u(t) = -F(\theta)x(t) \tag{5}$$

where the closed-loop system combining Eq. (5) with Eq. (1) is stable and the specified control performance is achieved in the parameter box. This paper proposes a method designing such gain scheduling state feedback laws in flight control problems. To do this in flight control design, the following section gives a general form of polytopic models for the LPV system in the descriptor form. A design of gain scheduling state feedback law based on the polytopic model is then presented in the frame of an LMI formulation. The proposed techniques are applied to the linearized equation of aircraft.

3 Descriptor Polytopic Model and Gain Scheduling State Feedback

3.1 Construction of descriptor polytopic model

The number of design constraints for the LPV system is, in general, infinite if the structure of $E(\theta)$, $A(\theta)$ and $B(\theta)$ in Eq. (1) is not particularly specified. One of techniques reducing the design constraints finitely is that LPV system (1) is transformed into a polytopic model which is constructed by linearly combining LTI models in the vertex set Θ.

$$\sum_{i=1}^{2^g} \alpha_i(\theta)E_i\dot{x}(t) = \sum_{i=1}^{2^g} \alpha_i(\theta)\{A_i x(t) + B_i u(t)\} \tag{6}$$

$$[E_i \ \ A_i \ \ B_i] \triangleq [E(\theta^i) \ \ A(\theta^i) \ \ B(\theta^i)]$$
$$\theta^i \in \Theta \quad (i = 1, \cdots, 2^g)$$

$$\alpha_i(\theta) \geq 0, \quad \sum_{i=1}^{2^g} \alpha_i(\theta) = 1 \tag{7}$$

Equation (6) is called the descriptor polytopic model in this paper. Whether Eq. (1) is exactly transformed into Eq. (6) or not depends on the structures of $E(\theta)$, $A(\theta)$ and $B(\theta)$ in Eq. (1). A general form of descriptor polytopic models is given as the following theorem.

Theorem 1 *Suppose that (a1-i) $E(\theta)$, $A(\theta)$ and $B(\theta)$ in Eq. (1) is affine with respect to functions $f_j(\theta_j)$ $(j = 1, \cdots, g)$ whose variable is θ_j, and (a1-ii) in the parameter box (3), $f_j(\underline{\theta}_j) \neq f_j(\overline{\theta}_j)$. Then, Eq. (1) is exactly transformed into Eq. (6) where $\alpha_i(\theta)$ $(i = 1, \cdots, 2^g)$ satisfying Eq. (7) are given by*

$$\alpha_i(\theta) = \prod_{j=1}^{g} \phi_j(\theta_j) \quad (i = 1, \cdots, 2^g) \tag{8}$$

where $\phi_j(\theta_j)$ is either $\underline{\phi}_j(\theta_j)$ or $\overline{\phi}_j(\theta_j)$ defined as

$$\underline{\phi}_j(\theta_j) \triangleq \frac{f_j(\overline{\theta}_j) - f_j(\theta_j)}{f_j(\overline{\theta}_j) - f_j(\underline{\theta}_j)}, \tag{9}$$

$$\overline{\phi}_j(\theta_j) \triangleq \frac{f_j(\theta_j) - f_j(\underline{\theta}_j)}{f_j(\overline{\theta}_j) - f_j(\underline{\theta}_j)} \tag{10}$$

$$(j = 1, \cdots, g)$$

□

The proof is given in Appendix A. Even if assumption (a1-i) does not hold, Eq. (1) can be transformed into a polytopic model by defining a new varying parameter. However, the following problems arise in this aspect: the number of the vertices is increased to twice if a new varying parameter is added. The obtained polytopic model includes needless operating regions which are not contained in the original LPV system (1). These may lead to conservative results in control analysis and design. Although some techniques for reducing the needless operating regions have been proposed [6], it is not possible to delete those regions completely. Furthermore, assumption (a1-ii) is needed for the definition of $\underline{\phi}_j$ and $\overline{\phi}_j$ in Eq. (10).

3.2 Variable transformation for polytope and gain scheduling state feedback

Once a descriptor polytopic model is obtained, a gain scheduling state feedback law is designed as will be shown in the next section. Assumptions (a1-i) and (a1-ii) of Theorem 1 is not always satisfied in general. Transforming variables is one of possibilities to satisfy the assumptions. This section gives a variable transformation and the gain scheduling state feedback gain. The variable transformation is defined as

$$x \triangleq T_x(\theta)\tilde{x}, \quad u \triangleq T_u(\theta)\tilde{u} \tag{11}$$

where $T_x(\theta)$ and $T_u(\theta)$ are nonsingular in the parameter box (3). Applying Eq. (11) to Eq. (1), the transformed LPV system is written as

$$\begin{aligned} \tilde{E}(\theta)\dot{\tilde{x}}(t) &= \tilde{A}(\theta)\tilde{x}(t) + \tilde{B}(\theta)\tilde{u}(t) \\ \tilde{E} \triangleq ET_x,\ \ &\tilde{A} \triangleq AT_x,\ \ \tilde{B} \triangleq BT_u. \end{aligned} \tag{12}$$

If the transformed LPV system satisfies assumptions (a1-i) and (a1-ii), a gain scheduling state feedback gain is obtained as the following theorem.

Theorem 2 *Suppose that (a2-i) the transformed LPV system (12) satisfies assumptions (a1-i) and (a1-ii) of Theorem 1, and (a2-ii) a gain scheduling state feedback law for the transformed LPV system (12) is obtained as*

$$\tilde{u}(t) = -\tilde{F}(\theta)\tilde{x}(t) \tag{13}$$

which stabilizes the closed loop of the transformed system and achieves the specified control performance. Then, the gain scheduling state feedback gain for the original LPV system (1) is given by

$$F(\theta) = T_u(\theta)\tilde{F}(\theta)T_x^{-1}(\theta). \tag{14}$$

□

The proof is given in Appendix B.

4 Design of Gain Scheduling State Feedback

This section describes a design of the gain scheduling state feedback gain $\tilde{F}(\theta)$ satisfying (a2-ii) of Theorem 2 by means of an LMI formulation. The LMI formulation proposed for fuzzy model [7] is modified to the descriptor form in this section.

Consider a Lyapunov function $V(\tilde{x}) = \tilde{x}^T P\tilde{x}$ for a descriptor polytopic system (12) where $P \in \Re^{n\times n} > 0$. Then, a sufficient condition for the quadratic stability is given by [7]

$$\dot{V}(t) < -\tilde{x}^T(Q + \tilde{F}^T R\tilde{F})\tilde{x} \tag{15}$$

where $Q \in \Re^{n\times n} \geq 0$ and $R \in \Re^{m\times m} > 0$ are weighting matrices and are remained to be designed. Defining $X \triangleq P^{-1}$ and introducing an auxiliary variable

$$\tilde{M}(\theta) \triangleq \tilde{F}(\theta)X\tilde{E}^T(\theta) \tag{16}$$

the condition (15) is transformed into the following matrix inequality.

$$\mathrm{He}(\tilde{A}X\tilde{E}^T - \tilde{B}\tilde{M}) + \tilde{E}XQX\tilde{E}^T + \tilde{M}^T R\tilde{M} < 0 \tag{17}$$

where $\mathrm{He}(A) \triangleq A + A^T$. Applying Schur Complement to the matrix inequality (17), we have the following LMI.

$$\begin{bmatrix} \mathrm{He}(\tilde{A}X\tilde{E}^T-\tilde{B}\tilde{M}) & \star & \star \\ HX\tilde{E}^T & -I_q & \star \\ \tilde{M} & 0 & -R^{-1} \end{bmatrix} < 0 \quad (18)$$

where $Q=H^TH$ and rank$H=q$. '$\star$' means the transpose of the elements located at diagonal position. LMI (18) must be satisfied over the entire parameter box. To express the LMI finitely, the polytopic form is used. Since Eq. (12) is a descriptor polytopic system, $\tilde{E}(\theta)$, $\tilde{A}(\theta)$ and $\tilde{B}(\theta)$ are written as

$$[\tilde{E}(\theta)\ \ \tilde{A}(\theta)\ \ \tilde{B}(\theta)] = \sum_{i=1}^{2^g}\alpha_i(\theta)[\tilde{E}_i\ \ \tilde{A}_i\ \ \tilde{B}_i] \quad (19)$$
$$[\tilde{E}_i\ \ \tilde{A}_i\ \ \tilde{B}_i] \triangleq [\tilde{E}(\theta^i)\ \ \tilde{A}(\theta^i)\ \ \tilde{B}(\theta^i)]$$

In this paper, the auxiliary variable $\tilde{M}(\theta)$ is also given by a polytopic form:

$$\tilde{M}(\theta) = \sum_{i=1}^{2^g}\alpha_i(\theta)\tilde{M}_i \quad (20)$$

Substituting the above relations into LMI (18), it is sufficient to satisfy the inequality in the vertex set Θ. Then, the following LMIs are derived.

$$\begin{bmatrix} \mathrm{He}(\tilde{A}_iX\tilde{E}_i^T-\tilde{B}_i\tilde{M}_i) & \star & \star \\ HX\tilde{E}_i^T & -I_q & \star \\ \tilde{M}_i & 0 & -R^{-1} \end{bmatrix} < 0 \quad (21)$$

$$\begin{bmatrix} \begin{array}{l}\mathrm{He}(\tilde{A}_iX\tilde{E}_j^T-\tilde{B}_i\tilde{M}_j \\ +\tilde{A}_jX\tilde{E}_i^T-\tilde{B}_j\tilde{M}_i)\end{array} & \star & \star \\ HX(\tilde{E}_i+\tilde{E}_j)^T & -2I_q & \star \\ \tilde{M}_i+\tilde{M}_j & 0 & -2R^{-1} \end{bmatrix} < 0 \quad (22)$$
$$(i=1,\cdots,2^g,\ \ j=i+1,\cdots,2^g)$$

After all, if $X>0$ and $\tilde{M}_i$ $(i=1,\cdots,2^g)$ satisfying LMIs (21) and (22) are found, the gain scheduling state feedback gain for the original LPV system (1) is obtained as

$$F(\theta) = T_u(\theta)\tilde{M}(\theta)\tilde{E}^{-T}(\theta)X^{-1}T_x^{-1}(\theta). \quad (23)$$

5 Descriptor Polytopic Model of Aircraft

This section derives descriptor polytopic models of aircraft using the proposed method.

5.1 LPV system of aircraft

Consider that the linearized equations of motion of aircraft are expressed as the following longitudinal and the lateral equations in the frame of the stability axes [8].

$\star$ Longitudinal equation

$$\left\{\begin{array}{l} \dot{u}-X_uu-X_\alpha\alpha+g\cos\Theta_0\theta=0 \\ -Z_uu+V\dot{\alpha}-Z_\alpha\alpha+(V+Z_q)q \\ \quad +g\sin\Theta_0\theta=Z_{\delta_e}\delta_e \\ -M_uu-M_{\dot{\alpha}}\dot{\alpha}-M_\alpha\alpha-M_qq=M_{\delta_e}\delta_e \\ \dot{\theta}=q \end{array}\right. \quad (24)$$

u is the x-axis velocity, α the angle of attack, q the pitch angular velocity, θ the pitch angle and δ_e the elevator angle.

$\star$ Lateral equation

$$\left\{\begin{array}{l} V\dot{\beta}-Y_\beta\beta-Y_pp+(V-Y_r)r \\ \quad -g\cos\Theta_0\phi=Y_{\delta_r}\delta_r \\ -L_\beta\beta+\dot{p}-L_pp-(I_{xz}/I_{xx})\dot{r}-L_rr= \\ \quad L_{\delta_a}\delta_a+L_{\delta_r}\delta_r \\ -N_\beta\beta-(I_{zx}/I_{zz})\dot{p}-N_pp+\dot{r}-N_rr \\ \quad =N_{\delta_a}\delta_a+N_{\delta_r}\delta_r \\ \dot{\phi}=p+r\tan\Theta_0 \end{array}\right. \quad (25)$$

β is the sideslip angle, p the roll angular velocity, r the yaw angular velocity, ϕ the roll angle, δ_a the aileron angle and δ_r the rudder angle. The notations used in Eqs. (24) and (25) are based on the symbols which have been usually used in flight dynamics [8]. The variables denoted by small letters mean the perturbed values. Θ_0 is the pitch angle in the steady-state. g is the center of gravity. I_{xx}, I_{xz}, etc. are the mass moments of inertia. X_u, M_α, etc. are the dimensional stability derivatives [8]. The definition of the dimensional stability derivatives is given in Appendix C. It is seen that the dimensional stability derivatives are functions with respect to the atmospheric density ρ and the flight velocity V. ρ is a monotonously decreasing function with respect to the altitude H. After all, Eqs. (24) and (25) can be regarded as LPV systems whose varying parameters are H and V. In this paper, the flight region; that is, the parameter box with respect to H and V is given by

$$\begin{aligned} H &\in [\underline{H}, \overline{H}], \quad \underline{H} < \overline{H} \\ V &\in [\underline{V}, \overline{V}], \quad \underline{V} < \overline{V}. \end{aligned} \tag{26}$$

As shown in Appendix C, all dimensional stability derivatives are linear with respect to ρ. On the other hand, X_u, Z_u, M_u, $M_{\dot\alpha}$, Z_q, M_q, Y_p, Y_r, L_p, L_r, N_p and N_r are linear with respect to V, while X_α, Z_α, M_α, Z_{δ_e}, M_{δ_e}, Y_β, L_β, N_β, Y_{δ_a}, Y_{δ_r}, L_{δ_a}, L_{δ_r}, N_{δ_a} and N_{δ_r} are linear with respect to V^2. Therefore, Eqs. (24) and (25) are written as the following descriptor LPV system

$$E(H,V)\dot{x}(t) = A(H,V,V^2)x(t) + B(H,V^2)u(t). \tag{27}$$

The variable vectors and the matrices in Eq. (27) are given as

⋆ Longitudinal equation

$$x(t) = [u \;\; \alpha \;\; q \;\; \theta]^T \quad u(t) = \delta_e \tag{28}$$

$$E(H,V) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & V & 0 & 0 \\ 0 & -M_{\dot\alpha} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{29}$$

$$A(H,V,V^2) = \begin{bmatrix} X_u & X_\alpha & 0 & -g\cos\Theta_0 \\ Z_u & Z_\alpha & V+Z_q & -g\sin\Theta_0 \\ -M_u & M_\alpha & M_q & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{30}$$

$$B(H,V^2) = [0 \;\; Z_{\delta_e} \;\; M_{\delta_e} \;\; 0]^T \tag{31}$$

⋆ Lateral equation

$$x(t) = [\beta \;\; p \;\; r \;\; \phi]^T \quad u(t) = [\delta_a \;\; \delta_r]^T \tag{32}$$

$$E(H,V) = \begin{bmatrix} V & 0 & 0 & 0 \\ 0 & 1 & -I_{xz}/I_{xx} & 0 \\ 0 & -I_{zx}/I_{zz} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{33}$$

$$A(H,V,V^2) = \begin{bmatrix} Y_\beta & Y_p & Y_r - V & g\cos\Theta_0 \\ L_\beta & L_p & L_r & 0 \\ N_\beta & N_p & N_r & 0 \\ 0 & 1 & \tan\Theta_0 & 0 \end{bmatrix} \tag{34}$$

$$B(H,V^2) = \begin{bmatrix} 0 & Y_{\delta_r} \\ L_{\delta_a} & L_{\delta_r} \\ N_{\delta_a} & N_{\delta_r} \\ 0 & 0 \end{bmatrix} \tag{35}$$

5.2 Transformed descriptor polytopic model

The matrices in Eq. (27) are affine with respect to $\rho(H)$ but not V. The proposed variable transformation is then applied to Eq. (27) to be affine with respect to ρ and V. The transformed matrices in Eq. (11) are given by

⋆ Longitudinal equation

$$T_x = \mathrm{diag}\{1 \;\; 1/V \;\; 1 \;\; 1\}, \quad T_u = 1/V \tag{36}$$

⋆ Lateral equation

$$T_x = \mathrm{diag}\{1/V \;\; 1 \;\; 1 \;\; 1\}, \quad T_u = \mathrm{diag}\{1/V \;\; 1/V\} \tag{37}$$

Then, the transformed descriptor LPV system is written as

$$\tilde{E}(H)\dot{\tilde{x}} = \tilde{A}(H,V)\tilde{x} + \tilde{B}(H,V)\tilde{u} \tag{38}$$

where

$$\tilde{E} \triangleq ET_x, \quad \tilde{A} \triangleq AT_x, \quad \tilde{B} \triangleq BT_u.$$

The matrices in Eq. (38) are affine with respect to $\rho(H)$ and V. That is, assumptions (a1-i) and (a1-ii) of Theorem 1 are satisfied. Then, the linearized equations of aircraft can be transformed into descriptor polytopic models which consist of four LTI models at the vertices of the parameter box (26).

According to Theorem 1, the matrices in Eq. (38) are written as follows.

$$[\tilde{E}\ \ \tilde{A}\ \ \tilde{B}](H,V) = \sum_{i=1}^{4} \alpha_i(H,V)[\tilde{E}_i\ \ \tilde{A}_i\ \ \tilde{B}_i] \tag{39}$$

where

$$[\tilde{E}_1\ \ \tilde{A}_1\ \ \tilde{B}_1] \triangleq [\tilde{E}\ \ \tilde{A}\ \ \tilde{B}](\underline{H},\underline{V})$$
$$[\tilde{E}_2\ \ \tilde{A}_2\ \ \tilde{B}_2] \triangleq [\tilde{E}\ \ \tilde{A}\ \ \tilde{B}](\underline{H},\overline{V})$$
$$[\tilde{E}_3\ \ \tilde{A}_3\ \ \tilde{B}_3] \triangleq [\tilde{E}\ \ \tilde{A}\ \ \tilde{B}](\overline{H},\underline{V})$$
$$[\tilde{E}_4\ \ \tilde{A}_4\ \ \tilde{B}_4] \triangleq [\tilde{E}\ \ \tilde{A}\ \ \tilde{B}](\overline{H},\overline{V})$$
$$\alpha_1(H,V) \triangleq \underline{\phi}(H)\underline{\phi}(V), \quad \alpha_2(H,V) \triangleq \underline{\phi}(H)\overline{\phi}(V)$$
$$\alpha_3(H,V) \triangleq \overline{\phi}(H)\underline{\phi}(V), \quad \alpha_4(H,V) \triangleq \overline{\phi}(H)\overline{\phi}(V)$$
$$\underline{\phi}(H) \triangleq \frac{\rho(\overline{H}) - \rho(H)}{\rho(\overline{H}) - \rho(\underline{H})}, \quad \overline{\phi}(H) \triangleq \frac{\rho(H) - \rho(\underline{H})}{\rho(\overline{H}) - \rho(\underline{H})}$$
$$\underline{\phi}(V) \triangleq \frac{\overline{V} - V}{\overline{V} - \underline{V}}, \quad \overline{\phi}(V) \triangleq \frac{V - \underline{V}}{\overline{V} - \underline{V}}$$

6 Numerical Example

This section presents a numerical example of the longitudinal equation [9] to illustrate the construction of the descriptor polytopic model and the design of the gain scheduling state feedback law. The flight region of H and V was given as

$$H \in [1000, 7000]\ \text{m}, \quad V \in [50, 150]\ \text{m/s} \tag{40}$$

The other numerical data were referred from Ref. [9]. For the flight region (40), the following three types of the state feedback laws were designed and were compared with each other.

- Fix-lqr: LQR state feedback law (fixed state feedback). The design model was an LTI model whose flight condition was given by $(H,V) = (7000, 50)$.
- GS-ss: gain scheduling state feedback law. The design model was given by a state-space polytopic model[1] which was essentially the same as fuzzy model described in Ref. [7].
- GS-dsc: gain scheduling state feedback law. The design model was given by a descriptor polytopic model which was the proposed model in this paper.

[1]When the LPV system is given as a state-space form; that is, $E(\theta) = I_n$ in Eq. (1), the polytopic model is also given as the state-space form; that is, $E_i = I_n$ in Eq. (6).

The weighting matrices of the quadratic index in LQR and the quadratic stability condition in Eq. (15) were given as

$$Q = 0.1 I_4, \quad R = 1 \tag{41}$$

First, the model error between the design model and the original LPV system was evaluated in the flight region by the ν-gap metric. Letting $P_d(s;H,V)$ and $P_{lpv}(s;H,V)$ be transfer functions of the design model and the original LPV system where the flight condition was (H,V), the ν-gap metric is defined as [10]

$$\delta_\nu(P_d, P_{lpv}) \triangleq \sup_{\omega} \kappa(P_d(j\omega;H,V), P_{lpv}(j\omega;H,V)) \tag{42}$$

where

$$\kappa(X,Y) \triangleq \overline{\sigma}\left[(I+YY^*)^{1/2}(Y-X)(I+XX^*)^{1/2}\right]$$

where $\overline{\sigma}$ means the maximum singular value. The range of $\delta_\nu(P_d, P_{lpv})$ is $\delta_\nu \in [0,1]$. A large δ_ν means that the model error is large. Figure 1 shows the plot of the $\delta_\nu(P_d, P_{lpv})$ where the design model was $P_d(s;7000,50)$ which was used for designing Fix-lqr. $\delta_\nu(P_d, P_{lpv})$ was monotonously increased when the flight condition was shifted from the design point $(H,V) = (7000, 50)$. The plots of ν-gap metric whose design models were the state-space polytopic model and the descriptor polytopic model are shown in Figs. 2 and 3, respectively. In Fig. 2, $\delta_\nu(P_d, P_{lpv})$ was zero at the four vertices, but not zero in the intermediate flight region. On the other hand, in Fig. 3, $\delta_\nu(P_d, P_{lpv})$ was zero over the entire flight region. That is, the obtained descriptor polytopic model was completely equivalent to the original LPV system.

The robustness of the designed state feedback laws was evaluated by the stability margin. The stability margin of the state feedback gain F for the original LPV system $P_{lpv}(s;H,U)$ is defined as [10]

$$b_{P_{lpv},F} \triangleq \inf_{\omega} \rho(P_{lpv}(j\omega), F) \tag{43}$$

where

$$\rho(X,Z) \triangleq 1/\overline{\sigma}\left(\begin{bmatrix} X \\ I \end{bmatrix}(I-ZX)^{-1}[-Z\ \ I]\right)$$

The range of $b_{P_{lpv},F}$ is $b_{P_{lpv},F} \in [0,1]$. According to Ref. [10], a sufficient condition for the robust stability is to satisfy the relation $\delta_\nu(P_d, P_{lpv}) < b_{P_{lpv},F}$ over the entire flight region. Figures 4, 5 and 6 show the plots of $b_{P_{lpv},F}$ in which the state feedback laws were Fix-lqr, GS-ss and GS-dsc, respectively. It is seen from Figs. 1, 2, 4 and 5 that the robust stability was guaranteed near the design point $(H,V) = (7000, 50)$ and the four vertices by Fix-lqr and GS-ss, respectively. On the other hand, in Figs. 3 and 6, it was guaranteed over the entire flight region by GS-dsc.

Figures 7, 8 and 9, furthermore, show the initial time responses of the closed-loop system at $(H,V) = (4000, 100)$ by using the three state feedback laws. The solid-lines indicate the responses whose plant was the original LPV system, while the dashed-line indicate the responses whose plant was the design model. In Figs. 7 and 8, the influence of the model error was appeared in the responses. On the other hand, when the descriptor polytopic model was used, there was no difference between both responses as shown in Fig. 9. These figures present the responses which were deviated from the trimmed states. The designed state feedback laws should be evaluated by nonlinear time-varying simulation. It is expected from the above results that GS-dsc may show acceptable control performance also in the nonlinear time-varying simulation; that is, be able to stabilize the responses of the state and track guidance commands.

7 Conclusions

This paper proposed an exact polytopic model for the LPV system of aircraft. The LPV system of the linearized equation of aircraft was represented by a descriptor form which reserves physical features of the equation. A descriptor polytopic model was derived through a variable transformation to satisfy conditions for the polytope. Using the descriptor polytopic model, a gain scheduling state feedback law was designed by means of an LMI formulation. It was shown in a numerical example of a longitudinal flight control that the proposed descriptor polytopic model had no model error and exactly represented the original LPV system without any needless operating region. It should be noted that the proposed method is effective in the case where assumptions (a1-i) and (a1-ii) of Theorem 1 hold. Nevertheless, the descriptor form and the variable transformation used in this paper may be useful for general LPV systems to reduce the model error and to be less conservative.

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Appendix

A. Proof of Theorem 1

If a matrix $A(\theta) \in \Re^{n\times n}$ is affine with respect to functions $f_j(\theta_j)$ $(j = 1, \cdots, g)$ whose variables are θ_j, it can be written as

$$A(\theta) = X + \sum_{j=1}^{g} f_j(\theta_j) Y_j \tag{A.1}$$

where $X, Y_j \in \Re^{n\times n}$ $(j = 1, \cdots, g)$ are constant matrices. Using $\underline{\phi}_j(\theta_j)$ and $\overline{\phi}_j(\theta_j)$ defined in Eq. (10), $f_j(\theta_j)$ is written as

$$f_j(\theta_j) = \underline{\phi}_j(\theta_j) f_j(\underline{\theta}_j) + \overline{\phi}_j(\theta_j) f_j(\overline{\theta}_j) \tag{A.2}$$

The following relation holds in the parameter box Eq. (3).

$$\underline{\phi}_j(\theta_j),\ \overline{\phi}_j(\theta_j) \geq 0, \quad \underline{\phi}_j(\theta_j) + \overline{\phi}_j(\theta_j) = 1, \quad \theta_j \in [\underline{\theta}_j, \overline{\theta}_j] \tag{A.3}$$

First, consider the single varying parameter case; that is, $g = 1$, $A(\theta)$ is

$$\begin{aligned} A(\theta) &= X + (\underline{\phi}_1 f_1(\underline{\theta}_1) + \overline{\phi}_1 f_1(\overline{\theta}_1))Y_1 \\ &= \underline{\phi}_1 (X + f_1(\underline{\theta}_1)Y_1) + \overline{\phi}_1 (X + f_1(\overline{\theta}_1)Y_1) \end{aligned} \tag{A.4}$$

Defining $\alpha_i(\theta)$ $(i = 1, 2)$ as

$$\begin{aligned} \alpha_1(\theta) &\triangleq \underline{\phi}_1, \quad A_1 \triangleq X + f_1(\underline{\theta}_1)Y_1 \\ \alpha_2(\theta) &\triangleq \overline{\phi}_1, \quad A_2 \triangleq X + f_1(\overline{\theta}_1)Y_1 \end{aligned}$$

$A(\theta)$ is then represented by Eq. (6). The matrices A_1 and A_2 are respectively $A(\theta)$ at $\theta_1 = \underline{\theta}_1$ and $\theta_1 = \overline{\theta}_1$. Similarly, when $g = 2$, $A(\theta)$ is written as

$$\begin{aligned} A(\theta) = X &+ \sum_{j=1}^{2} (\underline{\phi}_j f_j(\underline{\theta}_j) + \overline{\phi}_j f_j(\overline{\theta}_j))Y_j \\ &= \underline{\phi}_1 \underline{\phi}_2 (X + f_1(\underline{\theta}_1)Y_1 + f_2(\underline{\theta}_2)Y_2) \\ &+ \underline{\phi}_1 \overline{\phi}_2 (X + f_1(\underline{\theta}_1)Y_1 + f_2(\overline{\theta}_2)Y_2) \\ &+ \overline{\phi}_1 \underline{\phi}_2 (X + f_1(\overline{\theta}_1)Y_1 + f_2(\underline{\theta}_2)Y_2) \\ &+ \overline{\phi}_1 \overline{\phi}_2 (X + f_1(\overline{\theta}_1)Y_1 + f_2(\overline{\theta}_2)Y_2) \end{aligned} \tag{A.5}$$

Defining $\alpha_i(\theta)$ $(i = 1, \cdots, 4)$ as

$$\begin{aligned} \alpha_1(\theta) &\triangleq \underline{\phi}_1 \underline{\phi}_2, \quad A_1 \triangleq X + f_1(\underline{\theta}_1)Y_1 + f_2(\underline{\theta}_2)Y_2 \\ \alpha_2(\theta) &\triangleq \underline{\phi}_1 \overline{\phi}_2, \quad A_2 \triangleq X + f_1(\underline{\theta}_1)Y_1 + f_2(\overline{\theta}_2)Y_2 \\ \alpha_3(\theta) &\triangleq \overline{\phi}_1 \underline{\phi}_2, \quad A_3 \triangleq X + f_1(\overline{\theta}_1)Y_1 + f_2(\underline{\theta}_2)Y_2 \\ \alpha_4(\theta) &\triangleq \overline{\phi}_1 \overline{\phi}_2, \quad A_4 \triangleq X + f_1(\overline{\theta}_1)Y_1 + f_2(\overline{\theta}_2)Y_2 \end{aligned}$$

$A(\theta)$ is also represented by Eq. (6). This expression also holds for $g \geq 3$.

B. Proof of Theorem 2

The characteristic equation of the closed-loop LPV system combining Eq. (5) with Eq. (1) is given by $|sE - A + BF| = 0$. While, that of the transformed closed-loop LPV system combining Eq. (13) with Eq. (12) is given by

$$|s\tilde{E} - \tilde{A} + \tilde{B}\tilde{F}| = |(sE - A + \tilde{B}\tilde{F}T_x^{-1})T_x| = 0 \tag{B.1}$$

To meet both equations, the following relation must hold.

$$BF = \tilde{B}\tilde{F}T_x^{-1} = BT_u\tilde{F}T_x^{-1} \tag{B.2}$$

Equation (14) is thus obtained.

C. Dimensional Stability Derivative

The definition of the dimensional stability derivatives in the frame of the stability axes is given as follows.

⋆ Longitudinal equation

$$X_u = \frac{\rho VS}{2m}(C_{xu} + 2C_L \tan\Theta_0), \quad Z_u = \frac{\rho VS}{2m}(C_{zu} - 2C_L)$$

$$M_u = \frac{\rho VSc}{2I_{yy}}C_{mu}, \quad X_\alpha = \frac{\rho V^2 S}{2m}C_{x\alpha}$$

$$Z_\alpha = \frac{\rho V^2 S}{2m}C_{z\alpha}, \quad M_\alpha = \frac{\rho V^2 Sc}{2I_{yy}}C_{m\alpha}, \quad M_{\dot{\alpha}} = \frac{\rho VSc^2}{4I_{yy}}C_{m\dot{\alpha}}$$

$$Z_q = \frac{\rho VSc}{4m}C_{zq}, \quad M_q = \frac{\rho VSc^2}{4I_{yy}}C_{mq}$$

$$Z_{\delta_e} = \frac{\rho V^2 S}{2m}C_{z\delta_e}, \quad M_{\delta_e} = \frac{\rho V^2 Sc}{2m}C_{m\delta_e}$$

⋆ Lateral equation

$$Y_\beta = \frac{\rho V^2 S}{2m}C_{y\beta}, \quad L_\beta = \frac{\rho V^2 Sb}{2I_{xx}}C_{l\beta}, \quad N_\beta = \frac{\rho V^2 Sb}{2I_{zz}}C_{n\beta}$$

$$Y_p = \frac{\rho VSb}{4m}C_{yp}, \quad L_p = \frac{\rho VSb^2}{4I_{xx}}C_{lp}, \quad N_p = \frac{\rho VSb^2}{4I_{zz}}C_{np}$$

$$Y_r = \frac{\rho VSb}{4m}C_{yr}, \quad L_r = \frac{\rho VSb^2}{4I_{xx}}C_{lr}, \quad N_r = \frac{\rho VSb^2}{4I_{zz}}C_{nr}$$

$$Y_{\delta_a} = \frac{\rho V^2 S}{2m}C_{\delta_a}, \quad L_{\delta_a} = \frac{\rho V^2 Sb}{2I_{xx}}C_{\delta_a}, \quad N_{\delta_a} = \frac{\rho V^2 Sb}{2I_{zz}}C_{\delta_a}$$

$$Y_{\delta_r} = \frac{\rho V^2 S}{2m}C_{\delta_r}, \quad L_{\delta_r} = \frac{\rho V^2 Sb}{2I_{xx}}C_{\delta_r}, \quad N_{\delta_r} = \frac{\rho V^2 Sb}{2I_{zz}}C_{\delta_r}$$

Where m is the mass of aircraft, S the main wing area, c the main wing chord, and b the main wing span. C_L is the lift coefficient. C_{xu}, $C_{m\alpha}$, etc. are the non-dimensional stability derivatives [8] and are obtained from the structural parameters of aircraft.

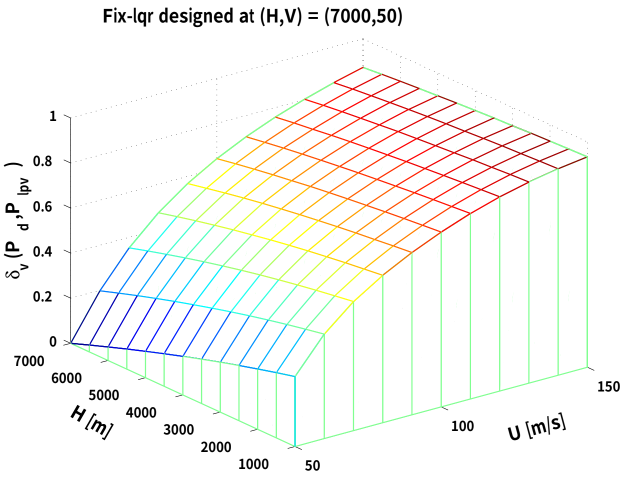

Fig. 1 ν-gap metric between LPV system and LTI model obtained at $(H,V) = (7000,50)$.

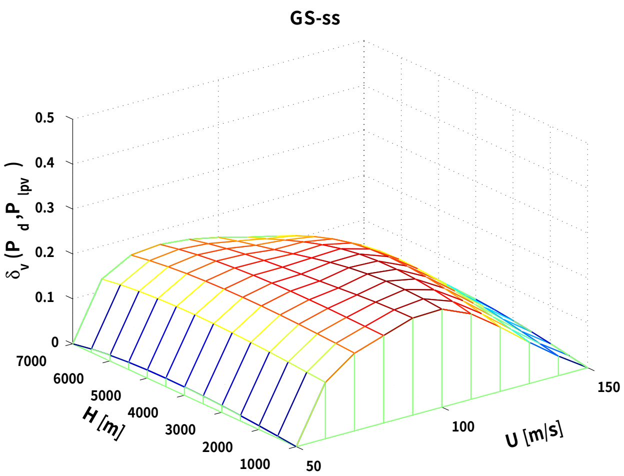

Fig. 2 ν-gap metric between LPV system and state-space polytopic model.

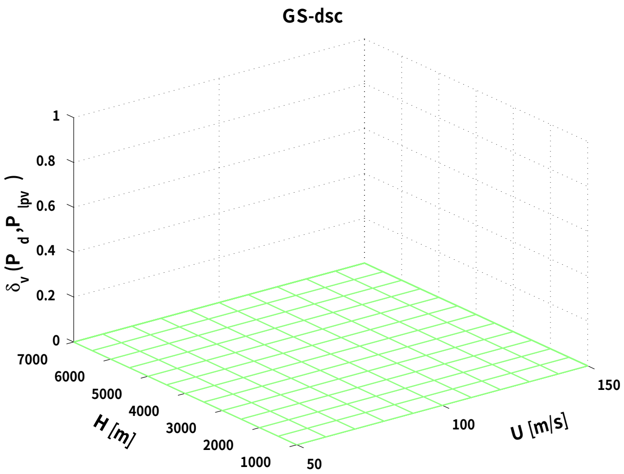

Fig. 3 ν-gap metric between LPV system and descriptor polytopic model.

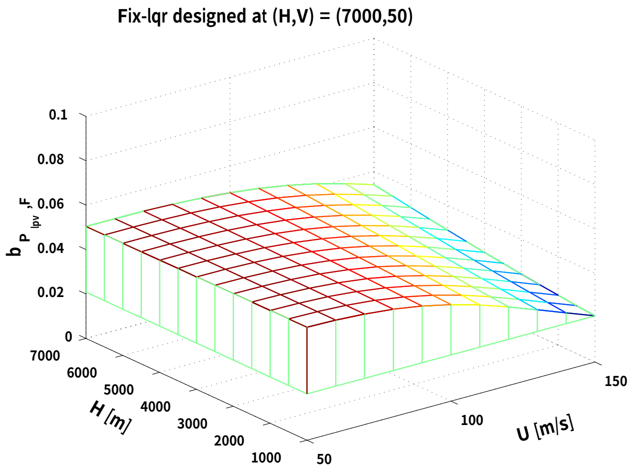

Fig. 4 Stability margin of Fix-lqr designed at $(H,V) = (7000,50)$ for LPV system.

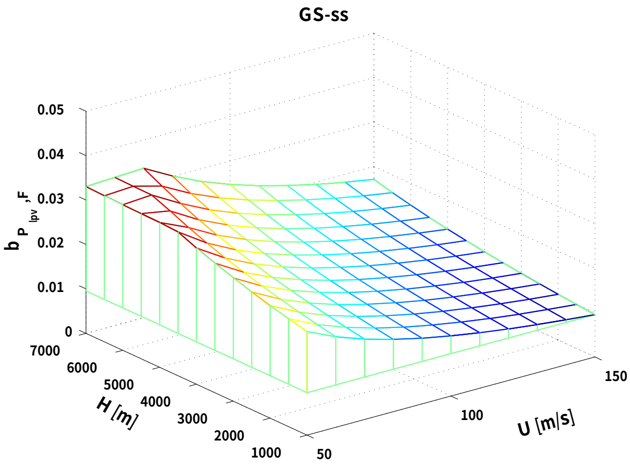

Fig. 5 Stability margin of GS-ss for LPV system.

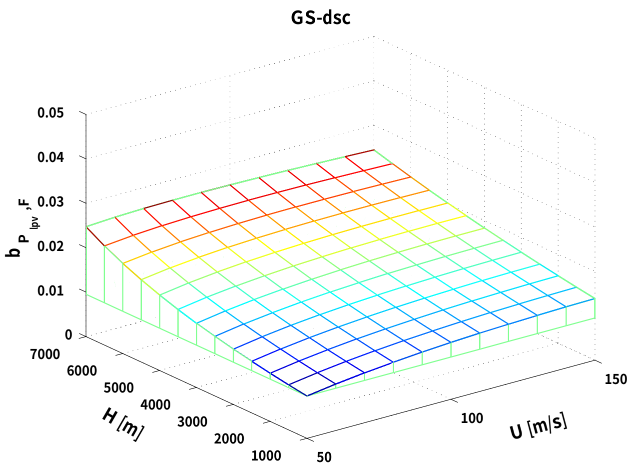

Fig. 6 Stability margin of GS-dsc for LPV system.

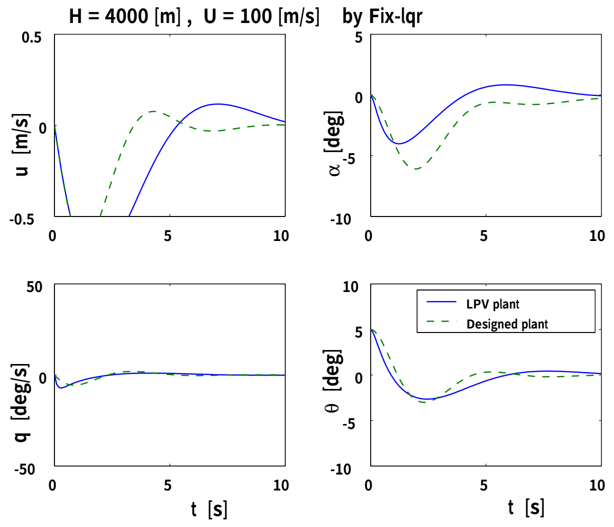

Fig. 7 Initial time response using Fix-lqr, $(H,V) = (4000,100)$.

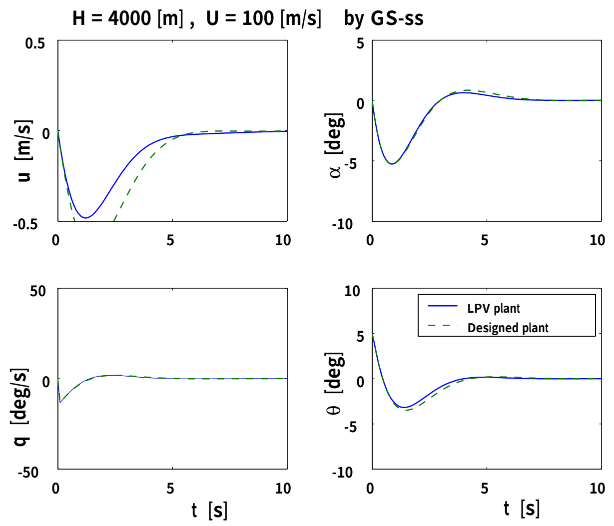

Fig. 8 Initial time response using GS-ss, $(H,V) = (4000,100)$.

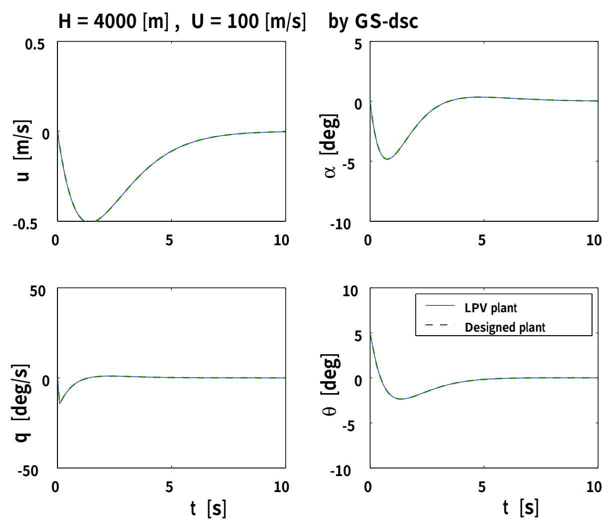

Fig. 9 Initial time response using GS-dsc, $(H,V) = (4000,100)$.