NEW GENERAL APPROACH TO THE AIRPLANE ROTATION ANALYSIS

Dr Milovan Zivanovic*, Dr Milan Bajovic*, Mr Predrag Stojakovic*, Prof Dr Bosko Rasuo**
***VTI – Aeronautical Institute, ** Faculty of Mechanical Engineering, University of Belgrade**

Keywords: *Airplane rotation, flight dynamics, stability, high angles of attack, non-linear model*

Abstract

The new approach to the analysis of the dynamic properties of the rotating airplane is presented in this paper. The analysis method is based on the rate of rotational energy and full non-linear model of airplane dynamics. The method is applicable within whole range of the values of angle of attack. To provide the relations for the stable rotation of the airplane it is needed to know only the functional relations of aerodynamic moments, inertial properties of airplane and mathematical apparatus of algebra. As the corollary, there has been proofed that within the area of validity of simplified model there exists the full congruency of the results of this method and classical approach.

1 Introduction

In the design of the dynamic properties of the fixed wing aircraft – airplane one of the crucial problems is adequate assessing of it behavior when performing all aspects of rotational motion. The tools available to them in the period when the airplane has been constructed have governed approach of the designers to this problem. Historically, initial approaches to this problem are belonging to the “bottom-up” category, the first examples being separate analysis of pitching and rolling motion of the airplane.

Basic characteristic of the “bottom-up” approach is the application of the approximate models with the limited scope of validity. One of the most widely used has been the application of the models based on the assumption that the aerodynamic loads, forces and moments, are linear functions. For the airplanes with coplanar geometric and inertial plane of symmetry it yields models of so-called ‘longitudinal’ and ‘lateral-directional’ motion. For the brevity, approach based upon separated modes of longitudinal and lateral directional motion shall be noted as the classical one. Alas, these models are unable to predict airplane behavior in the stall, deep-stall and spin flight regimes.

Therefore, integral observation and analysis of airplane rotation has not been usual. General approach has been governed by obtaining desired design aim, required airplane dynamics in the predefined flight regime. The whole scope of airplane flight regimes has been parsed into separate parts, and in each airplane dynamics has been described with separate approximate and usually linear mathematical model. Particular model has been constrained by validity scope of stream variables, angles and characteristic parameters, or kinematical values, i.e. angular velocities. Adequate answers could be obtained within this and only this scope where approximation assumptions in model generation has been valid. Number of attempts to find answers spreading through more than one flight regime have been small, and in itself limited to partial aspects of airplane dynamics, for example aerodynamic forces or inertial properties of the airplane. The most common designer error, often with catastrophic results, stemmed from the excursions out of the validity scope of particular model of airplane dynamics, related to the area of either initial assumptions or answer limitations.

As is often the case in the history of engineering practice there has been the simultaneous appearance of the requirements for the increased scope of the controlled airplane maneuvers, on the one side, and the design tool in the shape of adequate mathematical model of it dynamics and the means to apply this model in corresponding computational environment on the other. Airplane maneuvering requirements demanded that the evaluation of the dynamics of the airplane with the pilot-in-the-loop is performed in the design phase. This generated the approach in the design practice, which can be placed in the “top-down” category, resulting with the set of related practical mathematical models of airplane dynamics, generally with variations in the presentation of the aerodynamic forces and moments throughout the operational scope of flight regimes. They comprise full range of the stream variables, angles and characteristic parameters, on the one side and angular velocities on the other. Basic advantage of their application is that the quality of the answers they provide depends only of the quality of input data.

Furthermore, thus generated model of the airplane dynamics serves as the kernel for the model of the closed loop aircraft dynamics, with power plant and flight control system (FCS) models and true or modeled pilot as the main supplementary subsystems. The research in this area has been performed in three directions. The first has been aimed toward experimental determination of aerodynamic shapes, which are enabling flight control in as wide range of airplane state space as possible. In the second, the results have been searched by theory of bifurcations and the methodology of theory of catastrophes. The third has been the flight simulation in the regime of deep stall and spin, with input parameters determined by the first. The last method is the most widely spread, as the hardware-in-the-loop simulation for the whole range of flight regimes is mandatory during the airplane-FCS integration. Property of all three approaches is that the conclusion about airplane dynamics characteristics is made on the basis of the analysis of the elements of the set of the results.

The main disadvantage of previous methods is that the conclusions in the design process are made ‘a-posteriori’, when the shape of the airplane has been already determined, or, they validate already obtained aerodynamic parameters. What the designer really needs is to know what is the aerodynamic shape or what are the values of aerodynamic parameters that are providing the required quality of airplane rotational motion.

The designer of the desired airplane motion needs the method that can provide integral analysis of airplane rotation within the whole range of stream angles and angular velocities, where separated models are presenting only it special cases. Therefore, it must belong to the ‘top-down’ category of design procedures. The analysis method presented in this paper fulfills these requirements, with the aim to provide answers about the dynamics of airplane rotation within all stages of the design procedure, from the conceptual determination of airplane shape to the final stages of the hardware-in-the-loop simulation and in-flight testing. As is often the case, to obtain this, it has been necessary to make one backward step to the basic principles of mechanics.

The approach to the analysis of rigid body motion in classical mechanics is based on the principle of minimal action expressed through integral or differential mathematical interpretation and presented as the equation of motion. These equations of motion are describing the laws of continuity of momentum and energy. The analysis performed in this paper is based on the observation of system motion through the variations of the function describing integral interpretation of the principle of minimal action, avoiding the requirement to obtain the equations of motion. In the observation of the relation between Hamilton and Liapunov function has been proven that positively defined Hamilton function fulfills all the requirements of Liapunov function, and system solution all of the Liapunov stability conditions. This idea has been used as the basis in this paper so that the total airplane motion is considered as the complex one, consisting of translational motion of airplane center of inertia and relative airplane rotational motion about this center of inertia.

The total energy of rotation is only the kinetical one and it fulfills the requirements of positively definite function, whose independent variables are airplane angular velocities. That enables that the conclusions about properties of the solutions of vector differential equation describing airplane rotational motion in the vicinity of it minimum are made on the basis of the

algebraic properties of the derivative of rotational energy along the vector of angular velocity.
The problem to be solved can be set as follows. The rotation of the airplane with the non-rotating initial state (zero angular velocity) is observed. Assuming that the scalar value of rotational energy is known, it is necessary to define the methodology for determination and evaluation of those aircraft parameters that are providing it stable behavior within whole range of stream angles and angular velocities. It is necessary to define conditions that the increase of rotation or it stoppage are enabled by control surfaces.

2 Theoretical grounds

In developing theoretical grounds for rotation analysis method presented in this paper, it is assumed that the airplane is the rigid body, i.e. in the period of observation it inertial properties are invariant with time. The complete nonlinear mathematical model of the airplane dynamics is, then, obtained from the law of continuity of momentum and moment of momentum in the form

$$\begin{aligned}\mathbf{F} = \mathbf{F}_a + \mathbf{F}_g + \mathbf{F}_p = \frac{\partial(m\mathbf{V})}{\partial t} + \boldsymbol{\omega}\times m\mathbf{V}\\ \mathfrak{M} = \mathfrak{M}_a + \mathfrak{M}_p = \frac{\partial(\mathfrak{J}\boldsymbol{\omega})}{\partial t} + \boldsymbol{\omega}\times\mathfrak{J}\boldsymbol{\omega}\end{aligned} \tag{1}$$

where $\mathbf{F}_a, \mathbf{F}_g, \mathbf{F}_p$ are, respectively, the vectors of resulting aerodynamic, gravitational and propulzive force and $\mathbf{F}$ of resulting force, $\mathfrak{M}$ is the resulting moment of force, $\mathfrak{M}_p$ and $\mathfrak{M}_a = col(M_a, L_a, N_a)$, are the vectors of resulting moments of propulsive group and aerodynamic forces, m is the mass and $\mathfrak{J}$ is the general form of tensor of inertia of the airplane, $\mathbf{V} = col(u,v,w)$ and $\boldsymbol{\omega} = col(p,q,r)$ are the vectors of airplane translational and angular velocity. Variables L_a, M_a, N_a; u,v,w and p,q,r are, respectively, projections of vectors $\mathfrak{M}_a$, $\mathbf{V}$ and $\boldsymbol{\omega}$ on the X,Y,Z axes of the airplane body coordinate system. By solving(1) upon highest order derivatives, the following set of equation is obtained:

$$\begin{aligned}\dot{\mathbf{V}} = -\tfrac{1}{m}\left[(\boldsymbol{\omega}\times m\mathbf{V}) + \mathbf{F}_a + \mathbf{F}_g + \mathbf{F}_p\right]\\ \dot{\boldsymbol{\omega}} = -\mathfrak{J}^{-1}\left[(\boldsymbol{\omega}\times\mathfrak{J}\boldsymbol{\omega}) + \mathfrak{M}_a + \mathfrak{M}_p\right].\end{aligned} \tag{2}$$

Dot over variable denotes its rate, or first order time derivative.
Airplane on some point of its trajectory possesses total energy consisting of kinetic energy of translation $E_V = 1/2\, m\mathbf{V}^T\mathbf{V}$ and rotation, $E_R = 1/2\,\boldsymbol{\omega}^T\mathfrak{J}\boldsymbol{\omega}$, potential energy $E_H = mgH$ and gyroscopic energy of propulsive group $E_P = 1/2\sum_1^n (I_p\boldsymbol{\omega}_p)_i$, where I_p and $\boldsymbol{\omega}_p$ are moment of inertia and angular velocity and n is the total number of rotating elements of propulsive group that are generating significant gyroscopic moments.
Along its trajectory airplane traverses path s from initial point “0” at the moment t_0 to some point “1” at the moment t_1. During that motion propulsive group performed work which increased total energy of the system by the amount $W_T^{01} = \int_0^1 \mathbf{F}_p ds$, while in the same time aerodynamic forces caused energy dissipation equal to $W_a^{01} = \int_0^1 \mathbf{F}_a ds$. If with the upper indices “0” and “1” are denoted energies at the corresponding points on the trajectory, while “01” denotes work along path s, then the law of the conservation of energy is of the form

$$\begin{aligned}(E_V^1 - E_V^0) + (E_R^1 - E_R^0) + (E_H^1 - E_H^0) +\\ +(E_p^1 - E_p^0) + W_T^{01} - W_a^{01} = 0.\end{aligned} \tag{3}$$

If the initial condition at the moment t_0 are set for non-rotating airplane, then $\boldsymbol{\omega}^0 = 0$, $E_R^0 = 0$. That corresponds to any flight with straight flight-path, and the energy of airplane rotation can be than simply obtained by rearranging (3) to form

$$E_R = \Delta E_V + \Delta E_H + \Delta E_p + W_T^{01} - W_a^{01}. \tag{4}$$

If the maneuver condition are further constrained so that there are no changes in angular velocities of the elements of propulsive group that are generating gyroscopic moments and initially airplane is in steady, straight, horizontal, wing-level flight, then (4) becomes

$$E_R = \frac{1}{2} m \Delta \mathbf{V}^T \Delta \mathbf{V} + m\left(\mathbf{V}^0 + \Delta \mathbf{V}\right)^T \Delta \mathbf{V} + + mg\Delta H + W_T^{01} - W_a^{01}. \qquad (5)$$

Equations (4) and (5) present the transfer from the other modes into the kinetic energy of rotation. The method in presenting equations (3) to (5) is chosen only to clarify particular contributions to the energy of rotation E_R.

The airplane with the plane of inertial symmetry $0xz$ possesses kinetic energy of rotation $E_R = = 1/2\, \boldsymbol{\omega}^T \mathfrak{J} \boldsymbol{\omega}$, or briefly rotational energy, of the inherent form

$$2E_R = \boldsymbol{\omega}^T \mathfrak{J} \boldsymbol{\omega} = I_u \left\|\boldsymbol{\omega}\right\|^2 = J_1 p_J^2 + J_2 q_J^2 + J_3 r_J^2 = = I_x p^2 + I_y q^2 + I_z r^2 - 2I_{xz} pr = R^2, \qquad (6)$$

where I_u is axial moment of inertia around instantaneous axis of rotation, I_x, I_y, I_z are axial and I_{xz} is centrifugal moments of inertia around corresponding axes of airplane body coordinate system. Scalar $R > 0$ is defined as the *intensity of rotation*, square of its value being two times the amount of rotational energy.

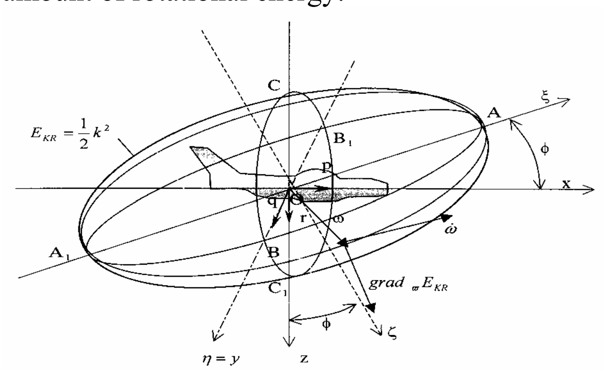

Figure 1. Ellipsoid of rotation.

Expression (6) presents the equation of ellipsoid of rotational energy, or briefly ellipsoid of rotation. Three main, orthogonal axis of ellipsoid of inertia are collinear with three main axis of inertia of the airplane and are intersecting at the airplane center of mass. Angular velocity presented through components in this system is $\boldsymbol{\omega} = = col\left(p_J, q_J, r_J\right)$ and tensor of inertia has the form $\mathfrak{J} = diag\left(J_1, J_2, J_3\right)$, J_1, J_2, J_3 being the main moments of inertia. For the inertially symmetric airplane, ellipsoid of rotation is turned about y -axis of airplane body coordinate system by the angle $\Phi = 1/2 \operatorname{arctg}[2I_{xz}/(I_x - I_z)]$. Values of the main half-axes of the ellipsoid of rotation are $\lambda_1 = R/\sqrt{J_1}$, $\lambda_2 = R/\sqrt{J_2}$ and $\lambda_3 = R/\sqrt{J_3}$.

During maneuvering flight ellipsoid of inertia is varying with time, i.e. $R = R(t)$. In the absence of rotation it becomes a point. With increase or decrease of intensity of rotation ellipsoid of rotation inflates or deflates. The origin of the vector of angular velocity $\boldsymbol{\omega}$ is at the intersection of the main axis of inertia, while its apex is touching the surface of the ellipsoid of rotation at the point T. Normal to the surface of the ellipsoid of rotation at T is $\vec{n} = -\text{grad}_\omega E_R$, i.e. the gradient of energy of rotation along the vector of angular velocity $\boldsymbol{\omega}$.

The vector of the rate (derivative relative to time) of angular velocity, $\dot{\boldsymbol{\omega}}$, is with origin at T. Kinetic energy of rotation E_R is scalar function of $\boldsymbol{\omega}$, therefore the rate of change of E_R, it derivative relative to time, is

$$\frac{d}{dt}\left(E_R\right) = \dot{E}_R = \left[\text{grad}_{\boldsymbol{\omega}} E_R\right]^T \cdot \dot{\boldsymbol{\omega}}, \qquad (7)$$

i.e. it rate along the angular velocity vector $\boldsymbol{\omega}$. To simplify geometric presentation, let denote relative angle between $\text{grad}_{\boldsymbol{\omega}} E_R$ and the rate of angular velocity $\dot{\boldsymbol{\omega}}$ with τ. When the intensity of rotation R is constant, the vector of the rate of angular velocity $\dot{\boldsymbol{\omega}}$ is tangent to the ellipsoid of rotation. Then the angle τ is right, making the inner product $\left[\text{grad}_{\boldsymbol{\omega}} E_R\right]^T \cdot \dot{\boldsymbol{\omega}} = 0$. When the intensity of rotation is increasing angle τ is acute with inner product $\left[\text{grad}_{\boldsymbol{\omega}} E_R\right]^T \cdot \dot{\boldsymbol{\omega}} > 0$. Opposite, when the intensity of rotation is decreasing, angle τ is obtuse with inner product $\left[\operatorname{grad}_{\boldsymbol{\omega}} E_R\right]^T \cdot \dot{\boldsymbol{\omega}} < 0$. Therefore, if it is required to prevent the increase of rotation, the condition $\left[\operatorname{grad}_{\boldsymbol{\omega}} E_R\right]^T \cdot \dot{\boldsymbol{\omega}} = 0$ must be satisfied, and if it is to be stopped, there must be fulfilled $\left[\operatorname{grad}_{\boldsymbol{\omega}} E_R\right]^T \cdot \dot{\boldsymbol{\omega}} < 0$. Let notify that the airplane control intentionally generates the condition $\left[\operatorname{grad}_{\boldsymbol{\omega}} E_R\right]^T \cdot \dot{\boldsymbol{\omega}} > 0$.

The next step is correlation of this condition with aerodynamic moments. Initially, it is established that

$$\operatorname{grad}_{\boldsymbol{\omega}} E_R = \operatorname{grad}_{\boldsymbol{\omega}} \left(1/2\, \boldsymbol{\omega}^T \mathfrak{J} \boldsymbol{\omega}\right) = \mathfrak{J}\boldsymbol{\omega}. \tag{8}$$

The vector of the rate of angular velocity $\dot{\boldsymbol{\omega}}$ is defined from second equation in (2). The moment of propulsive group has two parts. The first, $\mathfrak{M}_p^P$, is generated by propulsive force of engine(s), while the second, $\mathfrak{M}_p^G$, is generated by gyroscopic moments, so

$$\begin{aligned} \mathfrak{M}_p &= \mathfrak{M}_p^P + \mathfrak{M}_p^G = \\ &= \sum\nolimits_i \left(\mathbf{r}_i \times \mathbf{T}_i\right) + \sum\nolimits_j \left(I_p(\boldsymbol{\omega}_p \times \boldsymbol{\omega})\right)_j \end{aligned} \quad ; \quad \begin{aligned} i &= 1,2,...,n, \\ j &= 1,2,...,m \end{aligned} \tag{9}$$

where n is the total number of propulsors on the airplane, while m is the total number of rotating elements in these propulsors that are generating significant gyroscopic moments. Scalar product of vector with vector product where one of the multipliers is that vector is zero. When elements from (2) are multiplied by $\boldsymbol{\omega}^T \mathfrak{J}$, then $\boldsymbol{\omega}^T \left(\boldsymbol{\omega} \times \mathfrak{J}\boldsymbol{\omega}\right) = 0$ and $\boldsymbol{\omega}^T \mathfrak{M}_p^G = \boldsymbol{\omega}^T \left(I_p(\boldsymbol{\omega}_p \times \boldsymbol{\omega})\right) = 0$. If the active moments generated by aerodynamic and propulsive forces are denoted by $\mathfrak{M}^A$, then

$$\mathfrak{M} = \mathfrak{M}^A = \mathfrak{M}_a + \mathfrak{M}_p^P = \begin{bmatrix} L_a + L_p^P \\ M_a + M_p^P \\ N_a + N_p^P \end{bmatrix} = \begin{bmatrix} L \\ M \\ N \end{bmatrix}. \tag{10}$$

Consequently, it is obtained

$$\begin{aligned} \dot{E}_R &= \left[\operatorname{grad}_{\boldsymbol{\omega}} E_R\right]^T \dot{\boldsymbol{\omega}} = \boldsymbol{\omega}^T \mathfrak{J}\dot{\boldsymbol{\omega}} = \boldsymbol{\omega}^T \mathfrak{M}^A = \\ &= \boldsymbol{\omega}^T \left(\mathfrak{M}_a + \mathfrak{M}_p^P\right) = Lp + Mq + Nr. \end{aligned} \tag{11}$$

Therefore, the scalar equation defining condition for the prevention of the increase and the stoppage of the airplane rotation is defined by

$$\dot{E}_R = \mathfrak{M}^T \boldsymbol{\omega} = Lp + Mq + Nr \le 0. \tag{12}$$

This general condition has been obtained without any simplification in the description of the airplane dynamics. It stems out from the inherent form of the kinetic energy of rotation in (6). There is, however, the simplification in defining initial conditions for the maneuver in which transfer of the other modes of energy into the kinetic energy of rotation occurred, presented in (5). The physical meaning of (12) is simple, as $\dot{E}_R = \mathfrak{M}^T \boldsymbol{\omega}$ presents power necessary to apply to change the energy of rotation. The positive power increases the energy of rotation, while the negative power decreases it.

The expression (12) contains within its active moments all controls available to the airplane during the flight along the flight path, aerodynamic ($\delta_l, \delta_m, \delta_n$) and propulsive ($\delta_T$) ones. As the accent in this paper is on the *airplane design procedure*, consideration shall be further simplified by assuming that resulting thrust force T_r is acting through airplane center of inertia, so that $\mathfrak{M}_p^P = 0$. Then the condition in (12) reduces to

$$\dot{E}_R = \mathfrak{M}_a^T \boldsymbol{\omega} = L_a p + M_a q + N_a r \le 0. \tag{13}$$

To make conclusions about the capabilities of the control of the airplane rotation by the means of aerodynamic moments only, it is necessary to know functional dependence of it components L_a, M_a, N_a. Aerodynamic moment $\mathfrak{M}_a = \operatorname{col}\left(L_a, M_a, N_a\right)$ is nonlinear function of the airplane state variables and control surface deflection, as well as the rate of both, or

$$\mathfrak{M}_a = \mathfrak{M}_a\left(\mathbf{X}, \dot{\mathbf{X}}, \mathbf{U}, \dot{\mathbf{U}}\right). \tag{14}$$

Translatory components are defined and measured in the wind, or stream, axis coordinate system, for the rotation the same is done in body axis system. Then, $\mathbf{X} = \operatorname{col}(V, \beta, \alpha, p, q, r, \Phi, \Theta, \Psi, x_0, y_0, z_0)$ is airplane state vector, where the intensity of velocity is V, angle of sideslip and attack are α, β, Euler angles Φ, Θ, Ψ and Earth fixed coordinates x_0, y_0, z_0. Control vector $\mathbf{U} = \operatorname{col}(\delta_l, \delta_m, \delta_n)$ contains deflections of roll (ailerons) pitch (elevator) and yaw (rudder) commands. For the atmospheric flight there exists biunivoke correspondence between airplane state vector $\mathbf{X}$ and the stream parameters, Mach M and Reynolds Re number. Then when the stream parameters are known, the functional relation for the vector of aerodynamic moment, with the precision of up to the first order time derivatives, is of the form

$$\mathfrak{M}_a = \mathfrak{M}_a\left(\alpha, \dot{\alpha}, \beta, \beta, \dot{\beta}, p, q, r, \delta_l, \dot{\delta}_l, \delta_m, \dot{\delta}_m, \delta_n, \dot{\delta}_n\right). \quad (15)$$

Important aspect in the presentation of aerodynamic moment is that practically in most cases it can be modeled as additive regarding to the control vector $\mathbf{U}$, i.e. it consists of 'referent' $\mathfrak{M}_a^r$ and 'control' $\mathfrak{M}_a^c$ part, or

$$\mathfrak{M}_a = \mathfrak{M}_a^r + \mathfrak{M}_a^c = \\ = \mathfrak{M}_a^r\left(\mathbf{X}, \dot{\mathbf{X}}\right) + \mathfrak{M}_a^c\left(\mathbf{X}, \dot{\mathbf{X}}, \mathbf{U}, \dot{\mathbf{U}}\right) = \begin{bmatrix} L_a^r + L_a^c \\ M_a^r + M_a^c \\ N_a^r + N_a^c \end{bmatrix}. \quad (16)$$

During the design phase of the airplane, the easiest and most precise method to determine functional dependences of aerodynamic moments is by measurements in wind tunnels.

It is important to note that (13) is exact analytical expression, regardless of the means by which the components of aerodynamic moment, L_a, M_a, N_a are obtained. The equation (13) is condition derived for the arbitrary moment of time that is valid regardless to the history of the airplane motion. It is obtained out of the scope of the analysis of the differential equations of motion. The exactness of the analytical approach to the dynamic of airplane rotation based on (13) is constrained only by the exactness of the method of obtaining the components of aerodynamic moment $\mathfrak{M}_a$.

The significance of (13) is in it applicability. It is valid throughout whole range of airplane state space and can be used to assess the properties of dynamic of airplane rotation, providing thus analyzing tool. Furthermore, whenever aerodynamic moment $\mathfrak{M}_a$ can be modeled as in (16), equation (13) can be rendered to the form

$$\dot{E}_R = \mathfrak{M}_a^T \boldsymbol{\omega} = \\ = \left(L_a^r + L_a^c\right) p + \left(M_a^r + M_a^c\right) q + \left(N_a^r + N_a^c\right) r \leq 0. \quad (17)$$

Because (17) permits separate observation of the influence of airplane control, it can be readily used as design tool, applicable ab-initio in the design of desired properties of dynamics of airplane rotation.

3 Locally linearized model of aerodynamic moment

The choice of the shape of model of aerodynamic moment in the analysis of dynamics of airplane rotation makes the way that leads condition (13) or (17) toward approaches based on the separated longitudinal and lateral-directional motion. That shall be presented through the linearized models of the aerodynamic moment of decreasing complexity. The reason for the application of the simple linearized models is to proof that the analysis of dynamics of airplane rotation based on the condition (13) is, within domain of validity of the models of separated longitudinal and lateral-directional motion, fully congruent with the analysis based upon these models. Nevertheless, the condition (13) enables the analysis of dynamics of airplane rotation outside of above-mentioned areas of congruency, without any decrease in the quality of the results through the loss of the character of the solution by omitting part of it.

Within the set of linearized models of aerodynamic moment locally linearized one is with the highest complexity. It maintains validity within sufficiently small vicinity of any point within the range of stream angles where the non-stationary effects of flow field can be neglected. Then the elements of the first and higher order derivatives can be neglected and the model of aerodynamic moment is derived with the precision of up to the zero order derivatives. Let the linearized model of aerodynamic moment be derived in this vicinity of the referent values of the angle of attack and sideslip, $\alpha = \alpha^r, \beta = \beta^r$, where the above-mentioned conditions can be sustained. Then the components of the aerodynamic moment are of the form

$$
\begin{aligned}
&L_a^L = L_\beta\beta + L_p p + L_r r + L_{\delta_l}\delta_l + L_{\delta_n}\delta_n + L_0 = QSbC_l = \\
&= QSb\left(C_{l\beta}\beta + C_{lp}\frac{pb}{2V} + C_{lr}\frac{rb}{2V} + C_{l\delta_l}\delta_l + C_{l\delta_n}\delta_n + C_{l0}\right) \\
&M_a^L = M_\alpha\alpha + M_q q + M_{\delta_m}\delta_m + M_0 = QSbC_m = \\
&= QSb\left(C_{m\alpha}\alpha + C_{mq}\frac{qc}{2V} + C_{m\delta_m}\delta_m + C_{m0}\right) \\
&N_a^L = N_\beta\beta + N_p p + N_r r + N_{\delta_l}\delta_l + N_{\delta_n}\delta_n + N_0 = QSbC_n = \\
&= QSb\left(C_{n\beta}\beta + C_{np}\frac{pb}{2V} + C_{nr}\frac{rb}{2V} + C_{n\delta_l}\delta_l + C_{n\delta_n}\delta_n + C_{n0}\right)
\end{aligned}
\tag{18}
$$

where Q denotes dynamic pressure, V intensity of air velocity vector and S,c,b are, respectively, wing surface, mean aerodynamic chord and span. Dimensionless C_{ij} and dimensional L_j, M_j, N_j coefficients of aerodynamic moment are derived for moment components $i = L, M, N$ derived upon variables $j = \alpha, \beta, p, q, r, \delta_l, \delta_m, \delta_n$, whereas index $j = 0$ notifies free element in the developed sequence.

By substituting (18) into (13) and rearranging is obtained

$$
\begin{aligned}
\mathfrak{M}_a^T\boldsymbol{\omega} = &\left(p\,q\,r\right)\begin{bmatrix} L_p & 0 & L_r \\ 0 & M_q & 0 \\ N_p & 0 & N_r \end{bmatrix}\begin{bmatrix} p \\ q \\ r \end{bmatrix} + \\
&+\left(L_\beta\beta + L_{\delta_l}\delta_l + L_{\delta_n}\delta_n + L_0\right)p + \\
&+\left(M_\alpha\alpha + M_{\delta_m}\delta_m + M_0\right)q + \\
&+\left(N_\beta\beta + N_{\delta_l}\delta_l + N_{\delta_n}\delta_n + N_0\right)r.
\end{aligned}
\tag{19}
$$

Further on, the rearranged vector of angular velocity $\bar{\boldsymbol{\varpi}}^T = [q\,p\,r]$ is introduced, as well as the variables

$$
\bar{\boldsymbol{\varpi}}^T = [q\,p\,r]; \quad \bar{\boldsymbol{\varpi}}^T = [\boldsymbol{\omega}_L\,\boldsymbol{\omega}_D];
$$

$$
\boldsymbol{\omega}_L = [q]; \qquad \boldsymbol{\omega}_D^T = [p\,r];
$$

$$
\boldsymbol{\mathcal{M}} = \begin{bmatrix} \boldsymbol{\mathcal{M}}_L & 0 \\ 0 & \boldsymbol{\mathcal{M}}_D \end{bmatrix} = \begin{bmatrix} M_q & 0 & 0 \\ 0 & L_p & L_r \\ 0 & N_p & N_r \end{bmatrix};
$$

$$
\boldsymbol{\mathcal{M}}_L = \left[M_q\right]; \qquad \boldsymbol{\mathcal{M}}_D = \begin{bmatrix} L_p & L_r \\ N_p & N_r \end{bmatrix};
$$

$$
a = \left(M_\alpha\alpha + M_{\delta_m}\delta_m + M_0\right);
$$

$$
b = \left(L_\beta\beta + L_{\delta_l}\delta_l + L_{\delta_n}\delta_n + L_0\right);
$$

$$
c = \left(N_\beta\beta + N_{\delta_l}\delta_l + N_{\delta_n}\delta_n + N_0\right),
$$

where $\boldsymbol{\omega}_L$ is **L**ongitudinal and $\boldsymbol{\omega}_D$ lateral-**D**irectional part of the angular velocity vector, $\boldsymbol{\mathcal{M}}$ is matrix of the contributions of angular velocity to the aerodynamic moment and $\boldsymbol{\mathcal{M}}_L$ and $\boldsymbol{\mathcal{M}}_D$ its **L**ongitudinal and lateral-**D**irectional part. Then, the shortened form of (19) is

$$
\mathfrak{M}_a^T\boldsymbol{\omega} = \bar{\boldsymbol{\varpi}}^T\boldsymbol{\mathcal{M}}\bar{\boldsymbol{\varpi}} + aq + bp + cr, \tag{20}
$$

or, rearranged with contributions of angular velocity to the aerodynamic moment separated to longitudinal and lateral-directional part

$$
\begin{aligned}
\mathfrak{M}_a^T\boldsymbol{\omega} &= \underbrace{\boldsymbol{\mathcal{M}}_L q^2 + aq}_{\mathfrak{M}_L^T\boldsymbol{\omega}_L} + \underbrace{\boldsymbol{\omega}_D^T\boldsymbol{\mathcal{M}}_D\boldsymbol{\omega}_D + bp + cr}_{\mathfrak{M}_D^T\boldsymbol{\omega}_D} = \\
&= \mathfrak{M}_L^T\boldsymbol{\omega}_L + \mathfrak{M}_D^T\boldsymbol{\omega}_D
\end{aligned}
\tag{21}
$$

$$
\mathfrak{M}_L^T\boldsymbol{\omega}_L = \boldsymbol{\mathcal{M}}_L q^2 + aq
$$

$$
\mathfrak{M}_D^T\boldsymbol{\omega}_D = \boldsymbol{\omega}_D^T\boldsymbol{\mathcal{M}}_D\boldsymbol{\omega}_D + [b\ c]\boldsymbol{\omega}_D.
$$

It must be noted that in (21) separation is performed only upon the longitudinal and lateral-directional part of the aerodynamic moment, whereas the analysis of the dynamics of airplane rotation remains integral.

Geometrical interpretation of eq. (20) and (21) is simple. Equation (20) is observed in the four-dimensional space $O\dot{E}_R qpr$ with axes $O\dot{E}_R$, Oq, Op and Or. Let notify that $d/dt\left(E_R\right) = \mathfrak{M}_a^T\boldsymbol{\omega}$. The selection of $O\dot{E}_R$ for notification of the axis instead of $\mathfrak{M}_a^T\boldsymbol{\omega}$ in (20) and (21) is done to accent it physical meaning. It presents sum of the hyperparaboloid $\bar{\boldsymbol{\varpi}}^T\boldsymbol{\mathcal{M}}\bar{\boldsymbol{\varpi}}$ and plain $aq + bp + cr$ containing the point of coordinate beginning O. The plain can rotate about point O with the position determined by the values of coefficients a, b and c which are dependent upon the instantaneous values of state variables α, β and deflection of control surfaces δ_l, δ_m and δ_n. In the case of stationary, straight, wing-level flight of geometrically symmetric airplane geometrical interpretation of (20) reduces to coordinate hyper plane $Oqpr$.

In the analysis of the dynamics of airplane rotation, the transfer from the general form of condition (13) to the form presented in equations (20) and (21) is based upon the locally linearized model of airplane aerodynamic moments. Stationary states of airplane dynamics present equilibrium states from the flight reference point. The equilibrium states of flight are only the subset within the range of angles of attack and sideslip, α, β, where the local linearization can be performed.

4 Relation to the classical approach to the analysis of the dynamics of airplane rotation

One special case of flight equilibrium is stationary, straight, horizontal, wing-level flight. For the inertially and geometrically symmetric airplane these are conditions for the classical approach to the analysis of airplane dynamics, based upon the separated modes of longitudinal and lateral-directional motion. Let denote the equilibrium flight states by upper index 'e', so for the case of symmetric airplane coefficients a, b and c in (20) are zero, or

$$\begin{aligned} a^e &= M_\alpha \alpha^e + M_{\delta_m}\delta_m^e + M_0^e = 0; \\ b^e &= L_\beta \beta^e + L_{\delta_l}\delta_l^e + L_{\delta_n}\delta_n^e + L_0^e = 0; \\ c^e &= N_\beta \beta^e + N_{\delta_l}\delta_l^e + N_{\delta_n}\delta_n^e + N_0^e = 0. \end{aligned} \tag{22}$$

Then the values of stream angles and control surface deflections are $\alpha^e \neq 0$, $\delta_m^e \neq 0$, $\beta^e = 0$, $\delta_l^e = 0$ and $\delta_n^e = 0$ as well as $L_0^e = 0$ and $N_0^e = 0$, because last two equations must be identically satisfied. In the case of non-symmetric airplane is $\beta^e \neq 0$, $\delta_l^e \neq 0$ and $\delta_n^e \neq 0$, $L_0^e \neq 0$ and $N_0^e \neq 0$ so the analysis reverts to locally linearized model in (20) and (21).

To show the congruency of the analysis of the rotational dynamics of the airplane based on the condition (13) with the classical approach to this problem, it is necessary to subject it to the same set of limitations and constrains that has been used in developing decomposed linearized models of longitudinal and lateral-directional modes of airplane motion. Let these constrains be reviewed in brief. Models are derived as linear ones for the above mentioned initial condition of equilibrium state of symmetric airplane. Range of the values of airplane state variables is limited to the domain of their linear functional dependability, the most important being the range of angle of attack of up to $\alpha \leq 0.8\alpha_{crit}$ of it critical value α_{crit}. For the defined set of the stream parameters, Mach M and Reynolds Re number, dimensionless C_{ij} and dimensional L_j, M_j, N_j coefficients of aerodynamic moment are then within this range of linearity constant.

Stability analysis within each of separated modes is limited to the domain of validity of particular linear model. Overall stability of the airplane exists if and only if (iff) there simultaneously exists local stability in the each of the particular modes of airplane motion.

The meaning of these constrains applied to the condition (13) modified to the form in equations (20) and (21) is as follows. The sign definiteness of expression $\mathfrak{M}_a^T \boldsymbol{\omega}$ is achieved iff there is fulfilled the requirement that as the total expression elements $\mathfrak{M}_L^T \boldsymbol{\omega}_L$ and $\mathfrak{M}_D^T \boldsymbol{\omega}_D$ are achieving independently and simultaneously this sign definiteness. For example, non-positive sign definiteness $\mathfrak{M}_a^T \boldsymbol{\omega} \leq 0$ can be achieved if simultaneousl and independently there can be achieved $\mathfrak{M}_L^T \boldsymbol{\omega}_L \leq 0$ and $\mathfrak{M}_D^T \boldsymbol{\omega}_D \leq 0$. Applications of constrains applied in generation of classical approach permits that in the integral condition (21) each of the elements, $\mathfrak{M}_L^T \boldsymbol{\omega}_L$ and $\mathfrak{M}_D^T \boldsymbol{\omega}_D$, is considered separately.

4.1 Longitudinal motion analysis

Observation of expression

$$\begin{aligned} \frac{d}{dt}\left(E_q\right) &= \dot{E}_q = \mathfrak{M}_L^T \boldsymbol{\omega}_L = \boldsymbol{\mathcal{M}}_L q^2 + aq = \\ &= M_q q^2 + \left(M_\alpha \alpha + M_{\delta_m}\delta_m + M_0\right) q \leq 0, \end{aligned} \tag{23}$$

$$\mathfrak{M}_L^T \boldsymbol{\omega}_L = M_q \left[\left(q + \frac{a}{2M_q}\right)^2 - \left(\frac{a}{2M_q}\right)^2\right] \leq 0,\ M_q \neq 0$$

is sufficient to use condition (13) in the analysis of linear model of longitudinal mode of motion. In (23) E_q is separated kinetic energy of pitching (longitudinal) angular motion. The second form of condition (23) accents form and elements of parabola. The significance of simplification made in (23) can be best concluded if the comparison is made with ellipsoid of rotation (6) as the inherent form of kinetic energy of rotation. By assumption, separated observation of the longitudinal mode of airplane motion reduces to zero rolling and yawing component of angular velocity, i.e. $p = 0$ and $r = 0$, which in the analogy with the ellipsoidal rigid body is equivalent to it reduction to thin stick. Therefore, the variation of rotational energy is generated only along one fixed direction Oq.

Geometrical interpretation of condition (23) is performed in two-dimensional space $O\dot{E}_q q$ with axis $O\dot{E}_q$ and Oq. The function $\mathfrak{M}_L^T \boldsymbol{\omega}_L = = M_q q^2 + aq$ is the sum of parabola $M_q q^2$ and straight line aq containing coordinate beginning O and is defined for any value of M_q. The second form of condition in (23) determines the extreme of parabola and is defined only when $M_q \neq 0$. Condition $M_q = 0$ is extremely rare and practically impossible for the airplanes with empennage. The angular orientation of the straight line is defined by coefficient a. Direction coefficient from (22) is $a^e = 0$ for the airplane in the equilibrium flight and straight line aq is collinear with axis Oq. For decomposed motion is $M_0^e = M_0 |_{\alpha=0}$ because $\mathfrak{M}_L(\alpha)$ is linear.

For the above equilibrium initial state, the necessary condition for the stable rotation (pitching) of the airplane in the separated longitudinal mode is $\mathfrak{M}_L^T \boldsymbol{\omega}_L = M_q q^2 < 0$, wherefrom emerges requirement $M_q < 0$. Boundary stability condition is $M_q q^2 = 0$, while for the unstable pitching motion is $M_q q^2 > 0$. Any deviation out of equilibrium state causes increment Δa of coefficient a, so $a = a^e + \Delta a \neq 0$ and straight line is Δaq, i.e. rotated by Δa. Parabola extreme is then shifted from coordinate beginning along Oq axis by amount $\left(-\Delta a / 2M_q\right)$. Then the parabola presenting condition (23) is with two zeroes and two sets of values with opposite sign.

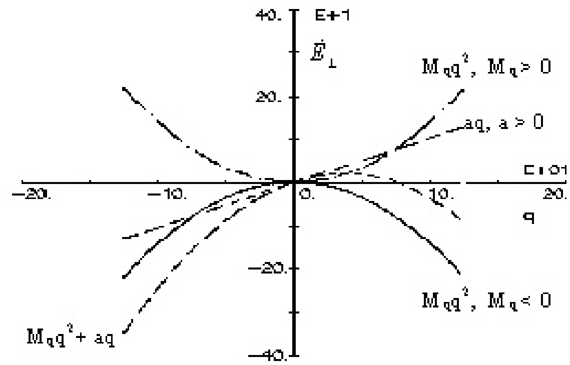

Figure 2. The rate of energy of rotation in the longitudinal mode.

For example, for stable airplane and above equilibrium initial state is $M_q < 0$ and element $M_q q^2$ is with nonpositive values. Deviation out of equilibrium causes that sum $M_q q^2 + \Delta aq$ has region of positive values. Deviation can be caused by required deflection of control surface $\Delta\delta_m$ or by environment disturbance in the form of either $\Delta\alpha$ or variations of coefficients ΔM_α, ΔM_{δ_m} and ΔM_0.

To prevent the increase of rotation in the longitudinal mode of motion for the known amount of the intensity of rotation R must be fulfilled boundary condition $\mathfrak{M}_L^T \boldsymbol{\omega}_L = M_q q^2 + aq = 0$ for value of pitching angular velocity $q = \pm R / \sqrt{J_2}$.

In the area of constant aerodynamic parameters, the necessary boundary increment of control surface deflection is determined from the condition $M_q \left(R / \sqrt{J_2}\right) + M_\alpha \left(\alpha^e + \Delta\alpha\right) + M_{\delta_m} \left(\delta_m^e + \Delta\delta_m\right) + M_0^e = 0$.

There from, the boundary increment of control surface deflection necessary to prevent the increase of rotation in longitudinal mode of airplane

motion is $(\Delta\delta_m)_b = \left(M_q\left(R/\sqrt{J_2}\right) - M_\alpha \Delta\alpha\right)/M_{\delta_m}$, which is in full compliance with [8].
The condition of the stability of the longitudinal motion stems out of the demand to fulfill the requirement $\mathfrak{M}_L^T \boldsymbol{\omega}_L < 0$ in the presence of the disturbance $\Delta\alpha$ relative to the existing equilibrium state (22). As the necessary condition is $M_q q^2 < 0$, there from emerges sufficient condition $\mathfrak{M}_L^T \boldsymbol{\omega}_L = \left(M_\alpha \Delta\alpha/q + M_q\right)q^2 < 0$ for $q \neq 0$. As there is $M_\alpha = QScC_{m\alpha}$, $M_q = QScC_{mq}(c/2V)$ and $q/\Delta\alpha = -Z_\alpha/mV = -C_{z\alpha}(QS/mV)$ previous condition becomes $C_{mq}C_{z\alpha}(c/2V) - C_{m\alpha}(mV/QS) > 0$, which is identical to the condition for the longitudinal motion stability in [9].

4.2 Lateral-directional motion analysis

To analyze lateral-directional mode of airplane motion on the basis of condition (13) is sufficient to separately observe expression

$$\begin{aligned} \frac{d}{dt}(E_D) = 2\dot{E}_D = \mathfrak{M}_D^T \boldsymbol{\omega}_D = \\ = \boldsymbol{\omega}_D^T \boldsymbol{\mathcal{M}}_D \boldsymbol{\omega}_D + bp + cr \leq 0, \end{aligned} \tag{24}$$

where E_D is separated kinetic energy of the lateral-directional angular motion and $\boldsymbol{\mathcal{M}}_D$ and $\boldsymbol{\omega}_D$ are in the same form as in the case of locally linearized model of aerodynamic moment. By assumption, separated observation of the lateral-directional mode of airplane motion reduces to zero pitching angular velocity, i.e. $q = 0$. Then the inherent form of kinetic energy of rotation presented as the ellipsoid of rotation (6) is reduced to ellipse in the Opr plane

$$2E_{R_D} = J_1 p^2 + J_3 r^2 = R_D^2. \tag{25}$$

Geometrical interpretation of condition (24) is performed in three-dimensional space $O\dot{E}_D pr$ with axis $O\dot{E}_D$, Op and Or. The function $\boldsymbol{\omega}_D^T \boldsymbol{\mathcal{M}}_D \boldsymbol{\omega}_D + bp + cr$ then presents the sum of paraboloid $\boldsymbol{\omega}_D^T \boldsymbol{\mathcal{M}}_D \boldsymbol{\omega}_D$ and plane $bp + cr$ containing coordinate beginning O. The plane $bp + cr$ can rotate around coordinate beginning O with orientation defined by coefficients b and c, i.e. by straight lines bp in coordinate plane $O\dot{E}_D p$ and cr in coordinate plane $O\dot{E}_D r$. The intersection of paraboloid $\boldsymbol{\omega}_D^T \boldsymbol{\mathcal{M}}_D \boldsymbol{\omega}_D$ and plane $\dot{E}_D = \text{const}$ parallel to the coordinate plane Opr is ellipse with the main axes parallel to the first and third main axes of inertia of the airplane as the rigid body.

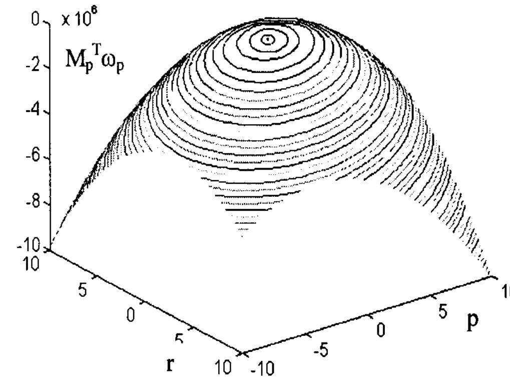

Figure 3. The rate of energy of rotation in the lateral-directional mode.

For the airplane in the equilibrium flight is from (22) $b^e = 0$ and $c^e = 0$, so plane $bp + cr$ and coordinate plane Opr are coplanar. For the above equilibrium initial state, the necessary condition for the stable rotation of the airplane in the separated lateral-directional mode is $\mathfrak{M}_D^T \boldsymbol{\omega}_D = \boldsymbol{\omega}_D^T \boldsymbol{\mathcal{M}}_D \boldsymbol{\omega}_D < 0$, wherefrom emerges require-ment $\boldsymbol{\mathcal{M}}_D < 0$ for negative sign definiteness of matrix $\boldsymbol{\mathcal{M}}_D$. This requirement is satisfied if all main minors of this matrix are negative [11]. Expressed through dimensionless and dimensional coefficients that is $L_p = \left(QSb^2/2V\right)C_{lp} < 0$, $C_{lp} < 0$ and $L_p N_r - N_p L_r = \left(QSb^2/2V\right)\left(C_{lp}C_{nr} - C_{np}C_{lr}\right) < 0$. The same requirement is defined in [8,9,12]. Boundary stability is in the case $\boldsymbol{\mathcal{M}}_D = 0$ and instability for $\boldsymbol{\mathcal{M}}_D > 0$.

Any deviation from the equilibrium state causes increments Δb and Δc, that is $b = b^e + \Delta b \neq 0$ and $c = c^e + \Delta c \neq 0$. Then, the plane equation becomes $\Delta bp + \Delta cr$ and the plane is rotated around coordinate beginning O by angle Δb in coordinate plane $O\dot{E}_D p$ and angle Δc in coordinate plane $O\dot{E}_D r$. Then, the apex of paraboloid is shifted from coordinate beginning generating three regions of condition (24). The first contains zero values of condition (24) presenting ellipse in coordinate plane Opr, with two other presenting surfaces of opposite sign.

For example, for stable airplane and above equilibrium initial state is $\mathcal{M}_D < 0$ and element $\boldsymbol{\omega}_D^T \mathcal{M}_D \boldsymbol{\omega}_D$ is with nonpositive values. Deviation out of equilibrium causes that sum $\boldsymbol{\omega}_D^T \mathcal{M}_D \boldsymbol{\omega}_D + \Delta bp + \Delta cr$ has region of positive values. Deviation can be caused by required deflection of control surfaces $\Delta\delta_l$ and $\Delta\delta_n$ or by environment disturbance in the form of either $\Delta\beta$ or variations of coefficients ΔL_β, ΔL_{δ_l}, ΔL_{δ_n}, ΔL_0, ΔN_β, ΔN_{δ_l}, ΔN_{δ_n} and ΔN_0.

The condition of the stability of the longitudinal motion stems out of the demand to fulfill the requirement $\mathfrak{M}_D^T \boldsymbol{\omega}_D < 0$ in the presence of the disturbance $\Delta\beta$ relative to the existing equilibrium state (22), so $\beta^e = 0$ and $\beta = \Delta\beta$. The necessary condition is always $\mathcal{M}_D < 0$, so the sufficient condition for the stability of the airplane in the lateral-directional motion in the presence of disturbance $\Delta\beta$ is that angular velocities p and r are such that the requirement $bp + cr < 0$ is fulfilled. Boundaries of area where previous inequality is valid can be determined from expression

$$bp + cr = \left(L_\beta \beta + L_{\delta_l}\delta_l + L_{\delta_n}\delta_n\right)p + \\ + \left(N_\beta \beta + N_{\delta_l}\delta_l + N_{\delta_n}\delta_n\right)r = 0,$$

because $L_0^e = L_0 = 0$ and $N_0^e = N_0 = 0$. One partial solution of this equation is its identical equality when $b = 0$ and $c = 0$. Computing β from $b = 0$ and substituting into $c = 0$ yields the condition

$$N_\beta - L_\beta \frac{N_{\delta_l} + N_{\delta_n}\dfrac{\delta_l}{\delta_n}}{L_{\delta_l} + L_{\delta_n}\dfrac{\delta_l}{\delta_n}} = 0, \tag{26}$$

which defines relations in stationary turning flight with sideslip.

From the same boundary condition equivalent conclusion can be made. Computing δ_l/β from $b = 0$ and substituting into $c = 0$ yields

$$N_\beta - L_\beta \frac{N_{\delta_l}}{L_{\delta_l}}\left(L_{\delta_n}\frac{N_{\delta_l}}{L_{\delta_l}} - N_{\delta_n}\right)\left(\frac{\delta_n}{\beta}\right) = 0. \tag{27}$$

For undeflected rudder, $\delta_n = 0$, previous equation reduces to

$$N_\beta L_{\delta_l} - L_\beta N_{\delta_l} = 0. \tag{28}$$

If in condition (24) are taken only elements multiplied by r and βp where $p = const$, the stability condition is $y = L_\beta \beta p + N_\beta \beta r + L_r rp + N_r r^2 < 0$. Function y is parabola with yawing angular velocity r as independent variable. The value of independent variable $r_{ax} = -\left(N_\beta \beta + L_r p/(2N_r)\right)$ yields the apex of parabola y as

$$y_{ax} = \frac{2\left(L_\beta N_r + N_\beta L_r\right)\beta p + N_\beta^2 \beta^2 + L_r^2 p^2}{2N_r} \tag{29}$$

For the airplane with empennage N_r is negative in the area of sub critical angles of attack, therefore apex of y is it maximum. Generally y is with two zeroes and two sets of values with opposite sign. If it is demanded that the function y is negative then must be $y_{ax} < 0$. One of the boundaries is

$$L_\beta N_r + N_\beta L_r = 0, \tag{30}$$

as in the condition (29) sum of squares is always positive. It must be noted that all obtained conditions are partial solutions of general condition $\mathfrak{M}_D^T \boldsymbol{\omega}_D < 0$ and are in full compliance with the results obtained in [9,12].

5 Airplane lateral-directional rotation due to the longitudinal command

Important property of an airplane is its lateral-directional behavior generated by the commanded longitudinal rotation. This behavior is of particular interest in the vicinity of critical values of angle of attack. Assessment of this phenomenon is necessary to understand the physical process related to the loss of the lateral-directional stability due to the commanded longitudinal rotation, the consequences being "wing-drop" or spin. To obtain the full understanding of the airplane lateral-directional rotation due to the commanded longitudinal rotation it is necessary to consider the complete model of the airplane dynamics. For the airplane with the plane of symmetry Oxz, the developed form of equations (2), with forces in the wind coordinate system and moments in the body coordinate system is [13]:

$$\dot{V} = -\frac{QS}{m} C_{D_{wind}} + g\left(c\varphi c\theta s\alpha + s\varphi c\theta\right) s\beta - $$
$$-gs\theta c\alpha c\beta + \frac{T}{m} c\left(\alpha + \zeta_0\right) c\beta$$
$$\dot{\alpha} = -\frac{QS}{mVc\beta} C_L + q - t\beta\left(pc\alpha + rs\alpha\right) +$$
$$+\frac{g}{Vc\beta}\left(c\varphi c\theta c\alpha + s\theta s\alpha\right) + \frac{T}{mVc\beta} s\left(\alpha + \zeta_0\right)$$
$$\dot{\beta} = \frac{QS}{mV} C_{Y_{wind}} + ps\alpha - rc\alpha + \frac{g}{V} c\beta s\varphi c\theta +$$
$$+\frac{s\beta}{V}\left[g\left(c\alpha s\theta - s\alpha c\varphi c\theta\right) + \frac{T}{m} c\left(\alpha + \zeta_0\right)\right]$$
$$\dot{p} = \frac{I_z L + I_{xz} N}{I_x I_z + I_{xz}^2} + \frac{I_{xz}\left(I_x - I_y + I_z\right)}{I_x I_z + I_{xz}^2} pq +$$
$$+\frac{I_z\left(I_y - I_z\right)}{I_x I_z + I_{xz}^2} qr - \frac{I_{xz}}{I_x I_z + I_{xz}^2}\left(\sum_1^n \left(I_p \omega_p\right)_i\right) q$$
$$\dot{q} = \frac{M}{I_y} + \frac{I_z - I_x}{I_y} pr + \frac{I_{xz}}{I_y}\left(r^2 - p^2\right) +$$
$$+\frac{1}{I_y}\left(\sum_1^n \left(I_p \omega_p\right)_i\right) r$$
$$\dot{r} = \frac{I_{xz} L + I_x N}{I_x I_z + I_{xz}^2} + \frac{I_x^2 - I_x I_y + I_{xz}}{I_x I_z + I_{xz}^2} pq +$$
$$-\frac{I_{xz}\left(I_x - I_y + I_z\right)}{I_x I_z + I_{xz}^2} qr - \tag{31}$$
$$-\frac{I_x}{I_x I_z + I_{xz}^2}\left(\sum_1^n \left(I_p \omega_p\right)_i\right) q$$

In equation (31) are, for brevity, with s, c and t notified sine, cosine and tangent functions of bank and pitching angles φ, θ and angle of attack and sideslip α, β. In (31) are also: $T-$resulting thrust force of propulsive group acting in plane of symmetry Oxz at relative angle ζ_0 to Ox axle, g – the acceleration of Earth gravity and $C_{D_{wind}}, C_L$ and $C_{Y_{wind}}$ are coefficients of aerodynamic forces in the wind axes coordinate system. Eq. (31) together with kinematical relations of orientation $col\left(\varphi, \theta, \psi\right)$ and relation of rate of this vector with angular velocities [13,9,10] are defining the complete nonlinear mathematical model of airplane dynamics.

In the stationary straight flight airplane rates $\dot{V}$, $\dot{\alpha}, \dot{\beta}, \dot{p}, \dot{q}, \dot{r}$ and angular velocity $\omega = col\left(p, q, r\right)$ are equal to zero. To disturb the airplane out of stationary flight it is necessary to make influence to the highest time derivative of some state variable in order to generate the variation of this and consequently other state variables.

The longitudinal command out of stationary straight flight shall generate the variation of the rates $\dot{V}, \dot{\alpha}$, and $\dot{q}$. The longitudinal command can be made on throttle, generating variations of T and consequently airspeed and all $(\omega_p)_i$, airbrakes, causing variations in the airspeed, and control surface displacement making direct variation of the angle of attack. Any mode of longitudinal command will generate variation in the angle of attack and thus change total aerodynamic moment, as well as the coefficients presenting it. The variation of the M component of aerodynamic moment shall cause pitching rotation with $\dot{q} \neq 0$ and $q \neq 0$. If on the airplane there are naturally unbalanced gyroscopic moments, then the q

variation generates $\dot{p} \neq 0$ and $\dot{r} \neq 0$ and therefore airplane bank and sideslip. The influence of the gyroscopic effects of the propulsive group on the rates $\dot{p}$ and $\dot{r}$ made through coefficient $\sum_i \left(I_p\right)_i I_x / \left(I_x I_z - I_{xz}\right)$ is weak and can often be neglected. As angular velocities are initially zero, products *pq* and *qr* are also zero and there are no influence of inertial coupling to the rates $\dot{p}$ and $\dot{r}$. Then, the variation of $\dot{p}$ and $\dot{r}$ can be made only through *L* and *N* components of aerodynamic moment being directly dependent upon the selected aerodynamic configuration and instantaneous value of angle of attack of the airplane. For example, the coefficient of directional stability N_β after the critical value of angle of attack is achieved significantly decreases and can change the sign. Coefficient of lateral stability L_β increases proportionally to the angle of attack up to it critical value and sharply decreases afterward. Similar behavior relative to the variation of the angle of attack are possessing damping and controlling aerodynamic lateral-directional coefficients.

Therefore, to observe lateral-directional rotation due to the longitudinal command, it is sufficient to observe developed form of locally linearized aerodynamic moment (18), taking into account the scope of it validity. Longitudinal command out of stationary straight flight with $p = q = r = 0$ generates $V \neq V^e$ or $Q \neq Q^e$, $\alpha = \alpha^e$ and $\delta_m \neq \delta_m^e$. Observing expressions for L and N components of aerodynamic moment in (18) can be concluded that their non-commandend variation can be generated if coefficients comprising them are multiplied with non-zero value of corresponding state or input variable, or if the coefficient with zero index is variable. That means that longitudinal maneuver is entered with:

- Residual equilibrium value of angle of sideslip $\beta^e \neq 0$,
- Residual equilibrium values of control surface deflections $\delta_l^e \neq 0$ and $\delta_n^e \neq 0$,
- Existence of non-zero free element coefficients $C_{l0} \neq 0$ and $C_{n0} \neq 0$, i.e. aerodynamic non-symmetry,
- Combination of previous cases.

ThisThe first case appears if equilibrium flight is performed with stationary value of angle of sideslip β^e. Moment generated by sideslip (elements $C_{l\beta}\beta^e, C_{n\beta}\beta^e$) is in stationary horizontal flight compensated by deflection of either one or both control surfaces δ_l, δ_n, as well as with action of air stream upon non-symmetric airplane ($C_{l\beta}\beta^e = C_{l0}$, $C_{n\beta}\beta^e = C_{n0}$). It must be noted that for the small values of angle of attack and well defined aerodynamic configuration the modulus of coefficients $C_{l\beta}$ and $C_{l\delta_l}$ is the same, which is also the case with coefficients $C_{n\beta}$, $C_{n\delta_l}$ and $C_{n\delta_n}$. Therefore, the unit angle deflection of control surface compensates approximately the unit angle of sideslip. With the increase of the angle of attack this correlation among these coefficients is not maintained.

The second case is the most complex one as it combines characteristics of the airplane, pilot as the human operator and piloting techniques. There are multiple causes for existence of residual control surface deflections. These are: the balancing in horizontal straight flight of drag difference on the left and right wing due to aerodynamic asymmetry ($C_{l0} \neq 0$, $C_{n0} \neq 0$), of the residual sideslip angle described in the previous case and of the bank below pilot sensibility threshold ($\varphi \neq 0$), as well as the compensation of the resulting force and moment of propulsive group. The last set of causes is related to the pilot – control system interaction due to inadequate force stimulance in one lateral command (stick or pedals) in the presence of small deflections of these control surfaces. This is the result of high friction in controls or the small gradient of commanding force relative to control surface deflection. Inadequacies in one lateral command are generally compensated by trimming the other.

The third case appears in flight as the combination of the first two. It is emphasized as the existence of elements $C_{l0} \neq 0$ and $C_{n0} \neq 0$ notifies the aerodynamic asymmetry that is exclusively the consequence of the design (modeling of the aerodynamic form) or fabrication of the airplane. The existence of moment elements $C_{l0} \neq 0$ and

$C_{n0} \neq 0$ in the area of small angles of attack is either unsensed or easily compensated. However, in the region of the high angles of attack due to the air stream separation and interference, the influence of these elements dominates the total amount of the aerodynamic moment components L and N, often without the possibility that it can be compensated by the control surfaces deflection.

The analysis of the weighting factors of the particular elements in the total values of the aerodynamic moment components as well as the modeling of the aerodynamic shape of the airplane that shall minimize the appearance of the lateral-directional rotation due to the longitudinal command is not the main topic of this paper.

6 The generalization of the condition to prevent the increase and stop the lateral-directional rotation of the airplane

General condition to prevent the increase and stop the lateral-directional rotation of the airplane is derived from the corresponding energy of rotation E_D and its rate $\dot{E}_D = \mathfrak{M}_D^T \boldsymbol{\omega}_D$. These two physical properties of the airplane performing lateral-directional rotation are necessary and sufficient to perform the analysis of this mode of motion. As previously said, the exactness of the analysis performed on the basis of known rotational energy E_R (4), (6) and its rate $\dot{E}_R = \mathfrak{M}^T \boldsymbol{\omega}$ (12), (13) is limited only by the exactness in the obtaining components of aerodynamic moment substituted into (13). Using developed form of locally linearized aerodynamic moment (18) provides the sufficient exactness for engineering purposes. As defined, the scope of validity of (13) is within sufficiently small vicinity of *any point* in the state space of the airplane. Thus, the analysis is based upon equations (6) and (13) reduced to the case when pitching angular velocity is zero, $q = 0$. Then the condition (13) on the basis of (21) becomes

$$\begin{aligned} \dot{E}_D &= \mathfrak{M}_D^T \boldsymbol{\omega}_D = \boldsymbol{\omega}_D^T \boldsymbol{\mathcal{M}}_D \boldsymbol{\omega}_D + bp + cr \leq 0, \\ \boldsymbol{\mathcal{M}}_D &= \begin{bmatrix} L_p & L_r \\ N_p & N_r \end{bmatrix} \\ b &= \left(L_\beta \beta + L_{\delta_l} \delta_l + L_{\delta_n} \delta_n + L_0 \right); \\ c &= \left(N_\beta \beta + N_{\delta_l} \delta_l + N_{\delta_n} \delta_n + N_0 \right) \end{aligned} \tag{32}$$

the main difference in relation to (24) of the same form is that it is derived under more general conditions than is the stringent case for (24). The ellipsoid of rotation then becomes ellipse in the Opr plane or plane of symmetry Oxz as Opr and Oxz are coplanar

$$I_x p^2 - 2I_{xz} pr + I_z r^2 = R^2, \tag{33}$$

with the analogous relation to (25) as is (32) to (24). Equation (33) is presented on figure 4, but the properties of the ellipse in the plane Opr are the same regardless to the value of pitching angular velocity q. When airplane is performing only lateral-directional rotation, then $q = 0$, $E_R = E_D$ and $R = R_D$.

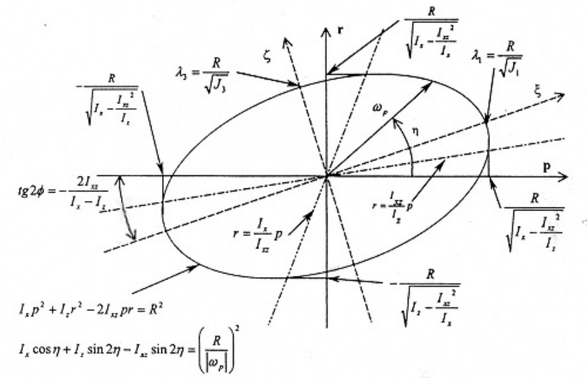

Figure 4. The ellipse of energy of rotation in Opr plane.

Let review briefly the analysis of the separated longitudinal mode of airplane motion. The inequality $\mathfrak{M}_L^T \boldsymbol{\omega}_L < 0$ is observed as the function of one argument q and two varying parameters: angle of attack α and pitching command δ_m. Additional condition has been obtained for the case of reducing ellipsoid of rotation to the Oq axes.

Similarly, the condition $\dot{E}_D = \mathfrak{M}_D^T \boldsymbol{\omega}_D < 0$ can be observed as the function of two arguments, p and r and four varying parameters α, β, δ_l and δ_n. Angle of sideslip β, and deflections of rolling δ_l and yawing δ_n command are explicitly figuring in (32), while angle of attack α is implicitly present as all of the aerodynamic coefficients are functions of instantaneous value of α. The equation (33) defines relation for the vector $\omega_D = col(p, r)$, as it tip is allways on the ellipse of rotation. Therefore, the number of independent arguments is reduced to one and any component of angular velocity can be selected as the independent one, either p or r. In other words, instead of analysing relations (32) and (33) in the space $O\dot{E}_D pr$ it is possible to analyse their projections to the planes $O\dot{E}_D p$ and $O\dot{E}_D r$.

Exact analysis of equations (32) and (33) can be made in several ways.

One approach is to solve (32) and (33) as the function of the relation r/p. That approach is consistent one if p is not changing sign and $p \neq 0$. Then from the boundary case $\mathfrak{M}_D^T \boldsymbol{\omega}_D = 0$ and condition $b + c\,r/p \neq 0$ the fourth order equation is obtained

$$
\begin{gathered}
H_4 h^4 + H_3 h^3 + H_2 h^2 + H_1 h^1 + H_0 = 0 \\
h = \frac{p}{r}, \quad p \neq 0, \quad b + c\frac{r}{p} \neq 0
\end{gathered} \tag{34}
$$

where

$$
\begin{aligned}
H_4 &= I_z c^2 - R^2 N_r^4, \\
H_3 &= 2\left[\left(I_z b - I_{xz} c\right)c - R^2 N_r \left(N_p + L_r\right)\right], \\
H_2 &= I_z b^2 + I_x c^2 - 4 I_{xz} bc - R^2 \left[\left(N_r L_p + N_p + L_r\right)^2\right], \\
H_1 &= 2\left[\left(I_x c - I_{xz} b\right)b - R^2 L_p \left(N_p + L_r\right)\right], \\
H_0 &= I_x b^2 - R^2 L_p^2.
\end{aligned}
$$

The roots of equation (34) are defining the conditions needed to prevent the increase of the lateral-directional rotation. Modeling of aerodynamic configuration and selection of the permitted intensity of rotation R adjust the distribution of the roots of this equotion. For example, the root close to the value I_x / I_{xz} defines the conditions to stop the spining of the airplane, while the root close to the value I_{xz} / I_z defines the conditions to stop the roll (fig. 4).

The second approach starts from the assumption that the value of angle of sideslip acts as the independently variable parametar in (32), when the equations (32) and (33) can be expressed either through rectangular coordinates p and r or through polar coordinates $\|\omega_D\| = \sqrt{p^2 + r^2}$ and η (fig. 4). In the maneuvers is $\beta = \beta(t)$, therefore as the analytical functional shape of the third equation in the (31) is unknown, this approach is siutable only in the flight conditions when $\beta = const$, or value of β is known.

The third approach starts from the assumption that the sideslip angle rate $\dot{\beta}$ is apriori known. According to the definition, the angle between velocity vector and its projection on the plane of symmetry Oxz is angle of sideslip β. Its rate $\dot{\beta}$ can be then assumed as the vector normal to the plane formed by β and lying in the Oxz. Its direction is normal to the projection of the velocity vector to Oxz, therefore, according to the definition of the stability coordinate system, $\dot{\beta}$ is always on its axes. As Oxz is coplanar with Opr, $\dot{\beta}$ can be expressed as the linear combination of angular velocities p and r, $\dot{\beta} = K_{\dot{\beta}p} p + K_{\dot{\beta}r} r$. The sideslip angle is rarely out of the range of $\pm 20^0$ so the approximations $\sin\beta \simeq \beta$ and $\cos\beta \simeq 1$ can be made. Putting the expression for $\dot{\beta}$ and approximations into the third equation in (31), the functional shape of sideslip angle becomes $\beta = \beta\left(V, \alpha, p, q, r, \theta, \varphi, \delta_l, \delta_m, \delta_n, T, K_{\dot{\beta}p}, K_{\dot{\beta}r}\right)$.

The unknown coefficients $K_{\dot{\beta}p}$ and $K_{\dot{\beta}r}$ can be determined from the assumption that value of $\dot{\beta}$ is known. Then the expression for β can be substituted into (32) to provide another analytical form of $\dot{E}_D$.

Let the third equation in (31) express in the form $\dot{\beta} = K_{\dot{\beta}\beta}\beta + K_{\dot{\beta}p}p + K_{\dot{\beta}r}r + K_{\dot{\beta}\delta_l}\delta_l + K_{\dot{\beta}\delta_n}\delta_n + \dot{\beta}^0$,

where the coefficients $K_{\dot{\beta}*}$ can be exactly calculated as the functions of variables used in the existing literature [13] in the form:

$$\begin{aligned}
K_{\dot{\beta}\beta} = &\left(\frac{QS}{mV}\right)[C_{Y\beta} + \left(C_{A\alpha}c\alpha + C_{N\alpha}s\alpha\right)\alpha \\
&+\frac{c}{2V}\left(C_{Aq}c\alpha + C_{Nq}s\alpha\right)q + \\
&+\left(C_{A\delta_m}c\alpha + C_{N\delta_m}s\alpha\right)\delta_m + \\
&+\left(C_{A0}c\alpha + C_{N0}s\alpha\right)] + \\
&+\frac{g}{V}\left(c\alpha s\theta - s\alpha c\varphi c\theta\right) + \frac{T}{mV}c\left(\alpha + \zeta_0\right)
\end{aligned}$$

$$K_{\dot{\beta}p} = \frac{QSb}{2mV^2}C_{Yp} + s\alpha; \quad K_{\dot{\beta}r} = \frac{QSb}{2mV^2}C_{Yr} - c\alpha;$$

$$K_{\dot{\beta}\delta_l} = \frac{QSb}{2mV}C_{Y\delta_l}; \quad K_{\dot{\beta}\delta_n} = \frac{QSb}{2mV}C_{Y\delta_n};$$

$$\dot{\beta}^0 = \frac{QS}{mV}C_{Y0} + \frac{g}{V}s\varphi c\theta.$$

In these expressions, the aerodynamic coefficients $C_{A\#}$ and $C_{N\#}$, $\# = \alpha, q, Y, \delta_m, 0$ are defined in the 'stability' axes coordinate system [13]. Then the angle of sideslip can be, by using this redefined expression for $\dot{\beta}$, written as

$$\beta = K_{\beta p}p + K_{\beta r}r + K_{\beta\delta_l}\delta_l + K_{\beta\delta_n}\delta_n + \beta^0. \tag{35}$$

The coefficients are $K_{\beta*} = K_{\beta*}\left(V,\alpha,q,\theta,\varphi,\delta_m,T\right) = -K_{\dot{\beta}*}/K_{\dot{\beta}\beta}$, $* = p, r, \delta_l, \delta_n$, whereas the free element $\beta^0 = \beta^0\left(V,\alpha,\dot{\beta},q,\theta,\varphi,\delta_m,T\right) = \left(\dot{\beta} - \dot{\beta}^0\right)/K_{\dot{\beta}\beta}$ is the function of the rate of sideslip angle, that is known by assumption. By substituting (35) into (32) the new form of the rate of rotatio-nal energy $\dot{E}_D$ along angular velocities p and r is:

$$\begin{aligned}
\dot{E}_D &= \mathfrak{M}_D^T\boldsymbol{\omega}_D = \\
&= \boldsymbol{\omega}_D^T\boldsymbol{\mathcal{M}}_D^\beta\boldsymbol{\omega}_D + b^\beta p + c^\beta r \le 0, \\
\boldsymbol{\mathcal{M}}_D^\beta &= \begin{bmatrix} L_p^\beta & L_r^\beta \\ N_p^\beta & N_r^\beta \end{bmatrix} \\
b^\beta &= \left(L_{\delta_l} + K_{\beta\delta_l}L_\beta\right)\delta_l + \\
&+\left(L_{\delta_n} + K_{\beta\delta_n}L_\beta\right)\delta_n + \\
&+\left(L_0 + L_\beta\beta^0\right) \\
c^\beta &= \left(N_{\delta_l} + K_{\beta\delta_l}N_\beta\right)\delta_l + \\
&+\left(N_{\delta_n} + K_{\beta\delta_n}N_\beta\right)\delta_n + \\
&+\left(N_0 + N_\beta\beta^0\right)
\end{aligned} \tag{36}$$

with coefficients in $\boldsymbol{\mathcal{M}}_D^\beta$ being

$$L_p^\beta = L_p + K_{\beta p}L_\beta; \quad L_r^\beta = L_r + K_{\beta r}L_\beta;$$
$$N_p^\beta = N_p + K_{\beta p}N_\beta; \quad N_r^\beta = N_r + K_{\beta r}N_\beta.$$

In the condition (36) angle of sideslip β is implicitly contained in the coefficients it contains, which is the main difference in relation to (32). Thus it permits more straightforward correlation between angular velocities p and r and airplane commands δ_l and δ_n, which together with (33) makes it suitable for the analysis of the stability conditions of the maneuvering airplane. When $\beta = 0$, as in coordinated turn, (36) becomes identical to (32).

The sign definiteness of (36) is subjected to the same rules, as is the case for (32). For the case when $b^\beta = c^\beta = 0$ the condition $\mathfrak{M}_D^T\boldsymbol{\omega}_D \le 0$ is fulfilled when the matrix $\boldsymbol{\mathcal{M}}_D^\beta$ is nonpositive, $\boldsymbol{\mathcal{M}}_D^\beta \le 0$, i.e. if and only if all main minors of this matrix are nonpositive. Therefore, the following set of conditions must be simultaneously fulfilled

$$L_p^\beta \le 0 \Rightarrow \quad \begin{cases} L_p + K_{\beta p}L_\beta \le 0 \\ L_p \le -K_{\beta p}L_\beta \end{cases}$$

$$L_p^\beta N_r^\beta - N_p^\beta L_r^\beta \le 0 \Rightarrow \begin{cases} L_pN_r - N_pL_r + \\ +K_{\beta r}\left(L_pN_\beta - N_pL_\beta\right) + \\ +K_{\beta p}\left(N_rL_\beta - L_rN_\beta\right) \le 0 \end{cases} \tag{37}$$

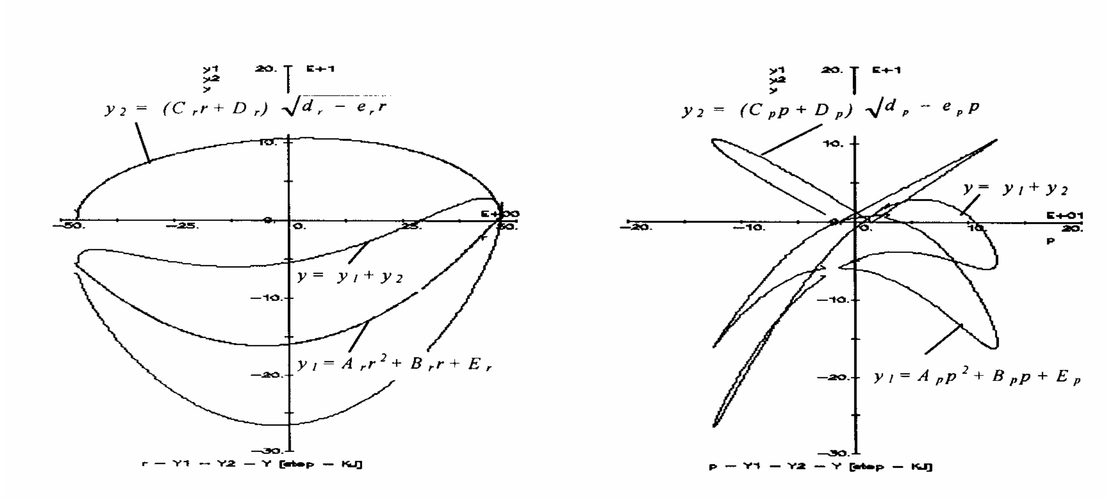

Figure 5. The variations of the rate of energy of rotation in planes $O\dot{E}_D p$ and $O\dot{E}_D r$.

In the same manner that was the case in (24) or (33) with matrix $\mathcal{M}_D$ the nonpositive sign definitness of matrix $\mathcal{M}_D^\beta$ defined in (37) presents only the necessary condition for the stable mode of lateral-directional rotation of the airplane, whereas the $\dot{E}_D \le 0$ or $\mathfrak{M}_D^T \boldsymbol{\omega}_D \le 0$ is the general one.

For the equilibrium straight stationary flight ($b=0$, $c=0$) and coordinated turn the condition (37) is simplified to $L_p \le 0$ and $L_p N_r - N_p L_r \le 0$. General stability condition for the lateral-directional motion (37) should be fulfilled within whole range of interest of the values of the angles of attack. Before defining stability conditions, let review that positive values of variables are: p - right wing down, r - nose to the right, β - velocity vector in the first octant of the airplane body system, $\delta_l, \delta_m, \delta_n$ - deflections generating local increment of lifting force. It must be noted that coefficients $K_{\beta p}$ and $K_{\beta r}$ are positive for adverse and negative for proverse motion. The second condition in (37) shall be absolutely fulfilled if all of the addends are simultaneously less than zero. The conditions than can be expressed as

$$
\begin{array}{ll}
\text{For adverse} & \text{For proverse} \\
L_p \le -K_{\beta p} L_\beta & L_p \ge -K_{\beta p} L_\beta \\
L_p N_r - N_p L_r \le 0 & L_p N_r - N_p L_r \le 0 \\
L_p N_\beta - N_p L_\beta \le 0 & L_p N_\beta - N_p L_\beta \le 0 \\
N_r L_\beta - L_r N_\beta \le 0 & N_r L_\beta - L_r N_\beta \le 0
\end{array}
\tag{38}
$$

The relations (37) and (38) are only necessary conditions obtained without taking into account relation (33). They are often met in the classical analysis of airplane rotation. Out of (33) the relations $p(r)$ or $r(p)$ can be obtained. Then, by substituting these relations into (36), the necessery and sufficient conditions for stoppage and prevention of the increase of the airplane rotation are obtained in the form of the following functions

$$
\begin{aligned}
\mathfrak{M}_D^T \boldsymbol{\omega}_D &= A_r^\beta r^2 + B_r^\beta r + E_r^\beta \pm \\
&\pm C_r^\beta r \sqrt{d_r - e_r r^2} \pm D_r^\beta \sqrt{d_r - e_r r^2} \\
&= A_p^\beta p^2 + B_p^\beta p + E_p^\beta \pm \\
&\pm C_p^\beta p \sqrt{d_p - e_p p^2} \pm D_p^\beta \sqrt{d_p - e_p p^2}
\end{aligned}
\tag{39}
$$

with diagrams given in the fig. 5 for the case $\delta_l \ne 0$. The coefficients in (39) are

$$
\begin{aligned}
A_r^\beta &= \frac{2I_{xz}^2 - I_x I_z}{I_x^2} L_p^\beta + \frac{I_{xz}}{I_x}\left(N_p^\beta + L_r^\beta\right) + N_r^\beta \\
B_r^\beta &= \frac{I_{xz}}{I_x} b^\beta + c^\beta \\
E_r^\beta &= \frac{R^2}{I_x} L_p^\beta \\
C_r^\beta &= 2\frac{I_{xz}}{I_x} L_p^\beta + N_p^\beta + L_r^\beta \\
D_r^\beta &= b^\beta \\
d_r &= \frac{R^2}{I_x} \ge 0 \\
e_r &= \frac{I_x I_z - 2I_{xz}^2}{I_x^2}
\end{aligned}
\tag{40}
$$

$$A_p^\beta = L_p^\beta + \frac{I_{xz}}{I_x}\left(N_p^\beta + L_r^\beta\right) + \frac{2I_{xz}^2 - I_x I_z}{I_z^2} N_r^\beta$$

$$B_p^\beta = b^\beta + \frac{I_{xz}}{I_x} c^\beta$$

$$E_p^\beta = \frac{R^2}{I_z} N_r^\beta$$

$$C_p^\beta = N_p^\beta + L_r^\beta + 2\frac{I_{xz}}{I_x} N_r^\beta$$

$$D_p^\beta = c^\beta$$

$$d_p = \frac{R^2}{I_z} \geq 0$$

$$e_p = \frac{I_x I_z - 2I_{xz}^2}{I_z^2}$$

For the cases of equilibrium horizontal flight and for the maneuvers in the vertical plane, coefficients b^β, c^β, B_r^β, D_r^β, B_p^β and D_p^β are approximately zero and then (39) becomes

$$\begin{aligned}\mathfrak{M}_D^T \boldsymbol{\omega}_D &= A_r^\beta r^2 + E_r^\beta \pm C_r^\beta r\sqrt{d_r - e_r r^2} = \\ &= A_p^\beta p^2 + E_p^\beta \pm C_p^\beta p\sqrt{d_p - e_p p^2}\end{aligned} \tag{41}$$

Because $\left\| A_*^\beta * + E_*^\beta \right\|$ is by order of magnitude greater than $\left\| C_*^\beta * \sqrt{d_* - e_* *^2} \right\|$, $* = p, r$, properties of $\mathfrak{M}_D^T \boldsymbol{\omega}_D$ are basically dominated by properties of expression $A_*^\beta * + E_*^\beta$, $* = p, r$, that is by the values of the moments of inertia, coefficients L_p^β, N_p^β, L_r^β, N_r^β and intensity of rotation R generated by longitudinal command. For the particular realization of the intensity of rotation R or R_D the values of angular velocities are constrained by equation (33). Then, out of the $\mathfrak{M}_D^T \boldsymbol{\omega}_D \leq 0$ the condition for the relation of the coefficients that are providing stable behavior of the airplane in the lateral directional mode of motion is obtained in the form

$$\begin{aligned}&\left[2\frac{I_z}{I_x} - 3\left(\frac{I_{xz}}{I_x}\right)^2\right] L_p^\beta - \frac{I_{xz}}{I_x}\left(N_p^\beta + L_r^\beta\right) \leq N_r^\beta \\ &N_r^\beta \leq \frac{1}{2\frac{I_x}{I_z} - 3\left(\frac{I_{xz}}{I_z}\right)^2} L_p^\beta + \frac{I_{xz}}{2I_x - 3\frac{I_{xz}^2}{I_z}}\left(N_p^\beta + L_r^\beta\right)\end{aligned} \tag{42}$$

The relation in (42) must be obtained simultaneously with $b^\beta = 0$ and $c^\beta = 0$ within whole range of interest of the values of the angles of attack.

If the relations $p(r)$ and $r(p)$ have been substituted into (32) instead of (36), the obtained expressions should be of the mathematical form identical to these in the equations (39) to (42), with the difference that the upper index $*^\beta$ would not exist. Then, for example, for the case of the airplane with centrifugal moment of inertia equal to zero, i.e. $I_{xz} = 0$, $I_x = J_1$ and $I_z = J_3$, the condition (42) is transformed to the simple form

$$2\frac{J_3}{J_1} L_p \leq N_r \leq \frac{1}{2}\frac{J_3}{J_1} L_p \tag{43}$$

that must bi fulfilled simultaneously with $b = 0$, $c = 0$ within whole range of interest of the values of the angles of attack.

Let remind that geometrical interpretation of (32) and (36) is the same as this for (24). For airplane fulfilling necessary stability condition for the lateral-directional mode of motion matrix $\boldsymbol{\mathcal{M}}_D$ is negative, as well as the matrix $\boldsymbol{\mathcal{M}}_D^\beta$. In maneuver of this airplane coefficients are $b \neq 0$, $c \neq 0$ or $b^\beta \neq 0$, $c^\beta \neq 0$ and if the energy of rotation is increasing, its rate is greater than zero, i.e. $\dot{E}_D = \mathfrak{M}_D^T \boldsymbol{\omega}_D > 0$. Then, the geometrical interpretation for $\dot{E}_D$ or $\mathfrak{M}_D^T \boldsymbol{\omega}_D$, regardless weather it is expressed through (32) or (36) is with three regions. First is positive, the second is the line of ellipse in plain Opr, while the third is negative. Therefore, the stoppage of rotation can be made if $\dot{E}_D = \mathfrak{M}_D^T \boldsymbol{\omega}_D = 0$ and simultaneously is fulfilled $b = 0$, $c = 0$ or $b^\beta = 0$, $c^\beta = 0$. Out of this condition is obtained the relation for aerodynamic coefficients

$$\begin{aligned}\left(L_{\delta_l} N_\beta - L_\beta N_{\delta_l}\right)\delta_l + \left(L_{\delta_n} N_\beta - L_\beta N_{\delta_n}\right)\delta_n + \\ + L_0 N_\beta - L_\beta N_0 = 0\end{aligned} \tag{44}$$

Deflecting control surface of either one or simultaneously both lateral-directional commands can fulfill this condition that the energy of rotation remains unchanged.

Let consider the case of extreme values of angular velocities p and r as they are defined on the ellipse of rotation (fig. 4) in the lateral-directional mode. For this case of angular velocities, the values of the deflections of the control surfaces δ_l and δ_n necessary to stop the increase of lateral-directional mode of rotation ($\mathfrak{M}_D^T \boldsymbol{\omega}_D = 0$) are easily determined by using expressions (32) and (33). In this case the relations in equations (39) and (40) are without upper index β, while the variables under square roots are zero, so the expression for $\mathfrak{M}_D^T \boldsymbol{\omega}_D$ is reduced to

$$\mathfrak{M}_D^T \boldsymbol{\omega}_D = A_r r + B_r r + E_r = A_p p + B_p p + E_p. \quad (45)$$

Assuming that the equilibrium flight is defined with zero values of lateral-directional control surface deflections and sideslip angle, after substitu-ting extreme values of angular velocities from (33) (fig. 4) and modified coefficients (40) into (45) and equalized with zero ($\mathfrak{M}_D^T \boldsymbol{\omega}_D = 0$) is obtained

$$\begin{aligned}
0 = &\underbrace{R\left(I_z - \frac{I_{xz}^2}{I_x}\right)^{-\frac{1}{2}}}_{r_{\max}} \left[\left(\frac{I_{xz}}{I_x}\right)^2 L_p + \frac{I_{xz}}{I_x}\left(N_p + L_r\right) + N_r\right] \pm \\
&\pm \left[\left(\frac{I_{xz}^2}{I_x} L_\beta + N_\beta\right)\beta + L_0 + N_0\right] \pm \\
&\pm \left[\left(\frac{I_{xz}^2}{I_x} L_{\delta_l} + N_{\delta_l}\right)\delta_l + \left(\frac{I_{xz}^2}{I_x} L_{\delta_n} + N_{\delta_n}\right)\delta_n\right] \\
0 = &\underbrace{R\left(I_x - \frac{I_{xz}^2}{I_x}\right)^{-\frac{1}{2}}}_{p_{\max}} \left[L_p + \frac{I_{xz}}{I_x}\left(N_p + L_r\right) + \left(\frac{I_{xz}}{I_z}\right)^2 N_r\right] \pm \\
&\pm \left[\left(L_\beta + \frac{I_{xz}^2}{I_z} N_\beta\right)\beta + L_0 + N_0\right] \pm \\
&\pm \left[\left(L_{\delta_l} + \frac{I_{xz}^2}{I_z} N_{\delta_l}\right)\delta_l + \left(L_{\delta_n} + \frac{I_{xz}^2}{I_z} N_{\delta_n}\right)\delta_n\right] \quad (46)
\end{aligned}$$

The first equation ($r_{\max}$) contains the relations that define the condition to prevent the increase of rotation in spin, while the second ($p_{\max}$) defines the condition to prevent the increase of rotation in roll. Within the boundaries of possible or permitted deflections of control surfaces, using either one, or simultaneously both lateral-directional commands can fulfill each of the conditions in (46). In the difference to (44), which defines only the relation among aerodynamic parameters, expression (46) defines the relation between aerodynamic parameters, inertial properties of the airplane and the intensity of rotation R. On the other side, for defined point within the state space of the airplane the efficiency of the control surfaces is constrained by aerodynamic shape and air stream properties, regardless to the possibilities of the airframe construction to deflect these surfaces. That means that control surfaces are generating limited the moments. Therefore, equation (46) enables reverse analysis. It can be used, for selected point in the airplane state space, to determine maximal value of intensity of rotation $R_{\max}$ or $R_{D\max}$ that can be compensated by any combination of control surface deflections of lateral-directional commands. For the case when $R > R_{\max}$, it is not possible to stop the increase of rotation by lateral-directional controls. Then, it is possible to prevent the increase of intensity of rotation by using longitudinal command to either reduce the intensity of rotation or to shift the airplane to the point within its state space where the constraints of lateral-directional command are providing greater limits to control moments. In most cases longitudinal command is performing both of these tasks simultaneously.

7 Conclusion

Presented new methodology to analyze dynamic behavior of the rotating airplane within whole range of angle of attack is based on the properties of the rate of kinetic energy of rotation. It has been shown that the character of the rotational motion of the airplane as the rigid body is determined by the rate of kinetic energy of rotation. The rate of kinetic energy of rotation is equal to the derivative of energy of rotation along the vector of angular velocity and by applying principles of classical mechanic and using full nonlinear model of airplane motion it is proofed that it presents scalar product of the vector of moment and vector of angular velocity of the airplane. Then, the exactness of the presented method depends only upon exactness of the source of moments that are applied in it. The concept of the model of locally linearized aerodynamic moment has been introduced with local applicability of this model within whole range of angles of attack. The application of this model enables simultaneous separated observation of longitudinal and lateral-directional motion. It has been also shown that, within the stringent constraints of the classical approach based on the separated modes of longitudinal and lateral-directional motion, there exists the full congruency of partial results of here presented method and the results of the classical one. On the other side, the general applicability of this method has been shown in the areas where the classical one fails to produce valid results. One is the analysis of the lateral-directional rotation generated by the longitudinal command. The other is the definitions of these conditions needed to prevent the increase and stop the lateral-directional rotation of the airplane that are applicable within the whole range of interest of the values of the angle of attack. The results in the second area are producing new conditions and relations between values of aerodynamic and inertial parameters of the airplane, on one side, and the intensity of the airplane rotation on the other. To apply exposed method, it is sufficient to know the mathematical apparatus of algebra and functional relations of the aerodynamic moments (values of the aerodynamic parameters) within the whole range of interest of the flight regimes.

References

[1] Beaurain,L., *Influence of light aircraft architecture on its spin characteristics*. Tech. Reo. AAAF-NT-80-11, Institute de Mechanique de Fluides de Lille, Lille (France), 1979.

[2] Kaus, W., *Aerodynamic development of the X-31 for the high angle of attack region*. Tech. Rep. MBB-FE211-S-PUB-452; ETN-92-90620, Messerschmit-Boelkow-Blohm G.m.b.H., Munich, 1991

[3] Mehre, R., Carroll, J., *Global stability and control analysis of aircraft at high angles-of-attack.* Tech. Rep. ONR-CR215-245-3, Scientific Systems Inc., Cambridge MA., USA, 1979.

[4] Tobak, M., Chapman, G., *Nonlinear problems in flight dynamics involving aerodynamic bifurcations*. Tech. Rep. NASA-TM-86706, NASA, Ames Research Center, Moffet Field, CA., USA, 1981.

[5] Stiefel, J., *Math model for NADC spin simulator and F-14 high angle of attack data base*. Tech. Rep. NADC-80220-60-VOL-1, Naval Air Development Center, Aircraft and Crew Systems Technology Directorate, Earminster, PA., USA, 1981.

[6] Hill, S., Martin, C., *Flight dynamics model of aircraft spining*. ARL-FLIGHT-MECH-R-180; DODA-AR-005-600, Aeronautical Reserch Labs., Melbourne, (Australia), 1990.

[7] Zivanovic, M., *Predlog postupka za projektovanje klase sistema automatskog upravljanja*. Zbornik radova III konferencije SAUM, Vrnjačka Banja, SRJ, oktobar 1989, p. 108-122.

[8] Etkin, B., *Dynamics of Atmospheric Flight*., John Wiley & Sons Inc, New York/London/Sidney, 1972.

[9] Blakelock, J., *Automatic Control of Aircraft and Missiles*. New York, John Wiley & Sons, 1965.

[10] Lebedev. A., Chernobrovkin, L., *Dinamika poleta bespilotnih letalnih aparatov*. Gosudarstvennoe nauchno-tehnicheskoe izdatelstvo, OBORONGIZ, Moskva, 1962.

[11] Bellman, R., *Introduction to Matrix Analysis*. Mc Graw-Hill Book Company, Inc. New York – Toronto – London, 1960.

[12] Bajovic, M., *Analiza kretanja aviona na velikim napadnim uglovima*. Tech. Rep. V3-1980, VTI Zarkovo, Zarkovo, Beograd, 1981.

[13] Maine, R., Iliff, K., *Application of parameter estimation to aircraft stability and control: the output-error approach*. Tech. Rep. NASA-RP-1186, NASA, 1986.

[14] Bajovic, M., *Novi prilaz u resavanju problema dinamike leta letelica sa stanovista primene kvaliteta leta*. PhD Thesis, Faculty for Mechanical Engeneering, University of Belgrade, Belgrade, 2002.