THE CONTINUATION DESIGN FRAMEWORK FOR NONLINEAR ENTOMOPTER CONTROL

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Abstract

The use of nonlinear dynamics theory for the analysis of flying vehicles motion and the assessment of aircraft control systems is well known. In this paper the continuation and bifurcation methods are applied to flapping wings Micro Air Vehicle nonlinear control design problems, particularly to the design of a global stability augmentation system for ensuring the steady flight in a turbulent atmosphere. The motion is performed by using bifurcation diagrams for equilibrium and periodical modes. The nonlinear control law is derived, taking into account both local stability characteristics of entomopter equilibrium states and domains of attraction, along with the requirement that all other attractors are eliminated.

1 General Introduction

The development of small (less than 6 inches, or hand-held) autonomous flying vehicles is motivated by a need for intelligent reconnaissance robots, capable of discreetly penetrating confined spaces and maneuvering in them without the assistance of a human telepilot. The ability to perform agile flight inside buildings, stairwells, ventilation systems, shafts and tunnels is of significant military and civilian value. The vehicles will fill the gap in the short-distance (less than ten miles) surveillance capabilities, not covered by today's satellites and spy planes. Such capabilities will be useful in battlefields (especially in urban warfare) and against terrorists. The vehicles can also be used in dull, dirty or dangerous (D^3) environments, where direct or remote human assistance is not feasible. Non-military uses of autonomous micro-air vehicles will, in time, exceed in scope and scale the defence applications. They will become standard equipment for law enforcement and rescue services. The ability to explore D^3 environments without human involvement will be of great interest for many industries - the vehicles will allow air quality sampling in nonattainment areas, utility inspection (power lines, oil pipes), examination of human-inaccessible confined spaces in buildings, installations and large machines.

Fig. 1 Shapes of butterfly wings during flight

Insects are capable of vertical take-off and landing and very agile maneuvers. To a first approximation, kinematic control of insect flight maneuvers is provided by changes in the tilt of the stroke plane, which is analogous to helicopter control (Ref. 21, 23, 24]. However, a detailed analysis reveals more subtle mechanisms at work, especially asymmetries in the wingbeats. This includes inter-wing differences in: the magnitude of force production, timing of the downstroke-to-upstroke wing rotation and the geometric position of the wings when the rotation occurs. This is abetted by the ability to control wingbeat amplitude and rotation timing almost independently, and by the presence of sensors on the wings [see Refs 2, 3, 9].

The remarkable maneuverability of insects is enabled by active control of the three-dimensional shape of the wings during the beat cycle. Insect wings combine the features of levers, oscillating aerofoils and cantilevered beams and must be able to perform and withstand shifting patterns of bending and twisting forces. This is achieved by deformability of the wings, realized through a composition of supporting (deformation-limiting, or stiff) and passively deformable (elastic) areas, whose behavior is monitored (and to some extent influenced) by a wide range of surface structures. The observed patterns of deformation (both active and passive) include: torsion, camber change and transverse bending. The insect wing is an integrated structure, combining sensors, materials and actuators in order to achieve structural control and adaptability required by highly maneuverable flapping flight [22].

The application of conventional aerodynamic theory to the flapping wing motion of insects predicts forces that are too low to keep the animal aloft. This failure of conventional steady-state theory has fuelled the search for unsteady mechanisms that might account for the elevated performance of insect wings. Direct measurement of the forces and flows produced by a flapping wing suggests that the aerodynamics of insect flight may be explained by the interaction of three distinct, yet interactive mechanisms: delayed stall, rotational circulation, and wake capture. While delayed stall is a translational mechanism, rotational circulation and wake capture depend explicitly on the rapid rotation of the wings during stroke reversal. The regulation of rotational phase provides insects with a potent means of controlling flight forces during steering maneuvers. A general theory of insect aerodynamics that incorporates both translational and rotational mechanisms shows promise in explaining the force generating mechanisms of many species as well providing insight for the design of biomimetic robots [10].

In the background of preparing this paper lies our believe that transferring ideas from the more matured disciple like aircraft technology to emerging animal technology should be beneficial for the later one and vice-versa. One integrated idea, of special interest to both disciplines, is the *active flexible wing* concept. This concept represents a return to the Wright Brothers' idea of wing warping or twisting by combining wing structures and flight controls to perform the desired maneuvers [22].

It is well known that the dynamics of entomopter over the flight envelope is highly nonlinear. The character of the loads acting on the vehicle - particularly the aerodynamics - vary substantially over the angle of attack operating range (which may include poststall incidences). The control of this type of plant can be achieved adequately via a variety of approaches, provided that the parameters of the controller (the gains in particular) are scheduled with flight condition. The nonlinearity of the system makes it difficult to implement a strategy of interpolating between gains derived from a few choice trim points. This is because the plant and the controller interact such that it is not clear precisely what the closed loop trim points are in vide flight regions, because aerodynamic loads often become asymmetric and where inertial coupling is significant.

What is required at the design stage is a means of continuously evolving the control laws in conjunction with the plant dynamics over a wide region of the flight envelope, whilst achieving some desired control goal.

In this paper it is used a methodology called *the continuation design framework* (CDF) (This methodology is precisely described in Refs [1, 4, 6, 7, 13, 25, 27, 28, 31, and 32].

Continuation methods are a class of numerical algorithm that can be used directly in the design of control law gain schedules, for example. Due to the versatility of the technique, this can be done in various ways, incorporating various analysis tools [5, 12, 14, 17, 19, 26, 29, and 30]:

- Direct inversion of the equations, e.g. to find variations in control surfaces to obtain desired state variable relationships. This has been used in the so-called "bifurcation tailoring" approach, in which control schedules are derived so as to effect a particular shape to the bifurcation diagram.
- Addition of constraint equations to the dynamical system. This is where the continuation method is implemented on the whole differential-algebraic system (although the bifurcation theory apply only to the differential equations). The additional equations may make reference to properties of the dynamical system (e.g. eigenvalues, handling quality parameters, or combinations of states to constrain the solution to, say, a velocity vector roll) or they may refer to auxiliary measures (e.g. a model reference database, frequency or time response criteria). Each additional equation allows an additional parameter to be solved for. Examples of this were reported in [ccc], applied to nonlinear dynamic inversion and also to direct eigenvalue placement.
- Combining the dynamical system with a controller design method that is traditionally applied to the linearized system at selected trim points. The method may be explicit, with a unique solution at each steady state point (such as the eigenstructure assignment example that follows) or implicit, requiring iterations until a specified condition is met. This offers an advantage relative to the addition of, for example, a cost function as an algebraic equation: the system may be redundant (allowing determination of a large number of unknowns, such as control law gains). The intention of CDF is to create a suite of interacting tools, linked to the continuation method, by which a multitude of design problems can be tackled in a modular and transparent manner. An important feature of this approach is that the model is defined only *once* and accessed by all the tools, thus obviating the difficulties experienced by practicing engineers when different computer packages use different model definitions.

The CDF approach used in this work uses a Fortran-based continuation method (basing on XPPAUT software) and MATLAB as the basis for most of the tools. The mathematical model of the dynamical system is defined within MATLAB and the graphical user interface tools will be also exploited.

2 Theoretical Background

2.1 Dynamical Systems Theory

In this paper we will study equations of the following form:

$$\dot{x} = f(x,t;\mu) \qquad (1)$$

and

$$x \mapsto g(x;\mu) \qquad (2)$$

with $x \in U \subset \Re^n$ $, t \in \Re^1$ $, and$ $\mu \in V \subset \Re^p$, where U and V are open sets in $\Re^n$ and $\Re^p$, respectively. We view the variables x as a vector of n state variables, the variables μ as a vector of m parameters (or controls), $\dot{x}$ is the time derivative of x and $f: \Re^n \times \Re^m \to \Re^n$ is the smooth vector field (the n non-linear functions). Note that both open loop (uncontrolled) and closed loop rigid-body flight dynamical systems can usually be represented in the form of equation (1), and referred to [15] as a *vector field* or ordinary differential equation and to (2) as a *map* or *difference equation*. Both are termed *dynamical systems*.

By a solution of Eq. (1) we mean a map, x, from some interval $\Im \subset \Re^1$ into $\Re^n$, which we represent as follows

$$x: \Im \to \Re^1, \qquad (3)$$
$$t \mapsto x(t)$$

such that $x(t)$ satisfies (1), i.e.,

$$\dot{x}(t) = f(x(t),t;\mu) \qquad (4)$$

Dynamical systems theory (DST) provides a methodology for studying systems of ordinary differential equations. The most important ideas of DST used in the paper will be introduced in the following sections. More information on DST can be found in the book of Wiggins [33]. The first step in the DST approach is to calculate the steady states of the system and their stability. Steady states can be found by setting all time derivatives equal to zero and solving the resulting set of algebraic equations. The Hartman-Grobman theorem (p. 234 in reference [33] proves that the local stability of a steady state can be determined by linearizing the equations of motion about the steady state and calculating the eigenvalues. The implicit function theorem (Ioos and Joseph [18], in Chap.2) proves that the steady states of a system are continuous function of the parameters of the system at all steady states where the linearized system is non-singular. A singular linearized system is characterised by a zero eigenvalue. Thus, the steady states of the equations of motion for an aircraft are continuous functions of the control surface deflections and/or vector of the thrust inclinations. Stability changes can occur as the parameters of the system are varied in such a way that the real parts of one or more eigenvalues of the linearized system change sign. Changes in the stability of a steady state lead to qualitatively different responses for the system and are called bifurcations. Stability boundaries can be determined by searching for steady states, which have one or more eigenvalues with zero real parts. There are many types of bifurcations and each has different effects on the vehicle response. Qualitative changes in the response of the entomopter can be predicted by determining how many and what types of eigenvalues have zero real parts at the bifurcations point. Bifurcations for which one real eigenvalue is zero lead to the creation or destruction of two or more steady states. Bifurcations for which one pair of complex eigenvalues has zero real parts can lead to the creation or destruction of periodic motion. Bifurcations for which more than one real eigenvalue or more than one pair of complex eigenvalues has zero real parts lead to very complicated behaviour. Continuation methods are a class of numerical algorithm used to follow a path of steady states in continuous or discrete dynamical systems as a parameter varies. They make use of the Implicit Function Theorem, which essentially states that if the Jacobian matrix **J** (5) of the system linearized at a stationary point is non-singular then this solution is *locally* unique, i.e. it is part of a unique curve of stationary points which is a continuous function of the parameters. Parametric continuation methods are used in the numerical application of bifurcation theory. The associated theorems involve properties of the eigenvalues at steady state solutions points (or Floquet multipliers for periodic orbit solutions), and it is therefore useful in bifurcation analysis to solve for all relevant solution branches within a state-parameter space whilst evaluating the eigensystem as the algorithm proceeds. It is this characteristic of continuation methods that make them suitable for the "global" control law design task at hand: the steady states provide a substantial amount of information about the mechanics governing system response - including that of the closed-loop controlled system. The Jacobian matrix of an equilibrium point x_0 of a vector field or the fixed point x_0 is the matrix

$$\mathbf{J} = Df(x_0) = \begin{Bmatrix} \frac{\partial f_1}{\partial x_1} & \cdot & \cdot & \frac{\partial f_1}{\partial x_1} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \frac{\partial f_n}{\partial x_1} & \cdot & \cdot & \frac{\partial f_n}{\partial x_n} \end{Bmatrix} \quad (5)$$

The eigenvalues of the Jacobian matrix are important for the stability analysis. The following notations are used:

Vector Fields: n_0=number of eigenvalues of Df_0 with zero real part; n_+=number of eigenvalues of Df_0 with positive real part; n_-=number of eigenvalues of Df_0 with negative real part.

Maps: n_0=number of eigenvalues of Df_0 on unit circle; n_+=number of eigenvalues of Df_0 outside the unit circle; n_-=number of eigenvalues of Df_0 inside the unit circle. An equilibrium or fixed point is called hyperbolic if n_0=0, that is, it has no overvalues on the imaginary axis.

A continuation method is used to explore the nature of the system steady states as a parameter varies. In this context, steady states may refer to standard *equilibria* (stationary trim points such as steady level flight, steady climbs and descents, steady turns and spins) or to *periodic orbits orbits* (limit cycles such as oscillatory motion). The evolution of branches of equilibria are computed by selecting one of the *m* controls/ parameters as the "free" parameter (or continuation parameter), and then solving for:

$$\dot{x} = f(x,\mu) = 0, \qquad x \in \Re^n \qquad \mu \in \Re \qquad (6)$$

where μ is one of the members of μ. In the work presented here, a continuation method based on that of [15] is used. Details on bifurcation theory can be found in, for example, Ref. [8, 11, 15, 20, 33].

2.1.1. Methodology scheme

Taking into account experience of many researches, one can formulate the following tree-step methodology scheme (being based on bifurcation analysis and continuation technique) for the investigation of nonlinear aircraft behaviour [5, 17]:

- During the first step it is supposed that all parameters are fixed. The main aim is to search for all possible equilibria and closed orbits, and to analyze their local stability. This study should be as thorough as possible. The global structure of the state space (or *phase portrait*) can be revealed after determining the asymptotic stability regions for all discovered attractors (stable equilibria and closed orbits). An approximate graphic representation plays an important role in the treating of the calculated results.
- During the second step the system behaviour is predicted using the information about the evolution of the portrait with the parameters variations. The knowledge about the type of encountered bifurcation and current position with respect to the stability regions of other steady motions are helpful for the prediction of further motion of the aircraft. The rates of parameters variations are also important for such a forecast. The faster the parameter change, the more the difference between steady state solution and transient motion can be observed.
- Last, the numerical simulation is used for checking the obtained predictions and obtaining transient characteristics of system dynamics for large amplitude state variable disturbances and parameter variations.

2.1.2. Steady state conditions

Bifurcation Theory is a set of mathematical results, which aims at the analysis and explanation of unexpected modifications in the asymptotic behaviour of non-linear differential systems when parameters are slowly varying. For a fixed control vector u, two types of asymptotic states are commonly encountered. The following relation gives the first:

$$f(x,\mu) = 0 \qquad (6)$$

This relation is named steady state. The second relation is given by the equation:

$$x(T) = x(0) + \int_0^T f(x,\mu)dt \qquad (7)$$

Starting with an approximation of a steady state for a given value of parameters, the computer code determines, by a continuation process, the solution curve $x(\mu)$ of a following set of non-linear algebraic equations, and determine type of bifurcation:

Equilibrium points	$f(x,\mu)=0$
Limit points	$f(x,\mu)=0$
	$\lambda=0$
Hopf points	$f(x,\mu)=0$
	$\lambda_{1,2}=\pm 2i\pi/T$
Periodic orbits	$x(T)=x(0)+\int_0^T f(x,\mu)dt$

The continuation process assumes that all functions for (6) are continuous and have derivatives.

The set of ordinary differential equations can be solved using the continuation and bifurcation software XPPAUT, a Windows version of a well known package AUTO97 [8, 11]. This very useful freeware gives all desired bifurcation points for different values of control vector components.

3 Mathematical model

Entomopter structures is continuous systems with spatially distributed dynamic properties. General mathematical model of such structures has the form of boundary value problem:

$$Aw = F, \quad w \in \Omega \tag{8}$$

$$B_j w\big|_{\partial\Omega} = g_j, j = 1,\dots,s \tag{9}$$

where: A – nonlinear, nonsteady differential operator, w – state variable; F – external loads; B – boundary value operator; g_j –functions defining the boundary conditions.

Such a general description is useful for problem formulation but it must be precised and simplified for obtaining useful results.

The standard model for elongated structures (such as airplane or animal wings) is a one dimension beam, described in the lack of damping by partial differential equation:

$$\mu(x)\ \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left[EI(x)\ \frac{\partial^2 w}{\partial x^2}\right] = P_z(x,t) \tag{15}$$

where $w(x,t)$ - beam deflection; μ - mass distribution along the span, $EI(x)$ - bending stiffness distribution along the beam span, $P_z(x,t)$ - external load distribution, t - time, x - spatial variable.

To complete the model the boundary conditions must be added to (10). For a cantilever model they take the form:

$$w(0,t) = 0\ , \frac{\partial w(0,t)}{\partial x} = 0\ , \qquad \frac{\partial^2 w(l,t)}{\partial x^2} = 0, \quad \frac{\partial^3 w(l,t)}{\partial x^3} = 0 \tag{16}$$

where l is the beam length.

As the control theory for continuous systems is not sufficiently developed for direct applications, a discretization for continuous models is applied, usually by using the Galerkin method. In this method, the resulting aeroelastic displacements at any time are expressed as a function of a finite set of selected modes:

$$w(x,t) = \sum_{i=1}^{N} \phi_i(x)\ \ q_i(t) \tag{12}$$

where: $\phi_i(x)$ - coupled mode shapes for all deformations beam eigenmodes; $q_i(t)$ - normal coordinates.

After discretization the final matrix form of the aeroelastic equations of motion is

$$\mathbf{M}\ \ddot{\mathbf{q}} + \mathbf{K}\ \mathbf{q} = \mathbf{F}(t) \tag{13}$$

where: $\mathbf{M}$ - matrix of generalized masses

$$M_j = \int_0^l \phi_j^2\ \mu\, dx \tag{14}$$

$\mathbf{K}$ - matrix of generalized stiffness

$$K_j = M_j\, \omega_j^2 \tag{15}$$

and $\mathbf{F}$ - vector of generalised forces

$$F_j(t) = \int_0^l P_z(x,t)\, \phi_j\, dx \tag{16}$$

This technique may be applied to more complex systems given, as we see in Sec. “Mathematical model of an entomopter”.

3.1. Control system design

The active control approach in this study is based on the deterministic linear optimal regulator problem. Results for this full state feedback controller are used as baseline against which other controllers are evaluated [xxx].

We present first the standard feedback design methodology. For control application the system (4) is transformed to a system of linear, first order differential equations in state and control variables:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{17}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} \tag{18}$$

where $\mathbf{x}(n \times 1)$ - vector of state variables, $\mathbf{u}(r \times 1)$ - vector of control variables, $\mathbf{y}(m \times 1)$ - the vector of system outputs, and **A, B** and **C** are state, control and outputs constant matrices of appropriate dimensions.
The objective is now to find control **u** that is the control input to the animal wings. Control system design is described as minimization of *performance index* (called sometimes the *quadratic cost function*) in the form:

$$I = \tfrac{1}{2}\int_{0}^{\infty}\left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right)\, dt \qquad (19)$$

where $\mathbf{Q}(n \times n)$ is non-negative and $\mathbf{R}(m \times m)$ is positive definite symmetric weighting matrix.
Applying calculus of variations for minimisation of performance index (24), the feedback control law is obtained in the form:

$$\mathbf{u}(t) = -\mathbf{R}^{-1}\mathbf{B}^T\mathbf{S}\mathbf{x}(t) \qquad (20)$$

A constant, positive-definite symmetric matrix **S** in the feedback gain matrix is obtained as a solution of matrix algebraic Riccati equation

$$\mathbf{S}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{S} - \mathbf{S}\mathbf{A} - \mathbf{A}^T\mathbf{S} - \mathbf{C}^T\mathbf{Q}\mathbf{C} = 0 \qquad (21)$$

Generally solution of Eq. (21) requires sophisticated numerical methods.
The resulting closed-loop dynamics equation is then defined as:

$$\dot{\mathbf{x}} = \mathbf{L}\mathbf{x},\ \ \mathbf{L} = \mathbf{A} + \mathbf{B}\mathbf{F} \qquad (22)$$

where the feedback control matrix has the form:

$$\mathbf{F} = -\mathbf{R}^{-1}\mathbf{B}^T\mathbf{S} \qquad (23)$$

For a controllable system such a solution yields to a stable closed-loop system, i.e. the eigenvalues $\lambda_j(\mathbf{L}),\ j = 1,\dots,n$ of **L**,

$$\operatorname{Re}\ \lambda_j(\mathbf{L}) < 0 \qquad (24)$$

lie in left-half plane of the complex plane.
This method is referred as the *linear regulator problem*. By this method systems may be stabilised in the range of parameters essential for the system application.

4 Mathematical model of an entomopter

The dynamical description of a flexible structure must accurately represent all structural characteristics by relating dynamic responses at specific locations throughout the structure to forces acting on the system. Mathematically speaking, the motion of animals with distributed elastic parts can be described by a set of ordinary differential equations for the rotational motion of a given reference frame, and a set of partial differential equations for the elastic motion relative to that frame. Such a system of differential equations is known as a *hybrid dynamical system*.

Often, in the space industry, the analysis of unmanned spacecraft with flexible appendages begins with the assumption that the attitude and the vibrational motions of the spacecraft are uncoupled. Such analyses are performed for bounding the spacecraft jitter due to instrument disturbances.

For our study the motion of the animalopter will be represented in the most general way by the differential equations:.

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}) \qquad (25)$$

where : **x** – is the state vector, **u** – is the control vector. For rigid MAV **x**=[*U, V, W, P, Q, R, Θ, Φ, Ψ*]^T where *U, V, W* are the forward, side, and yawing velocities of the animalopter; (*P., Q, R* are the angular velocities, roll, pitch, and yaw, *Θ, Φ,* and *Ψ* are roll and pitch angels. In Ref. 20, 21, 22 vector **u** was taken in the form $\mathrm{u} = [\delta, \gamma, \omega, \lambda]^T$ where: γ — feathering angle of wings, δ — flapping angle of wings, ω — frequency of wing motion respect to the body, λ - phase shifting between feathering and flapping.

We propose to derive the system equations of motion by means of Gibbs-Appel equations procedure. Those equations have the following form:

$$\frac{d}{dt}\left(\frac{\partial S}{\partial \ddot{\mathbf{q}}}\right) = \mathbf{Q} \qquad (26)$$

where: **q** - is the vector of generalized co-ordinates; $S(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}, t)$ - is so called Appel function, or functional of accelerations.

Functional S for i-th element of the mechanical system is given by the equation [16]:

$$S = \frac{1}{2}\iiint_{V^i} \dot{\mathbf{v}}^i \circ \dot{\mathbf{v}}^i dm^i \qquad (27)$$

where: $\dot{\mathbf{v}}^i$ means the vector of absolute acceleration of elementary mass dm^i of i-th body of the dynamical system considered (Fig. 2).

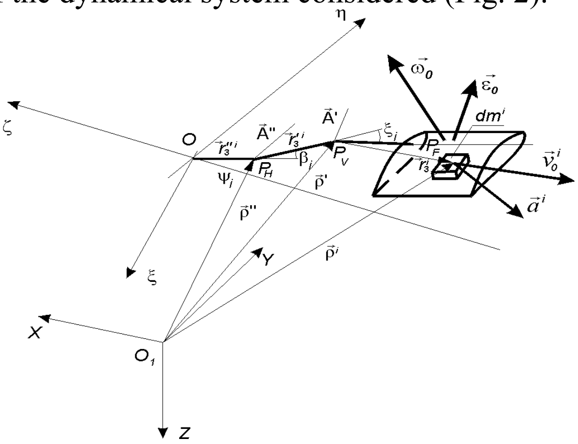

Fig. 2 Location of points, radius vectors and vectors of velocities and accelerations

$$\dot{\mathbf{v}}^i = \dot{\mathbf{v}}_0^i + \boldsymbol{\varepsilon}_0^i \times \boldsymbol{\rho}^i + \boldsymbol{\omega}_0^i \times \left(\boldsymbol{\omega}_0^i \times \boldsymbol{\rho}^i\right) \qquad (28)$$

Assuming that:

$$\mathbf{r}^i = \mathbf{r}_3^{\prime\prime i} + \mathbf{r}_3^{\prime i} + \mathbf{r}_3^i \qquad (29)$$

and

$$\mathbf{M}^i = \begin{bmatrix} m^i\mathbf{I} & m^i\tilde{\mathbf{r}}^{iT} \\ m\tilde{\mathbf{r}}^i & \mathbf{J}_O^i \end{bmatrix} \qquad (30)$$

$$\mathbf{h}^i = \begin{bmatrix} m^i\tilde{\boldsymbol{\omega}}_0 + m^i\tilde{\boldsymbol{\omega}}_0^2\mathbf{r}^i \\ m^i\tilde{\mathbf{r}}^i\tilde{\boldsymbol{\omega}}\mathbf{v}_0^i + \tilde{\boldsymbol{\omega}}_0\mathbf{J}_O^i\boldsymbol{\omega}_0 \end{bmatrix} \qquad (31)$$

where m^i – mass of the i-th element, J_O^i tensor of inertia of the i-th element, ω vector of the angular velocity, v_0^i vector of the velocity of the i-th element, and assuming that:

if $\mathbf{a} = \left[a_\xi, a_\eta, a_\varsigma\right]^T$, than $\tilde{\mathbf{a}} = \begin{bmatrix} 0 & -a_\varsigma & a_\eta \\ a_\eta & 0 & -a_\xi \\ -a_\eta & a_\xi & 0 \end{bmatrix}$

the term (31) can be expressed in the following matrix form:

$$S^i = \frac{1}{2}\left[\dot{\mathbf{v}}^i + \left(\mathbf{M}^i\right)^{-1}\mathbf{h}^i\right]^T \mathbf{M}^i \left[\dot{\mathbf{v}}^i + \left(\mathbf{M}^i\right)^{-1}\mathbf{h}^i\right] \qquad (37)$$

Calculating the matrixes $\mathbf{M}^i$, $\mathbf{h}^i$, the Appel function S^i for all k bodies of the system, and defining matrixes:

$$\mathbf{M} = diag\left[\mathbf{M}^1, \mathbf{M}^2, \ldots \ \ldots, \mathbf{M}^k\right] \qquad (32)$$

$$\mathbf{v} = \left[\left(\mathbf{v}^1\right)^T, \left(\mathbf{v}^2\right)^T, \ldots \ \ldots, \left(\mathbf{v}^k\right)^T\right]^T \qquad (34)$$

$$\mathbf{h} = \left[\left(\mathbf{h}^1\right)^T, \left(\mathbf{h}^2\right)^T, \ldots \ \ldots, \left(\mathbf{h}^k\right)^T\right]^T \qquad (35)$$

functional S for the whole mechanical system is given by the equation:

$$S = \frac{1}{2}\left(\dot{\mathbf{v}} + \mathbf{M}^{-1}\mathbf{h}\right)^T \mathbf{M}\left(\dot{\mathbf{v}} + \mathbf{M}^{-1}\mathbf{h}\right) \qquad (35)$$

Assuming that $\mathbf{q}$ is vector generalized coordinates of mechanical system, the relations between $\mathbf{q}$ and $\mathbf{v}$ are given by equation:

$$\mathbf{v} = \mathbf{D}\left(\mathbf{q}, t\right)\dot{\mathbf{q}} + \mathbf{f}\left(\mathbf{q}, t\right) \qquad (36)$$

hence:

$$\dot{\mathbf{v}} = \mathbf{D}\left(\mathbf{q}, t\right)\ddot{\mathbf{q}} + \boldsymbol{\varphi}\left(\mathbf{q}, \dot{\mathbf{q}}, t\right) \qquad (37)$$

where:

$$\varphi = \dot{\mathbf{D}}\dot{\mathbf{q}} + \dot{\mathbf{f}}$$

Therefore the Appel function can be expressed by following relation:

$$S\left(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}, t\right) = \\ = \frac{1}{2}\left(\mathbf{D}\ddot{\mathbf{q}} + \varphi + \mathbf{M}^{-1}\mathbf{h}\right)^T \mathbf{M}\left(\mathbf{D}\ddot{\mathbf{q}} + \varphi + \mathbf{M}^{-1}\mathbf{h}\right) \qquad (38)$$

Assuming, that:

$$\mathbf{M}_g = \mathbf{D}^T\mathbf{M}\mathbf{D} \text{ and } \mathbf{h}_g = \mathbf{D}^T\left(\mathbf{M}\varphi + \mathbf{h}\right)$$

and remembering, that: $\mathbf{D}\left(\mathbf{D}^T\mathbf{M}\mathbf{D}\right)^{-1}\mathbf{D}^T = \mathbf{M}^{-1}$

the equation (14) can be expressed in the form:

$$S(\mathbf{q},\dot{\mathbf{q}},\ddot{\mathbf{q}},t)= =\frac{1}{2}\left(\ddot{\mathbf{q}}+\mathbf{M}_g^{-1}\mathbf{h}_g\right)^T\mathbf{M}\left(\ddot{\mathbf{q}}+\mathbf{M}_g^{-1}\mathbf{h}_g\right) \tag{39}$$

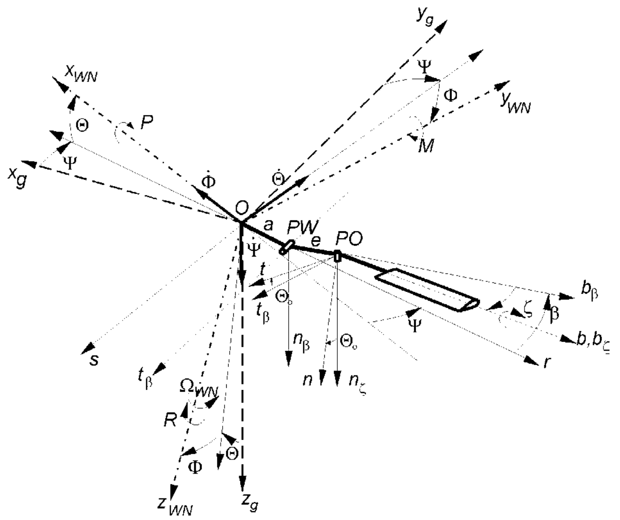

Fig. 3 Systems of co-ordinates attached to flapping wing of MAV

Non-linear equations of motion of rotorcraft-slung load system and the kinematics relations are expressed using moving co-ordinate systems. It is applied the following systems of co-ordinates (Fig. 3):

- Systems of co-ordinates attached to the aircraft, the common origin of which is located at the arbitrary accept point inside aircraft body.
- System of co-ordinates $Ox_{wn}y_{wn}z_{wn}$, the origin of those system overlap a wings roots, all axis are parallel to the system of co-ordinates *Oxyz* attached to aircraft.
- Systems of co-ordinates attached to the flapping wing. The wing is mounted to the hub on a universal join – free to flab (flapping hinge *PW*, system of co-ordinates $PWt_\beta b_\beta n_\beta$), lead or lag (lag hinge *PO*, system of co-ordinates $POt_\zeta b_\zeta n_\zeta$), but fixed in pitch (feathering hinge, system of co-ordinates *POtbn*).

In case when we consider model of a flapping wings MAV treated as mechanical system containing rigid fuselage and 2 rigid wings fixed to the fuselage by means of two hinges, (considering the flapping and feathering hinges only), the vector generalized co-ordinates has following form:

$$\mathbf{q}=[x_s,y_s,z_s,\Phi,\Theta,\Psi,\ \beta_L,\beta_R,\theta_L,\theta_R]^T \tag{40}$$

vector of quasi-velocities can be expressed by the following equation:

$$\mathbf{w}=[U,V,W,P,Q,R,\ \dot{\beta}_L,\dot{\beta}_R,\dot{\theta}_L,\dot{\theta}_R]^T \tag{41}$$

For the holonomic dynamical system the relation between generalized velocities $\dot{\mathbf{q}}=\left[\dot{q}_1\dot{q}_2,........\dot{q}_n\right]$ and quasi velocities $\mathbf{w}=\left[\dot{w}_{1,}\dot{w}_2,........\dot{w}_n\right]$ is following:

$$\dot{\mathbf{q}}=\mathbf{A}_T(\mathbf{q})\mathbf{w} \tag{42}$$

The matrix $\mathbf{A}_T$ has a construction:

$$\mathbf{A}_T=\begin{bmatrix}\mathbf{A}_G & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{C}_T & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{I}\end{bmatrix} \tag{43}$$

The matrices $\mathbf{A}_G$ and $\mathbf{C}_T$ are classical matrices of transformations of kinematics relations and can be found in Ref. [19], the unit matrix $\mathbf{I}$ has dimension $(3n+1)$ x $(3n+1)$, n – number of the main rotor blades.

From (18) we have the following relation:

$$\ddot{\mathbf{q}}=\mathbf{A}_T(\mathbf{q})\dot{\mathbf{w}}+\dot{\mathbf{A}}_T\mathbf{w} \tag{44}$$

Finally, the Appel function has following form:

$$S^*(\mathbf{q},\mathbf{w},\dot{\mathbf{w}},t)= =\frac{1}{2}\left(\dot{\mathbf{w}}+\mathbf{M}_w^{-1}\mathbf{h}_w\right)^T\mathbf{M}_w\left(\dot{\mathbf{w}}+\mathbf{M}_w^{-1}\mathbf{h}_w\right) \tag{45}$$

where:

$$\mathbf{M}_w(\mathbf{q})=\mathbf{A}_T^T\mathbf{M}_q\mathbf{A}_T,$$

and:

$$\mathbf{h}_w(\mathbf{q},\mathbf{w})=\mathbf{A}_T^T\left(\mathbf{M}_q\dot{\mathbf{A}}_T+\mathbf{h}_q\right)$$

At last the Gibbs-Appel equations of motion, written in quasi velocities has the following form:

$$\left(\frac{\partial S^*}{\partial \dot{\mathbf{w}}}\right)^T=\left[\frac{\partial S^*}{\partial \dot{w}_1},..........,\frac{\partial S^*}{\partial \dot{w}_k}\right]^T= =\mathbf{M}_w(\mathbf{q},t)\dot{\mathbf{w}}+\mathbf{h}_w(\mathbf{q},\mathbf{w},t)=\mathbf{Q}^*(\mathbf{q},\mathbf{w},t) \tag{46}$$

The vector $\mathbf{Q}^*$ is the sum of aerodynamic loads, potential forces acting on the MAV, and another non-potential forces acting on system. At last the equations of motion can be presented in the following form:

$$\begin{aligned}&\mathbf{M}\dot{\mathbf{V}}+\mathbf{M}\left(\dot{\mathbf{J}}_{\Omega}+\mathbf{J}_{\Omega}\mathbf{J}_{\Omega}\right)\mathbf{R}_{C}+\\&+\mathbf{M}\mathbf{J}_{\Omega}\dot{\mathbf{R}}_{C}+\mathbf{M}\mathbf{J}_{\Omega}=\mathbf{F}+\mathbf{G}\end{aligned} \qquad (47)$$

$$\begin{aligned}&\mathbf{J}_{B}\dot{\mathbf{\Omega}}+\mathbf{J}_{B}^{R}\dot{\mathbf{O}}_{R}+\mathbf{J}_{B}^{L}\dot{\mathbf{O}}_{L}+\\&+\left(\mathbf{J}_{s}+\mathbf{J}_{S}^{w}+2\mathbf{J}_{\Omega}\mathbf{J}_{S}+\dot{\mathbf{J}}_{S}^{w}\right)\mathbf{V}+\\&+\left(\dot{\mathbf{J}}_{B}^{R}+\mathbf{J}_{\Omega}\mathbf{J}_{B}^{R}\right)\mathbf{O}_{R}+\left(\dot{\mathbf{J}}_{B}^{L}+\mathbf{J}_{\Omega}\mathbf{J}_{B}^{L}\right)\mathbf{O}_{L}+\\&+\mathbf{J}_{\Omega}\mathbf{J}_{B}\mathbf{\Omega}=\mathbf{M}_{0}+\mathbf{R}_{C}\times\mathbf{G}\end{aligned} \qquad (48)$$

where: $\mathbf{M}=m\mathbf{I}$, m – mass of MAV, $\mathbf{I}$ – unit matrix, $\mathbf{F}=[F_x, F_y\ F_z]^T$– vector of aerodynamic forces, $\mathbf{M}_0=[L,M,N]^T$ – vector of aerodynamic moments, $\mathbf{V}=[U,V,W]^T$ – velocity vector; $\mathbf{\Omega}=[P,Q,R]^T$ – vector of angular velocity, $\mathbf{O}_R=\left[P-\dot{\beta},Q+\dot{\theta},R\right]^T$ - vector of right wing angular rates, $\mathbf{O}_L=\left[P+\dot{\beta},Q+\dot{\theta},R\right]^T$ - vector of left wing angular rate; $\mathbf{R}_c=[x_c, y_c, z_c]^T$ vector of the center of mass;

$$\mathbf{J}_{\Omega}=\begin{bmatrix}0&-R&Q\\R&0&-P\\-Q&P&0\end{bmatrix},\ \mathbf{J}_{S}=\begin{bmatrix}0&-S_z&S_y\\S_z&0&-S_x\\-S_y&S_x&0\end{bmatrix};$$

S_x, S_y, S_z - static moments entomopter without wings; $\mathbf{J}_S^w$ - matrix of static moments of MAV's wing; $\mathbf{J}_B$- inertial moment of MAV without wings; $\mathbf{J}_B^R$, $\mathbf{J}_B^L$ - inertial moments of right and left wing, respectively. The control vector is defined as follows:

$$\mathbf{u}=\left[\beta,\theta,\omega,\psi\right]^T \qquad (49)$$

ωηερε: β - flapping angle of wings; θ - feathering angle of wings, ω - frequency of wing motion respect to the body; ψ - phase shifting between feathering and flapping; and: $\beta=\beta_0\sin\omega t$, $\dot{\beta}=\beta_0\omega\cos\omega t$ $\theta=\theta_0\sin(\omega t+\psi)$, $\dot{\theta}=\theta_0\omega\cos(\omega t+\psi)$.

We assume that aerodynamic forces are nonlinear functions of angle of attack α, feathering angle θ, flapping angle β, and their derivatives:

$$\begin{aligned}F_x&=\frac{1}{2}\rho V_0^2SC_D\left(\alpha,\theta,\beta,\dot{\alpha},\dot{\theta},\dot{\beta}\right)\\F_y&=\frac{1}{2}\rho V_0^2SC_y\left(\alpha,\theta,\beta,\dot{\alpha},\dot{\theta},\dot{\beta}\right)\\F_z&=\frac{1}{2}\rho V_0^2SC_L\left(\alpha,\theta,\beta,\dot{\alpha},\dot{\theta},\dot{\beta}\right)\\L&=\frac{1}{2}\rho V_0^2SbC_l\left(\alpha,\theta,\beta,\dot{\alpha},\dot{\theta},\dot{\beta}\right)\\M&=\frac{1}{2}\rho V_0^2ScC_m\left(\alpha,\theta,\beta,\dot{\alpha},\dot{\theta},\dot{\beta}\right)\\N&=\frac{1}{2}\rho V_0^2SbC_n\left(\alpha,\theta,\beta,\dot{\alpha},\dot{\theta},\dot{\beta}\right)\end{aligned} \qquad (50)$$

where: C_L - lift coefficient; C_D - drag coefficient; C_y – side force coefficient; C_l, C_m, C_n – coefficients of aerodynamic moments.

4. Results

Control-augmentation systems of flapping wings Micro Air Vehicle involve both the direct interconnections between the control-surface deflections and different kinds of feedback. The direct interconnections can significantly improve the controllability of an aircraft and avoid possible departures due to aircraft-motion coupling.

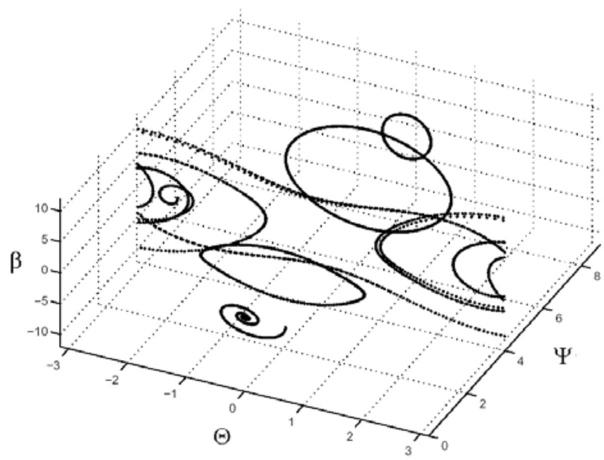

Fig. 4. Bifurcation diagram entomopter control.

The continuation technique can be applied to compute the nonlinear interconnection laws between stabilator, and stroke plane deflections to provide decoupling of longitudinal, directional and roll equilibrium states. Such decoupling may be useful during stabilization of flight in turbulent atmosphere with strong aerodynamic and inertia interaction between longitudinal and lateral dynamics.

Conclusions

The results presented demonstrate the efficiency of qualitative computational methods of nonlinear dynamics analysis for the design of control laws during flight in turbulent atmosphere. They are especially important in cases where there is strong nonlinear behaviour due to nonlinearities in aerodynamics and the control system.

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