# ON THE CALCULATION OF COLLISION PROBABILITIES AS AN ASSURANCE OF SAFE SEPARATION BETWEEN AIRCRAFT

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## Abstract

ICAO safety standards require low probabilities of collision of the order of $5\times10^{-9}$ per hour. Such low probabilities are virtually impossible to confirm through simulations, implying the need for alternative, and related safety metrics, which are easier to use. Several such alternative safety metrics are discussed in the present paper. For example it is shown that if the r.m.s. position error $\sigma$ is less than about one-tenth $L/\sigma\sim10$ of the minimum separation distance L, then the probabilities of collision will be less than the ICAO standard, in the case of aircraft flying in opposite directions on parallel tracks. This result is derived from an analytical formula based on the statistical application of a Gaussian probability distribution to a worst case collision scenario and a rather different result $L/\sigma\sim20$ arises for an Laplace probability distribution. The implication is that the ICAO standard of low probability of collision can be satisfied, by checking that the r.m.s. position error does not exceed a given value, though that value is dependent on the shape of the "tail" of the probability distribution. The formulas concern a variety of possible safety metrics, including the maximum and cumulative probabilities of coincidence, probabilities of overlap, collision rates and collision probabilities; the tables apply to horizontal, and vertical separation in controlled and transoceanic airspace. The sensivity of the results to the probability distribution assumed (Gaussian or Laplace), suggests the introduction of a parametric family exponential of probability distributions, of which these are particular cases.

## 1 Introduction

One of the important aspects of future Air Traffic Management (ATM) scenarios [1], is to increase capacity without reducing safety [2]. This requires consideration of smaller separations [3], together with measures ensuring that the risk of collision is reduced and the need for collision avoidance maneuvers [4] is not increased. The methods of calculation of probability of collision have been developed in considerable detail [5,6,7], involving both collection [8,9,10] and analysis [11,12,13] of traffic data. A good example is the Reduced Vertical Separation Minima (RVSM), halving the vertical separation in controlled air space from 2000ft to 1000ft, based on a careful study of collision probabilities [14]. The latter was based on flight data on aircraft altitude deviations fitted by appropriate probability distributions [15-23], generally non-gaussian [24-26]. For the purpose of establishing a safety metric, the probability of collision is the most obvious choice, but it is not only one. Other related parameters may be used as safety metrics [27], which may be advantageous if they are easier to measure.

## 2 Safety requirements for aircraft collision avoidance

To each aircraft may be associated a "safety volume", so that a collision between aircraft occurs when their safety volumes first touch. Thus the probability of collision depends on the intended flight paths of the aircraft, and the deviations from them, which could lead to their safety volumes overlapping. A simple approximation to the safety volume of an aircraft is a rectangle with sides equal to the length $R_x$, span $R_y$ and height $R_z$ of the aircraft. The exact safety volume would depend on the shape of the aircraft and its angular position (heading and bank angle) relative to other aircraft; the rectangular safety volume is a simple approximation, which overestimates slightly the collision risk. The collision rate between two aircraft is given by the probability that its safety volume be penetrated on any side [5], viz.:

$$P_r = P_x P_y N_z + P_y P_z N_x + P_x P_z N_y \,, \qquad (1)$$

where the $(P_x, P_y, P_z)$ are the probabilities of separations of less than $(R_x, R_y, R_z)$ respectively along track, across track and in altitude, and $(N_x, N_y, N_z)$ the frequency with these separation reduce to less than $(R_x, R_y, R_z)$. The frequencies of penetration $(N_x, N_y, N_z)$ are the probabilities of deviation $(P_x, P_y, P_z)$ divided by the time periods $(t_x, t_y, t_z)$ when the deviations exceed $(R_x, R_y, R_z)$, viz.:

$$i \equiv x, y, z: \qquad N_i = P_i / t_i \,, \qquad (2)$$

and thus:

$$P_r = P_x P_y P_z \left( \frac{1}{t_x} + \frac{1}{t_y} + \frac{1}{t_z} \right) = \prod_{j=1}^{3} P_j \sum_{i=1}^{3} \frac{1}{t_i} . \qquad (3)$$

In order to obtain a collision rate per aircraft pair this must be summed over the safety volume of the aircraft:

$$P_a = \int_0^{R_x} dx \int_0^{R_y} dy \int_0^{R_z} dz \, P_r(x, y, z). \qquad (4)$$

Note that the probabilities of deviation $(P_x, P_y, P_z)$ have the dimensions of inverse of length $L^{-1}$, the frequency of penetration $(N_x, N_y, N_z)$ has dimensions of inverse of length and time $L^{-1}T^{-1}$, the collision rate (3) has dimensions $L^{-3}T^{-1}$, and the collision probability (4) has the dimensions of the inverse time $T^{-1}$, and thus can be compared directly to the ICAO Target Level of Safety (TLS) standard.

If the collision rate (3) varies slowly over the aircraft size, the collision rate per aircraft pair (4) simplifies to:

$$P_a = R_x P_x R_y P_y R_z P_z \left( \frac{1}{t_x} + \frac{1}{t_y} + \frac{1}{t_z} \right). \qquad (5)$$

The frequency of penetration is [5] approximately

$$N_i = P_i \bar{V}_i / 2R_i \,, \qquad (6)$$

where $\bar{V}_i$ is the average rate of change of relative position between aircraft, and relates to the time spent at separation larger than $R_i$ by (2) viz.:

$$t_i = 2R_i / \bar{V}_i \,. \qquad (7)$$

Substituting (7) in (5) yields for the collision rate per aircraft pair is given by:

$$\begin{aligned} P_a &= \frac{1}{2} P_x P_y P_z \left( \bar{V}_x R_y R_z + \bar{V}_y R_x R_z + \bar{V}_z R_x R_y \right) \\ &= \frac{1}{2} \prod_{l=1}^{3} P_l \sum_{ijk}^{xyz} \bar{V}_i R_j R_k \end{aligned} \qquad (8)$$

The collision probability is the collision rate per pair multiplied by the time the aircraft spend in close proximity

$$P_c = \sum_{prox} P_a T_a \,, \qquad (9)$$

summed for all cases where aircraft fly by each other. This sum is dimensionless and will depend on the ATM scenario, viz. the geometry of flight paths and traffic flows along them. There general formulas will be illustrated in same simple cases in the sequel.

## 3 Use of general probability distributions and statistics

Consider next two aircraft flying on parallel paths at a constant distance $L$ equal to the minimum separation distance. In this collision scenario, the aircraft can at all times drift into positions less than a minimum separation distance apart. The minimum separation distance is in general smaller in the vertical direction than in the lateral or longitudinal direction; a similar analysis would apply to aircraft on the same flight path with a given longitudinal separation. In general, it will be assumed that along track, across track and altitude errors are statistically independent. Thus the three-dimensional collision problem is decoupled into three one-dimensional collision problems. Each may have different parameters, e.g. separation distances but the basic analysis is the same.

A convenient assumption would be that the position error satisfies Gaussian statistics for both aircraft. Note that the central limit theorem of the theory of probability [25] indicates that a long sequence $N$ of statistically independent events, in this case position errors, tends to a Gaussian distribution, with an accuracy of order $1/\sqrt{N}$, if the Lindeberg condition [26] it met, that events with large separation make a small contribution to the total variance. These two conditions, viz. (i) Lindeberg and (ii) large number of events can be questioned: aircraft collisions are extremely rare events, involving large deviations from the mean. Thus the number of statistically independent events may not be enough to justify a law of large numbers. Also, collisions correspond to the "tails" of the probability distribution, i.e. the large deviations, which the Lindeberg condition assumes to make a small contribution to the variance. The theoretical counter-arguments to the Gaussian distribution seem to be supported by observations of navigation errors [14], which suggest [15-23] that some form of generalized exponential distribution could be more appropriate. In order to assess the sensivity of results to the assumed probability distribution, the Gaussian is considered first, then the Laplace (§4.3), and then an exponential parametric family including both is considered (§7). Starting with the Gaussian case, the probability of the first aircraft having a lateral position error $x$ is:

$$P_1(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{x^2}{2\sigma^2}\right], \qquad (10)$$

where $\sigma$ is the r.m.s. position error.

## 4 Maximum and cumulative probabilities of coincidence or overlap

The highest probability of coincidence is not the only relevant result, because coincidences can also occur for other positions $x \neq L/2$, even if they are less likely; in fact, the probability of coincidence decays rapidly for $x$ greater than $L$, but remains close to the maximum for $x$ close to $L/2$. One way to address this aspect is to consider the total of cumulative probability of coincidence, summed or integrated over all possible positions transverse to the flight paths and in the same plane, viz.:

$$\overline{P} \equiv \int_{-\infty}^{+\infty} P_{12}(x)\, dx, \qquad (11)$$

which is specified by:

$$\overline{P} = \frac{1}{2\pi\sigma^2} \int_{-\infty}^{+\infty} \exp\left[-\frac{2x^2 + L^2 - 2xL}{2\sigma^2}\right] dx \qquad (12)$$

and thus depends only on separation distance $L$ and r.m.s. position error $\sigma$. A change of variable and the well-known Gaussian integral, leads to:

$$\overline{P} = \frac{1}{2\sigma\sqrt{\pi}} \exp\left[-\left(\frac{L}{2\sigma}\right)^2\right], \qquad (13)$$

namely, the cumulative probability of coincidence $\overline{P}$ as a function of separation distance $L$ and r.m.s. position error $\sigma$ due to all causes. For a fixed separation distance $L$, it is possible to use as safety metric, instead of the cumulative probability of coincidence $\overline{P}$, the r.m.s. position error $\sigma$.

There are both theoretical and observational [14,24] counter-arguments to the use of a Gaussian probability distribution for position errors, and a Laplace distributions has been used [5-7] instead:

$$P_0(x) = \frac{1}{\sigma\sqrt{2}} \exp\left[-\sqrt{2}\,\frac{|x|}{\sigma}\right], \qquad (14)$$

where $\sigma$ is again the r.m.s. position error.

## 5 Application to separation in controlled and uncontrolled airspace

The difference between the Gaussian and Laplace distributions will become apparent in the application vertical (§5.2) separation and horizontal separation in controlled (§5.1) and uncontrolled (§5.3) airspace.

### 5.1 Application to lateral separation in controlled airspace

Gaussian cumulative probability of coincidence (13) is most convenient use in logarithmic form:

$$\log \overline{P} = -[L/(2\sigma)]^2 - \log\sigma - \log 2 - \frac{1}{2}\log\pi\,, (15)$$

where the constant values can be inserted:

$$\log \overline{P} = -0.25(L/\sigma)^2 - \log\sigma - 1.2655\,. \quad (16)$$

Taking $L_h = 5$ nm for the separation distance specified by ICAO in controlled airspace leads to

$$\log \overline{P}_h = -6.25/\sigma_h{}^2 - \log\sigma_h - 1.2655\,, \quad (17)$$

as the relation between cumulative probability of coincidence $\overline{P}_h$ and r.m.s. position error $\sigma_h$ in *Table I*. It seen from the table that large r.m.s. horizontal position errors $\sigma_h$ of 2 to 3 nm, still bellow the minimum separation distance $L_h = 5\,\text{nm}$, give high cumulative probabilities of coincidence. A smaller r.m.s. position error of the order 0.7 to 1 nm would lead to lower probabilities of coincidence $(10^{-4}\ \text{to}\ 10^{-6})$, which could be tested in simulations; the aim of these simulations would be to check that the formula (17) for the cumulative probability of coincidence.

**Table I-** Assuming the ICAO standard horizontal separation $L_h = 5\,\text{nm}$ in controlled air space, the r.m.s. horizontal position error $\sigma_h$ due to all causes can be used as an alternative safety metric to the (17) cumulative probability of coincidence $\overline{P}_h$ or the (19) maximum probability of coincidence $P_{mh}$ using Gaussian statistics, compared with Laplace statistics for (23) the cumulative joint probability $\widetilde{P}_h$ and (21) the exponential $\widetilde{P}_{fh}$ factor in it.

| Probability Distribution | Gaussian | | Laplace | |
|---|---|---|---|---|
| r.m.s. position error | Cumulative probability of coincidence | Maximum probability of coincidence | Exponential factor | Cumulative joint probability |
| $\sigma_h$ (nautical miles) | $\overline{P}_h$ (per-nautical mile) | $P_{mh}$ (per-square nautical mile) | $\widetilde{P}_{fh}$ (per nautical mile) | $\widetilde{P}_h$ (per nautical mile) |
| 3 | 4.70x10$^{-2}$ | 8.83x10$^{-3}$ | 4.46x10$^{-2}$ | 6.92x10$^{-4}$ |
| 2 | 2.96x10$^{-2}$ | 8.34x10$^{-3}$ | 2.06x10$^{-2}$ | 4.79x10$^{-4}$ |
| 1 | 5.45x10$^{-4}$ | 3.07x10$^{-4}$ | 1.20x10$^{-3}$ | 5.58x10$^{-5}$ |
| 0.7 | 1.16x10$^{-6}$ | 9.38x10$^{-7}$ | 8.29x10$^{-5}$ | 5.51x10$^{-6}$ |
| 0.6 | 1.36x10$^{-8}$ | 1.28x10$^{-8}$ | 1.80x10$^{-5}$ | 1.40x10$^{-6}$ |
| 0.5 | 7.83x10$^{-12}$ | 8.84x10$^{-12}$ | 2.04x10$^{-6}$ | 1.90x10$^{-7}$ |

The main use of the formula (17) is for smaller r.m.s. position errors, as it shows that a value of $\sigma \leq 0.5\,\text{nm}$ leads to a cumulative probability of coincidence of less than $\overline{P}_h \leq 7.83\times10^{-12}$ per nautical mile; for an aircraft cruising at a speed not exceeding $V = 600\,\text{kt}$, this leads to a cumulative probability of collision $\overline{P}_h V \leq 4.7\times10^{-9}$ per hour, which meets the ICAO requirement of less than 5x10$^9$ in the conditions for which (18) holds. Specifying a larger r.m.s. position error quickly increases probability of collision, e.g. to $\overline{P}_h \leq 1.36\times10^{-8}$ for $\sigma_h = 0.6\,nm$. A smaller r.m.s. position error does decrease coincidence probability to minute levels, e.g. $\overline{P}_h \leq 6.51\times10^{-31}$ for $\sigma = 0.3\,nm$, but this places an unnecessarily severe demand in position accuracy, which could be costly to meet in terms of aircraft on-board equipment and hard to comply with by the ground based ATM system. The r.m.s. position error $\sigma_h = 0.5\,nm$ of one-tenth the separation distance $L_h = 5\,nm = 10\sigma_h$, is a fairly robust result as concerns meeting the ICAO target safety level (TLS), since: (i) a larger r.m.s. position error will rapidly increase the cumulative probability of coincidence to unacceptable levels; (ii) a smaller r.m.s. position error does reduce the cumulative probability of coincidence, but is unnecessary.

The ICAO target level of safety (TLS) of low probability of collision (5x10$^{-9}$ per hour) can be obtained, with a five nautical mile minimum separation distance $L_h = 5\,nm$, by requiring a $\sigma_h = 0.5$ $nm$ r.m.s. position error; the latter leads to a cumulative probability of coincidence not exceeding $\overline{P}_h \leq 7.83\times10^{-12}$ per nautical mile flown, or $\overline{P}_h D \leq 1.69\times10^{-7}$ for a $D = 40000km = 21587\,nm$ flight around the earth on a great circle. This low upper bound for the cumulative probability of coincidence makes it unnecessary to demand a higher r.m.s. position accuracy. To degrade the r.m.s. position accuracy would quickly lead to much higher probabilities of coincidence. The safety standard of 0.5 *nm* r.m.s. position error should include all causes for position error, e.g. inaccuracy of the navigation system, effects of atmospheric disturbances, trajectory drift between updates of position fixes, etc.. The 0.5 *nm* r.m.s. position error is easy to use as a safety metric in simulations: it just requires calculation of the r.m.s. deviation from the desired flight path. The preceding discussion has been based on the Gaussian cumulative probability of coincidence (17) per nautical mile flown by one aircraft. It also possible to use the Gaussian maximum probability of coincidence (36) per square nautical mile, i.e. per nautical mile flown by each aircraft:

$$P_m = \left(0.1592/\sigma^2\right)\exp\left[-0.25(L/\sigma)^2\right], \quad (18)$$

or for five nautical mile minimum separation distance

$$P_{mh} = \left(0.1592/\sigma_h^2\right)\exp\left(-6.25/\sigma_h^2\right). \quad (19)$$

The *Table I* also includes in the fourth column the factor in the exponential joint probability (51) which is independent of aircraft size:

$$\widetilde{P}_f = \frac{1.4142}{\sigma}\exp\left[-1.4142\frac{L}{\sigma}\right], \quad (20)$$

in the case of five nautical mile lateral separation:

$$\widetilde{P}_{fh} = \frac{1.4142}{\sigma}\exp\left[-7.0711/\sigma\right]. \quad (21)$$

The aircraft size appears in the exponential joint probability (51):

$$\widetilde{P} = \widetilde{P}_f \sinh\left[1.4142R/\sigma\right], \quad (22)$$

indicated in the fifth column of *Table I* for an aircraft span $R_y = 200\,\text{ft} = 3.29\times10^{-2}\,\text{nm}$:

$$\widetilde{P}_h = \widetilde{P}_{fh}\sinh\left[4.6525\times10^{-2}/\sigma\right]. \quad (23)$$

## 5.2 Application to reduced vertical separation minima

The values indicated in *Table I* are calculated from (17), (19), (21) and (23), and apply to a five nautical mile minimum horizontal separation distance in controlled airspace; the formulas (16),(18), (20) and (22) could also be applied to other minimum separation distances, e.g. to the vertical instead of the horizontal separation distance. The horizontal separation distance $L_h = 5\,nm$ used in (56,58,60,62) is that which applies to flight in controlled air space, for which the vertical separation is $L_v = 1000\text{ ft} = 0.1645\text{ nm}$ at lower flight levels (below FL 290); the same vertical separation distance is being applied for higher flight levels [14] where the earlier value of 2000 *ft* is being replaced by the RVSM of $L_v = 1000\text{ ft} = 0.1645\text{ nm}$. Using this value in (16) specifies the cumulative probability of coincidence due to error in vertical position $\sigma_v$:

$$\log \overline{P}_v = -6.7631 \times 10^{-3} / \sigma_v^{\,2} - \log \sigma_v - 1.2655, (24)$$

which is indicated in *Table II*. For a vertical separation error of $\sigma = 100\,ft$, the cumulative probability of coincidence does not exceed $\overline{P}_v \leq 2.38 \times 10^{-10}$ per nautical mile or $P_v D \leq 5.14 \times 10^{-6}$ for a great circle tour of the earth. The recommended r.m.s. error for vertical separation is smaller $\sigma_v = 90\,ft$, i.e. about one-eleventh of the minimum vertical separation $L_v = 1000\,ft$, i.e. $\sigma_v / L_v = 0.09$. Note that the recommended r.m.s. horizontal position error $\sigma_h = 0.5\,nm$, was one-tenth $\sigma_h / L_h = 0.1$ of the minimum horizontal separation $L_h = 5\,nm$. The reason for a smaller relative value here is that a r.m.s. vertical position error $\sigma_v = 90\,ft$ leads to an upper bound for the cumulative probability of coincidence $\overline{P}_v \leq 7.51 \times 10^{-13}$ per nautical mile; for the fastest commercial aircraft (Concorde), which cruises at a speed of $V = 1166\,kt$, the cumulative probability of collision, does not exceed $\overline{P}_v V \leq 8.76 \times 10^{-10}$ per hour which meets the ICAO TLS standard of less than $5 \times 10^{-9}$ per hour, in the condition for which (18) holds.

**Table II**- The cumulative $\overline{P}_v$ (24) and maximum $P_{mv}$ (25) probabilities of coincidence for Gaussian statistics and cumulative joint probability $\tilde{P}_v$ (27) and its exponential factor $\tilde{P}_{fv}$ (26) can be similarly related to the r.m.s. altitude error $\sigma_v$, for the fixed vertical minimum separation $L_v = 1000\,ft$, applying in controlled and uncontrolled air space, at lower flight level at present (and higher flight levels in the future).

| Probability Distribution | Gaussian | | Laplace | |
|---|---|---|---|---|
| Vertical separation error | Cumulative probability of coincidence | Maximum probability of coincidence | Exponential factor | Cumulative joint probability |
| $\sigma_v$ (*ft*) | $\overline{P}_v$ (per-nautical mile) | $P_{mv}$ (per-square nautical mile) | $\tilde{P}_{fv}$ (per nautical mile) | $\tilde{P}_v$ (per nautical mile) |
| 100 | $2.38 \times 10^{-10}$ | $8.17 \times 10^{-9}$ | $6.20 \times 10^{-5}$ | $4.76 \times 10^{-5}$ |
| 90 | $7.51 \times 10^{-13}$ | $2.86 \times 10^{-11}$ | $1.43 \times 10^{-5}$ | $1.24 \times 10^{-5}$ |
| 80 | $2.33 \times 10^{-16}$ | $9.97 \times 10^{-15}$ | $2.26 \times 10^{-6}$ | $2.27 \times 10^{-6}$ |
| 70 | $1.70 \times 10^{-21}$ | $8.35 \times 10^{-20}$ | $2.07 \times 10^{-7}$ | $2.47 \times 10^{-7}$ |
| 60 | $1.98 \times 10^{-29}$ | $1.13 \times 10^{-27}$ | $8.32 \times 10^{-9}$ | $1.22 \times 10^{-8}$ |
| 50 | $1.28 \times 10^{-42}$ | $8.75 \times 10^{-41}$ | $8.95 \times 10^{-11}$ | $1.73 \times 10^{-10}$ |

The Gaussian maximum probability of coincidence per square mile (18), can also be calculated for the reduced vertical separation minima of $L_v = 1000\,ft = 0.1645\,nm$:

$$P_{mv} = \left(0.1592/\sigma_v^2\right)\exp\left(-6.7631\times10^{-3}/\sigma_v^2\right), \quad (25)$$

and is also indicated in *Table II*. For the recommended r.m.s. vertical position error of $\sigma_v = 90\,ft$, the maximum probability of coincidence is $P_{mv} \le 2.\times10^{-11}$ per nautical mile flown by each aircraft, or $P_{mv}\,D^2 \le 1.33\times10^{-2}$ for a great circle tour of the earth suggesting that a smaller r.m.s. position error be considered. For two aircraft with a cruise speed not exceeding $V \le 547\,kt$, the maximum probability of coincidence $P_{mv} \le 9.97\times10^{-15}$ per square nautical mile, for a $\sigma_v = 80\,ft$ vertical r.m.s. position error, leads to an upper bound for the maximum probability of coincidence of $P_{mv}\,V^2 \le 2.98\times10^{-9}$ per flight hour squared, below to the ICAO modified TLS value of $5\times10^{-9}$ per hour squared.

The Laplace exponential factor (20) for the same vertical separation is given:

$$\tilde{P}_{fv} = \frac{1.4142}{\sigma}\exp\left[-\frac{0.2326}{\sigma}\right], \quad (26)$$

per nautical mile, not affected by aircraft size; the values in *Table II* show that the Gaussian cumulative probability of coincidence is much smaller than for the Laplace exponential factor. The aircraft size enters through the factor (22) in the cumulative joint probability:

$$\tilde{P}_v = \tilde{P}_{fv}\sinh\left[8.2245\times10^{-3}/\sigma\right], \quad (27)$$

where the aircraft size was taken to be the height $R_z = 50\,\text{ft} = 15.2\,\text{m} = 8.2245\times10^{-3}\,\text{nm}$.

## 6 Three-dimensional separation with Gaussian or Laplacian statistics

The preceding analysis of one-dimensional separation, can be combined for two and three-dimensional separation, for example, for aircraft staggered along parallel tracks. Consider: (i) a $L_y = 60\,\text{nm}$ lateral separation in transoceanic airspace, with $\sigma_y = 5\,nm$ r.m.s. position error, leading by *Table III* to a Gaussian cumulative probability of coincidence $P_y = 1.31\times10^{-17}$ (Laplace joint cumulative probability $1.13\times10^{-10}$) per nautical mile; (ii) a $L_x = 5\,nm$ along track stagger, with a $\sigma_x = 0.7\,nm$ longitudinal position error, leading by *Table I* to a Gaussian cumulative probability of coincidence $P_x = 1.16\times10^{-6}$ (Laplace joint cumulative probability $1.40\times10^{-6}$) per nautical mile. Then the combined two-dimensional probability of coincidence is $P_x P_y = 1.52\times10^{-23}$ ($1.58\times10^{-16}$) per nautical mile squared. If there is an altitude difference $L_z = 1000\,ft$ and vertical r.m.s. position error $\sigma_z = 90\,ft$, the *Table II*, gives the corresponding Gaussian cumulative probability of coincidence $P_z = 7.51\times10^{-13}$ (Laplace joint cumulative probability $1.24\times10^{-5}$) per nautical mile; hence, the combined three-dimensional probability of collision is $P_3 = P_x P_y P_z = 1.14\times10^{-35}$ ($2.25\times10^{-21}$) per nautical mile cubed. This shows that stagger and altitude difference combined with lateral separation lead to very low probabilities of coincidence. Taking for the aircraft "size" the span $R_y = 200\,ft = 60.8m = 3.29\times10^{-2}\,nm$, the length $R_x = 150\,\text{ft} = 45.7\text{m} = 2.47\times10^{-2}\,\text{nm}$ and height $R_z = 50\,\text{ft} = 15.2\,\text{m} = 8.22\times10^{-3}\,\text{nm}$, the aircraft volume is $R_3 = R_x R_y R_z = 6.68\times10^{-6}\,(\text{nm})^3$, and the Gaussian (Laplace) cumulative probability of overlap $\overline{P} = P_3 R_3 = 7.62\times10^{-41}$ ($1.50\times10^{-26}$) is dimensionless. It has been assumed that the aircraft remain always at the minimum separation distance, but if this happened say only a fraction 40% of the time $f = 0.4$, the probability of collision would be further reduced to $f\overline{P} = 3.04\times10^{-41}$ ($6.00\times10^{-27}$).

## 7 The generalized exponential family of probability distributions

The Gaussian (10) and Laplace (14) probability distributions are respectively the particular cases *k*=2 and *k*=1 of the generalized probability distribution:

$$F_k(x) = A\exp\left(-a|x|^k\right), \tag{28}$$

where A, a are two constants, viz.: (§7.1) the normalization constant $A$ is determined by the condition of unit total probability, leads to the family of probability distributions:

$$F_k(x) = \frac{1}{2\sigma}\frac{1}{\Gamma(1+1/k)}\sqrt{\frac{\Gamma(3/k)}{\Gamma(1/k)}}\times \times\exp\left\{-\left[\frac{\Gamma(3/k)}{\Gamma(1/k)}\right]^{k/2}\left(\frac{|x-\mu|}{\sigma}\right)^k\right\}, \tag{29}$$

as the generalized exponential distribution with mean $\mu$ and variance $\sigma^2$. The more interesting instances of the new family of probability distributions (83), for ATM applications, should be $1 < k < 2$ and $0 < k < 1$.

In *Figure 1* the distributions of large altitude errors in real flight [14] is shown to be consistent with the extended exponential probability distribution (83) with *k*=0.5, viz.

$$F_{1/2}(x) = \frac{1}{\sigma}\sqrt{\frac{15}{2}}\exp\left\{-\sqrt[4]{120}\sqrt{\frac{|x-\mu|}{\sigma}}\right\} \tag{30}$$

is a simple and relatively accurate probability distribution for position errors.

**Table III**- For flight in transoceanic regions the minimum horizontal separation is $L_t = 60nm$, and the r.m.s. horizontal position error $\sigma_t$ specifies the cumulative $P_t$ (67) and maximum $P_{mt}$ (68) probabilities of coincidence for Gaussian statistics, and the cumulative joint probability $\tilde{P}_t$ (70) and its exponential factor $\tilde{P}_{ft}$ (69) using Laplace statistics.

| Probability Distribution | Gaussian | | Laplace | |
|---|---|---|---|---|
| Vertical separation error | Cumulative probability of coincidence | Maximum probability of coincidence | Exponential factor | Cumulative joint probability |
| $\sigma_t$ (nautical miles) | $\overline{P}_t$ (per nautical mile) | $P_{mt}$ (per square nautical mile) | $\tilde{P}_{ft}$ (per-square nautical mile) | $\tilde{P}_t$ (per-square nautical mile) |
| 10 | $3.48\text{x}10^{-6}$ | $1.96\text{x}10^{-6}$ | $2.92\text{x}10^{-5}$ | $1.36\text{x}10^{-7}$ |
| 9 | $4.68\text{x}10^{-7}$ | $2.94\text{x}10^{-8}$ | $1.27\text{x}10^{-5}$ | $6.56\text{x}10^{-8}$ |
| 8 | $2.75\text{x}10^{-8}$ | $1.94\text{x}10^{-9}$ | $4.38\text{x}10^{-6}$ | $2.55\text{x}10^{-8}$ |
| 7 | $4.25\text{x}10^{-10}$ | $3.43\text{x}10^{-11}$ | $1.10\text{x}10^{-6}$ | $7.31\text{x}10^{-9}$ |
| 6 | $6.53\text{x}10^{-13}$ | $6.14\text{x}10^{-14}$ | $1.70\text{x}10^{-7}$ | $1.32\text{x}10^{-9}$ |
| 5 | $1.31\text{x}10^{-17}$ | $1.48\text{x}10^{-18}$ | $1.21\text{x}10^{-8}$ | $1.13\text{x}10^{-10}$ |

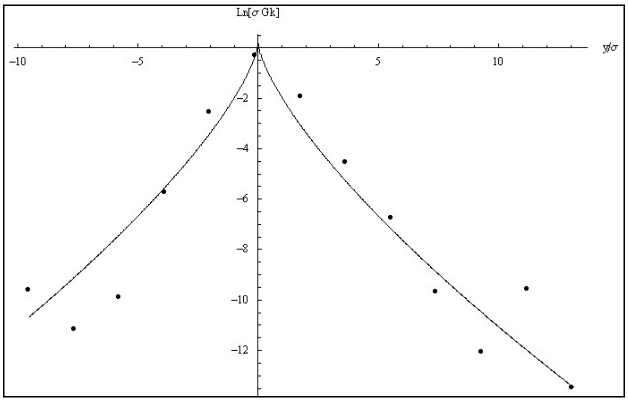


**Figure1** The exponential probability distribution (29) with weight $k = 0.53$ close to one-half (30) approximates the altitude deviations measured [14] for aircraft in flight.

## 8 Discussion

It has been pointed out [5,6,7] that the Laplace distribution $k$=1 underestimates the "tails" of the probability distribution, and the uniform distribution $k$=0 overestimates, so that a more accurate assessment of collision risk lies somewhere in between. The value $k$=0.5 is consistent with these observations, and arises out a comparison with altitude deviations of aircraft measured from flight data [14]. This data has been closely fitted [15-23] using double exponential or Gaussian probability distributions, with five parameters, allowing a close match both to the "body" and "tails" of the probability distribution. The choice of a generalized exponential distribution with weight $k$=0.5 is much simpler, in that it involves a single parameter (besides the mean), viz. the r.m.s. deviation $\sigma$, which is readily estimated from the data. Given the various sources of error involved in estimation of collision risk, this simple one-parameter probability distribution may do nearly as well as more complex multi-parameter models.

The probability distribution for large rare deviations is the key input is assessing collision risk. The actual calculation, for a simple or complex ATM scenario, involves several other probabilities, all related to the probability of deviation of a single aircraft from its flight path. The difference between simple and complex ATM scenarios depend on the number of aircraft involved and their relative paths, which determine how many proximities have to be considered; the calculations become more complex for higher traffic densities and crossings from many different directions. Based on the (i) probability of deviation from the flight path of a single aircraft, the calculation of collision rates or assessment of collision risks, involves several others probabilities which could serve as intermediate safety metrics; (ii) the probability of coincidence of two aircraft at the same position; (iii) the maximum probability of coincidence, at the most likely position of coincidence; (iv) the cumulative probability of coincidence, at all possible positions; (v) the probability of overlap, taking into account finite aircraft size. These can be used to calculate (vi) collision rates, which can be compared to the ICAO TLS standard per unit time or (vii) per unit distance. The dimensionless (viii) collision probability for a given traffic system over a given time is the final safety metric, which depends on many parameters, since most of the preceding are used as building blocks.

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