FLIGHT TRAJECTORIES OPTIMIZATION

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Abstract

System optimization is a process of translating the dynamics of a system and its desired objectives into the mathematical language, which give rise to what is called a control problem and then to find the solution of this problem. Such a solution is called optimal control and the path it follows to achieve the desired objectives is called optimal trajectory. Trajectory optimization is an optimal transfer problem. For any specified end condition and performance index, the problem of determining the optimal trajectory in powered flight of an aircraft in atmospheric conditions, subject to certain physical constraints, is very complex problem. In general it cannot be solved without using numerical computation based on a specified model of the atmosphere and aircraft aerodynamic and engine characteristics. In the past an intensive research has been carried out in the area of system optimization and optimal trajectories. In the work presented in this paper, emphasis is made on generalization of the optimal trajectories of aircraft, the basic ingredients of the optimization problem and formulating the precise statement of the optimization problem. The definition of optimal control problem, formulation of a control problem and solution of such a control problem are presented. Different optimization techniques are discussed and compared to show their merits and limitations. The optimal trajectories in different phases of flight, i.e. trajectories in horizontal plane, vertical climb trajectories, are analyzed using the Pontryagin's Maximum Principle.

1 Control Problem

In a general way a control problem is controlling of a dynamic system. Optimization of a system is a process of translating the system dynamics and its desired objectives into the abstract language of mathematics, which give rise to what is called a control problem, and then to find the solution to this problem. Such a solution is also called optimal control, and the path it follows to reach the desired goal is called optimal trajectory. The mathematical model, which represents the physical system, consists of a set of relations between state of the system and input to the system. Constraints are incorporated in this set of relations, which are usually expressed in the form of equations called the state equations. The requirement of obtaining a certain output is replaced with the requirement of hitting a certain target set in the state space of the system. The physical restrictions or constraints upon the set of inputs lead to a set of admissible inputs or controls. The desired objectives can be attained by many admissible inputs, each of which results in a different response. So it is required to evaluate each response and if possible pick up the best one. This requires the use of performance criterion, which is a measure of performance or cost of control. Such a performance criterion is called performance index or cost function (or functional). The solution of a control problem is to determine the admissible inputs which generate the desired output and which, in doing so, minimize (optimize) the cost functional.

The system is governed by a set of differential equations or the state equations

$$\dot{x} = f(x,u,t) \tag{1}$$

where x is state vector, u is control vector and t is time. It is associated with a performance index of general form:

$$J(u) = K[x(t_f), t_f] + \int_{t_i}^{t_f} L[x(t), u(t), t]dt \tag{2}$$

where $K[x(t_f),t_f]$ is the terminal cost and $L[x(t),u(t),t]$ is cost between the time interval $[t_i, t_f]$. The optimum control problem is to find $u(t)$ that minimizes the performance index along the optimal trajectory. Based on how to find $u(t)$, an optimal control problem can be handled in two possible ways:

- Parameter or Static Optimization
- Process or Dynamic Optimization

2 Parameter Optimization

The method where the control vector $u(t)$ can be found by a set of parameters a_1, a_2, a_3, .. a_n., is called parameter optimization and the performance index, which is to be minimized, becomes a function of these parameters. Such a performance index is also called cost function. This method is applied to trajectory optimization problem where the trajectory is expressed in terms of a finite number of parameters and a set of parameters is optimized to get the optimum performance. It is used to maximize or minimize a cost function

$$J = K(x(t_f), t_f) + \int_{t_i}^{t_f} L(x,u,t)dt \tag{3}$$

As trajectory x depends on $u(t)$, therefore, J depend on $u(t)$. If $u(t)$ can be expressed in terms of a finite number of parameters a_1, a_2, …a_n, then in the final analysis, J can be written as a function of finite number of parameters:

$$J = f(a_1, a_2, \ldots a_n) \tag{4}$$

The limitation of parameter optimization is that it becomes difficult to solve the problem if the number of parameters increases or the complexity of the problem increases where the performance criterion is a cost functional, i.e. it is function of different functions. The dominant optimization techniques used in parameter optimization are: the Theory of Maxima and Minima, the Direct Methods of optimization, such as Simplex, Rosenbrock-Powel Method, Method of Gradients or Steepest Descent, Conjugate Gradient Method, etc.

3 Process Optimization

The process optimization is the generalization of parameter optimization problem and is concerned with finding a maximum or minimum for a quantity (performance index) which depends upon n independent functions x_1(t), x_2(t), …x_n(t). This is a generalization of the concept of a function and is called functional. For more complicated control problems, where the performance criterion is cost functional, or where more accurate solution is required, process optimization is used, e.g. in case of trajectories in unsteady flight conditions. The cost functional is of the type:

$$J = K[x(t_f), t_f] + \int_{t_i}^{t_f} L[x(t), u(t), t]dt \tag{5}$$

J depends on $f(x(t), u(t))$, but $x(t)$ also depends on $u(t)$. Therefore, we can say that in final analysis, the performance index J is a functional of different controls and we can write

$$J = J(u) \tag{6}$$

The famous optimization techniques used in process optimization are: the Calculus of Variations, the Belman's Dynamic Programming, the Pontryagin's Maximum Principle, etc. The main theoretical approaches, which had a major impact upon research in optimization, are based on the basic theory of ordinary Mixima and Minima.

4 Theory of Maxima and Minima

The theory of ordinary maxima and minima is concerned with the problem of finding the values of each of n independent variables $a_1, a_2, \ldots a_n$ at which some function of variable $f(a_1, a_2, \ldots a_n)$ reaches either a maximum or a minimum. The existence of a solution to such a problem is guaranteed by the theorem of Weierstrass as long as the function is continuous.

4.1 Necessary conditions for maxima or minima

The location of local extrema for a real-valued function $f(x)$, which is defined and continuous on the closed interval [a, b] in the interior of a region 'R' may be determined by the two necessary conditions, i.e. if x is an extremum then

$$f'(x) = \frac{\partial f}{\partial x} = 0 \tag{7}$$

Such a point is also called stationary point or critical point. If there is a point where $f'(x)$ does not exists but it is piecewise continuous, i.e. $f'(x-)$ and $f'(x+)$ exists for all x but not necessarily $f'(x)$. The necessary conditions in such a case is

$$\lim_{x \to x+} f'(x) \ge 0 \quad \text{and} \quad \lim_{x \to x-} f'(x) \le 0 \quad \text{(local minimum)} \tag{8}$$

or

$$\lim_{x \to x+} f'(x) \le 0 \quad \text{and} \quad \lim_{x \to x-} f'(x) \ge 0 \quad \text{(local maximum)} \tag{9}$$

for the end points a, b

$$\lim_{x \to a+} f'(x) \ge 0 \text{ (local minimum)}$$

$$\text{or} \quad \lim_{x \to b-} f'(x) \le 0 \text{ (local maximum)} \tag{10}$$

We may conclude that the potential candidates for the absolute extremum of $f(x)$ are the stationary points, the points where one or more first partial derivatives of $f(x)$ are discontinuous, and the end points. However, neither stationary point nor discontinuities have to be extrema. In order to determine a local extremum of all the potential candidates for extremum of a function, the sufficient conditions are:

4.2 Function of one independent variable.

In case of a function $f(x)$ of one independent variable x, if x is a point at which $f'(x)$ is zero and if the derivative $f'(x)$ changes its sign from positive to negative (or negative to positive) when it passes through zero, then x is maximum (or minimum) of $f(x)$. This is the necessary condition for extremum of a function of one independent variable, which may be phrased as:

if x is interior point at which

$$f'(x) = 0 \text{ and } f''(x) = \frac{\partial^2 f}{\partial x^2} > \text{or} < 0 \tag{11}$$

then x is local extremum.
x is local minimum if $f''(x)$ is positive, i.e.

$$\frac{\partial^2 f}{\partial x^2} > 0 \tag{12}$$

and is local maximum if

$$f''(x) \text{ is negative, i.e. } \frac{\partial^2 f}{\partial x^2} < 0 \tag{13}$$

4.3 Function of two independent variables.

For a function of two independent variables $f(x_1, x_2)$, extremum is relative minimum if

$$f_{x_1x_1} > 0 \quad \text{and} \quad f_{x_1x_1} f_{x_2x_2} - (f_{x_1x_2})^2 > 0 \tag{14}$$

and is relative maximum if

$$f_{x_1x_1} < 0 \quad \text{and} \quad f_{x_1x_1} f_{x_2x_2} - (f_{x_1x_2})^2 > 0 \tag{15}$$

4.4 Function of n independent variables.

The corresponding necessary conditions for local extrema of a function of n independent variables $f(x_1, x_2, \ldots x_n)$ may be expressed as:

a stationary point will be local minimum if

$$D_i > 0 \quad (i = 1, 3, 5, \ldots\ldots)$$

$$\text{and} \quad D_i < 0 \quad (i = 2, 4, 6, \ldots\ldots) \tag{16}$$

a stationary point will be local maximum if

$D_i < 0$ (i = 1, 3, 5, …….)

and $D_i > 0$ (i = 2, 4, 6, …….)

(17)

where

$$D_i = \begin{vmatrix} f_{x_1x_1} & f_{x_1x_2} & . & . & f_{x_1x_i} \\ f_{x_2x_1} & f_{x_2x_2} & . & . & f_{x_2x_i} \\ . & . & . & . & . \\ . & . & . & . & . \\ f_{x_ix_1} & f_{x_ix_2} & . & . & f_{x_ix_i} \end{vmatrix}$$

4.5 Solutions subject to constraints.

If the variables x_1, x_2, …. x_n are subject to certain relations called constraints, of the form:

$$g_1(x_1, x_2, \ldots x_n) = 0,\ g_2(x_1, x_2, \ldots x_n) = 0,\ . \ . \ . \ .$$

$$g_m(x_1, x_2, \ldots x_n) = 0 \quad (18)$$

with m < n , the number of independent variables is reduced to n-m. The problem is to find an extremum of f(x_1, x_2, …. x_n) for one or more independent variables subject to m constraint equations. There are different methods to find the solution of this problem subject to constraints, such as Substitution Method, Lagrange Multiplier Method, Calculus of Variations. etc.

4.6 Lagrange Multiplier Method.

Consider a function of two independent variables $f(x_1, x_2)$ with constraint

$$g(x_1, x_2) = 0 \quad (19)$$

The derivative $\frac{\partial(f,g)}{\partial(x_1,x_2)}$ also called the Jacobian can be written [3] as:

$$\frac{\partial(f,g)}{\partial(x_1,x_2)} = \det \begin{vmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \end{vmatrix} \quad (20)$$

$$= \frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_2} - \frac{\partial f}{\partial x_2}\frac{\partial g}{\partial x_1}$$

The necessary condition for the existence of stationary point is also a point where the Jacobian determinant must vanish, i.e.

$$\frac{\partial(f,g)}{\partial(x_1,x_2)} = 0$$

or $$\frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_2} - \frac{\partial f}{\partial x_2}\frac{\partial g}{\partial x_1} = 0 \quad (21)$$

This equation may be rewritten as:

$$\frac{\partial f / \partial x_1}{\partial g / \partial x_1} = \frac{\partial f / \partial x_2}{\partial g / \partial x_2} = -\lambda \quad (22)$$

where λ is a constant called Lagrange multiplier. Equation (22) can also be written as:

$$\frac{\partial f}{\partial x_1} + \lambda \frac{\partial g}{\partial x_1} = 0$$

and $$\frac{\partial f}{\partial x_2} + \lambda \frac{\partial g}{\partial x_2} = 0 \quad (23)$$

Equation (23) can be considered as the necessary conditions for the existence of a stationary point of the function $f + \lambda g$ without the constraints. Therefore, the necessary condition for a stationary point of $f(x_1, x_2)$ with constraint $g(x_1, x_2) = 0$ is found by forming the augmented function $f + \lambda g$, and treating the problem as one without constraints. This result can be extended to the general case of a function of n variables with m constraint equations, with the introduction of a set of constants, Lagrange multipliers, $\lambda_1, \lambda_2, . \ . \ \lambda_m$. The use of augmented function allows a problem with constraints to be replaced by a problem without constraints. This new problem can be solved by any technique used for solving problems without constraints.

5 The Calculus of Variations

The calculus of variations is that branch of calculus in which extremal problems are investigated under more general conditions than those considered in the ordinary theory of maxima and minima. More specifically, the calculus of variations is applied to process optimization problems, where the maxima or minima of functional expressions are investigated. The most general problems of the calculus of variations in one independent variable are the problem of Bolza, Mayer, and Lagrange. These problems are theoretically equivalent and any one of them can be transformed into another by change of coordinates.

5.1 Problem of Bolza

In order to formulate the problem of Bolza, consider a class of functions $x_k(t)$, (k = 1, 2, …n) satisfying the constraints $g(x,\dot{x},t)=0$ which involve n-m degree of freedom. Assuming that these functions must be consistent with the end conditions:

$\psi_r(x_i,t_i)=0$ (r =1, 2, . . . q)

$\psi_r(x_f,t_f)=0$ (r = q+1, q+2, . . . s ≤ 2n+2) (24)

where (2n+2) are the total boundary conditions, q are the initial conditions, and (s-q) are the final conditions.

Find that special set that minimizes the functional form

$$J = K[x(t_f),t_f]+\int_{t_i}^{t_f} L(x,\dot{x},t)dt \qquad (25)$$

The above formulated problem can be treated in a simple manner if a set of variables, called Lagrange multiplies, $\lambda_1,\lambda_2,\ldots\lambda_m$ is introduced and if the following expression, called the augmented function, is formed:

$$F = L + \sum_{j=1}^{m}\lambda_j g_j \qquad (26)$$

or $F = L + \lambda_1 g_1 + \lambda_2 g_2 + \ldots + \lambda_m g_m$

The extremal arc must satisfy constraints and the Euler-Lagrange equation is

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}_k}\right) - \frac{\partial F}{\partial x_k} = 0 \quad (k = 1, 2, \ldots n) \qquad (27)$$

A mathematical consequence of the Euler-Lagrange equations is the differential relationship

$$\frac{d}{dt}\left(-F + \sum_{k=1}^{n}\frac{\partial F}{\partial \dot{x}_k}\dot{x}_k\right) + \frac{\partial F}{\partial t} = 0 \qquad (28)$$

For the problems where the augmented function is independent of t, the following integral, called the first integral is valid:

$$-F + \sum_{k=1}^{n}\frac{\partial F}{\partial \dot{x}_k}\dot{x}_k = C \qquad (29)$$

where C is an integration constant.

5.2 The Problem of Mayer

The problem of Mayer is that particular case of Bolza in which the integral part of cost functional J is zero, i.e. when

$$F = 0 \qquad (30)$$

As the constraint equation is $g_j(x_k,\dot{x}_k,t)=0$, therefore, the augmented function in Mayer problem is equal to zero.

$$F = L + \sum_{j=1}^{m}\lambda_j g_j = 0 \qquad (31)$$

As $F=0$, therefore, the cost functional is

$$J = K[x(t_f),t_f] \qquad (32)$$

5.3 The Problem of Lagrange

The problem of Lagrange is that particular case of the problem of Bolza in which the cost functional J is expressed in the integral form only, i.e.

$$K = 0 \qquad (33)$$

$$J = \int_{t_i}^{t_f} L(x_k,\dot{x}_k,t)dt \qquad (34)$$

the transversality condition is simplified to

$$\left. -Cdt + \sum_{k=1}^{n}\frac{\partial F}{\partial \dot{x}_k}dx_k \right]_i^f = 0 \qquad (35)$$

5.4 Problems Involving Inequalities

In many physical problems, there are various inequality constraints on the control vector for example, the maximum throttle setting, maximum control deflecting, etc. So in a problem with minimizing a cost functional

$$J = \int_{t_i}^{t_f} L(\dot{x},x,t)dt \qquad (36)$$

with equality constraints of the form:

$$g(\dot{x},x,t) = 0 \qquad (37)$$

and inequality constraints of the form:

$$\Gamma_{\min} \le \Gamma(\dot{x},x,t) \le \Gamma_{\max} \qquad (38)$$

In order for a trajectory to be admissible, it must satisfy both the equality and the inequality constraints. The slack-variable method, convert the inequality constraint equation (38) to an equality constraint by introducing new state variables z, satisfying the equation

$$(\Gamma - \Gamma_{\min})(\Gamma_{\max} - \Gamma) = z^2 \qquad (39)$$

The problem under consideration becomes identical with that finding the extremal arc, in class x(t), z(t), which satisfy the equality constraints, and Lagrange multiplier can now be used to adjoin equations (36 to 39) and used the usual necessary conditions are the applicable. Therefore, a problem with two-sided inequality constraint is replaced by the equality constraint in the new problem. Similarly, a problem with one-sided inequality constraint, $\Gamma(\dot{x}, x, t) \le \Gamma_{\max}$ is replaced by the equality constraint $(\Gamma_{\max} - \Gamma) = z^2$.

5.5 Formulation of Optimal Control Problem in Calculus of Variations

The formulation of the optimal control problem using the Hamiltonian function is set of the following expressions:

System dynamics

$$\dot{x} = f(x, u, t) \qquad t \ge t_i \quad t_i = \text{fixed} \qquad (40)$$

Performance index

$$J = K[x(t_f), t_f] + \int_{t_i}^{t_f} L(x, u, t)\,dt \qquad (41)$$

Final state constraint

$$\Psi(x(t_f), t_f) = 0 \qquad (42)$$

Hamiltonian

$$H(x, u, t) = L(x, u, t) + \lambda^T f(x, u, t) \qquad (43)$$

State equation

$$\dot{x} = \frac{\partial H}{\partial \lambda} = f(x, u, t) \qquad t \ge t_i \qquad (44)$$

Costate equation

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -\frac{\partial L}{\partial x} - \left(\frac{\partial f}{\partial x}\right)^T \lambda \qquad (45)$$

Stationary condition

$$\frac{\partial H}{\partial u} = \frac{\partial L}{\partial u} + \left(\frac{\partial f}{\partial u}\right)^T \lambda = 0 \qquad (46)$$

Boundary conditions $x(t_i)$ given

$$(K_x + \Psi_x^T \mu - \lambda)^T dx(t_f) + (K_t + \Psi_t^T \mu + H)\big|_{t_f} dt_f = 0 \qquad (47)$$

Therefore, the optimization depends on the solution to a two-point boundary value problem. Normally the initial state $x(t_i)$ is known and the final costate $\lambda(t_f)$ is determined by equation (45). It is generally very difficult to solve the two-point boundary value problems. The most serious limitation of the Calculus of Variations is that if there are constraints on the control input as in almost all the practical problems, it is not going to work, because the necessary conditions for the extremum would not apply.

6 Dynamic Programming

Dynamic programming is based on the Bellman's principle of optimality, can be viewed as an outgrow of the Hamilton-Jacobi approach to variational problems. The Dynamic programming approach to optimal control problem is basically meant for discrete-time systems. However, it can be used for continuous-time systems.

Consider a system with plant dynamics $\dot{x} = f(x, u, t)$ and associated with a performance index

$$J = K[x(t_f), t_f] + \int_{t_i}^{t_f} L(x, u, t)\,dt \qquad (48)$$

We are interested in determining a continuous optimal control $u^*(t)$ on a given interval $[t_i, t_f]$ that minimizes J and drives a given initial state $x(t_i)$ to a final state satisfying constraint $\Psi[x(t_f), t_f] = 0$

Suppose t is the current time and $t+\Delta t$ is a future time close to t. Then the cost functional can be written as:

$$J(x,t) = K[x(t_f), t_f] + \int_{t+\Delta t}^{t_f} L(x,u,\tau)\,d\tau + \int_{t}^{t+\Delta t} L(x,u,\tau)\,d\tau \qquad (49)$$

Comparing equation (48) and (49)

$$J(x,t) = \min_{\substack{u(\tau) \\ t<\tau<t+\Delta t}} \left(\int_{t}^{t+\Delta t} L(x,u,\tau)\,d\tau + J(x+\Delta x, t+\Delta t) \right) \qquad (50)$$

where $x + \Delta x$ is the state at time $t + \Delta t$, therefore, and $\Delta x = f(x, u, t)\Delta t$

Hence,

$$J^*(x,t) = \min_{\substack{u(\tau)\\ t<\tau<t+\Delta t}} \left(\int_t^{t+\Delta t} L(x,u,\tau)d\tau + J^*(x+\Delta x, t+\Delta t) \right) \quad (51)$$

The Taylor series expansion of $J^*(x+\Delta x, t+\Delta t)$ about x, t and taking an approximation to the integral in equation (51), we can write to first order

$$J^*(x,t) = \min_{\substack{u(\tau)\\ t<\tau<t+\Delta t}} \left(L\Delta t + J^*(x,t) + J_x^{*T}\Delta x + J_t^*\Delta t \right) \quad (52)$$

where $J_x^* = \frac{\partial J^*}{\partial x}$ and $J_t^* = \frac{\partial J^*}{\partial t}$

$$J^*(x,t) = J^*(x,t) + J_t^*\Delta t + \min_{\substack{u(\tau)\\ t<\tau<t+\Delta t}} \left(L\Delta t + J_x^{*T} f\Delta t \right) \quad (53)$$

or $$-J_t^*\Delta t = \min_{\substack{u(\tau)\\ t<\tau<t+\Delta t}} \left(L\Delta t + J_x^{*T} f\Delta t \right) \quad (54)$$

Letting $\Delta t = 0$

$$-\frac{\partial J^*}{\partial t} = \min_{u(t)} \left(L + \left(\frac{\partial J^*}{\partial x} \right)^T f \right) \quad (55)$$

This partial differential equation for the optimal cost $J^*(x,t)$ is called the Hamilton-Jacobi-Bellman (HJB) equation. It is solved backward in time from $t = t_f$, by setting $t_i = t_f$ in equation (55), its boundary conditions seen to be

$$J^*\left(x(t_f), t_f\right) = K\left(x(t_f), t_f\right) \quad (56)$$

on hyper surface $\Psi\left(x(t_f), t_f\right) = 0$

If we define the Hamiltonian function as:

$$H(x,u,\lambda,t) = L(x,u,t) + \lambda^T f(x,u,t)$$

and $$\lambda = \frac{\partial J^*}{\partial x}$$

Then the HJB equation can be written as:

$$-\frac{\partial J^*}{\partial t} = \min_u \left(H(x,u,J_x^*,t) \right) \quad (57)$$

The HJB equation provides the solution to the optimal control problem for general nonlinear time-varying systems; however, in most cases it is impossible to solve analytically. When it can be solved, it provides an optimal solution.

7 The Pontryagin's Maximum Principle

The mathematical modeling of a flight trajectory optimization using the Pontryagin's Maximum Principle can be done with the assumption that the optimal control exists and it is unique. Suppose that the optimal control $u^*(t)$ exists, it is unique, and that $x^*(t)$ is the generated optimal trajectory. Then, corresponding to $u^*(t)$ and $x^*(t)$, there exist a costate vector $p^*(t)$ such that the following relations hold:

Plant Dynamics

$$\dot{x}(t) = f[x(t), u(t), t] \quad x \text{ is n-dimensional} \quad (58)$$

Control Constraints

$$g[u(t),t] \geq 0 \qquad u \text{ is m-dimensional} \quad (59)$$
$$u(t) \in \Omega$$

Cost Functional

$$J(u) = K[x(t_f), t_f] + \int_{t_o}^{t_f} L[x(t), u(t), t]dt \quad (60)$$

The Necessary Conditions for Optimal Control

$$\frac{dx_i^*(t)}{dt} = \left.\frac{\partial H}{\partial p_i(t)}\right|_* \quad \text{state equations} \quad (61)$$

$$\frac{dp_i^*(t)}{dt} = -\left.\frac{\partial H}{\partial x_i(t)}\right|_* \quad \text{costate equations} \quad (62)$$

where $|_*$ means that the partial derivative must be evaluated at $u^*(t)$, $x^*(t)$ and $p^*(t)$.

7.1 Boundary Conditions

Initial Conditions

$$x^*(t_i) = x_0 \text{ (normally given)} \quad (63)$$

Terminal Conditions

$$\psi[x(t_f), t_f] = 0 \quad \psi \text{ is k-dimensional} \quad (64)$$

Transversality Conditions

$$p(t_f) = \frac{\partial K}{\partial x_f} + \left(\frac{\partial \psi}{\partial x_f}^T \right)\mu \quad (65)$$

μ is k-dimensional

The Hamiltonian

$$H = H[x(t), p(t), u(t), t] = p_0 L[x(t), u(t), t] + p^T f[x(t), u(t), t] \quad (66)$$

Minimization of the Hamiltonian

$$H\left[x^*(t), u^*(t), p^*(t), t\right] \leq H\left[x^*(t), u(t), p^*(t), t\right] \quad (67)$$

for every t, $t_i \leq t \leq t_f$, and all admissible values of $u(t)$, i.e. $u(t) \in \Omega$

7.2 Selection of state equations

In case of an aircraft, with the assumption that there is no wind, no sideslip and all the moments are in equilibrium, the general equations of motion over spherical and non rotating earth are the state equations. Usually, the optimal trajectory problems are sought under these conditions; otherwise, the problem becomes complicated and can't be solved. Therefore, the state equations are expressed as:

$$\left.\begin{aligned}
\dot{x}_1 &= \dot{v}(t) = \frac{dv}{dt} = \frac{P}{m}\cos\alpha - \frac{D}{m} - g \sin\gamma \\
\dot{x}_2 &= \dot{\gamma}(t) = \frac{d\gamma}{dt} = \frac{1}{mv}(P\sin\alpha + L)\cos\varphi - \frac{1}{v}\left(g - \frac{v^2}{R+h}\right)\cos\gamma \\
\dot{x}_3 &= \dot{\chi}(t) = \frac{d\chi}{dt} = \frac{1}{mv\cos\gamma}(P\sin\alpha + L)\sin\varphi \\
\dot{x}_4 &= \dot{x}_g(t) = \frac{dx_g}{dt} = v \cos\gamma \cos\chi \\
\dot{x}_5 &= \dot{y}_g(t) = \frac{dy_g}{dt} = v \cos\gamma \sin\chi \\
\dot{x}_6 &= \dot{h}(t) = \frac{dh}{dt} = v \sin\gamma \\
\dot{x}_7 &= \dot{m}(t) = \frac{dm}{dt} = -q_s
\end{aligned}\right\} \quad (68)$$

7.3 Selection of Control variables

We may chose a set of variables as control variable, which will determine the trajectory, such as throttle setting η, angle of attack α, angle of roll φ etc. Depending on the analysis of a desired trajectory any one or combination of the control variables can be selected, but these controls are subjected to physical limitations or constraints as mentioned below:

Control Constraints

$$u(t) \in \quad g[u(t), t] \geq 0 \quad u \text{ is m-dimensional} \quad (69)$$

1. $\eta_{\min} \leq \eta \leq \eta_{\max}$
2. $\alpha_{\min} \leq \alpha \leq \alpha_{\max}$
3. $C_{L\min} \leq C_L \leq C_{L\max}$
4. $|n_L| = n_{L\,perm}(M)$
5. $|\varphi| \leq \varphi_{perm}$
6. $M \leq M_{perm}$

7.4 Derivation of Costate Equations

Depending on the number of state variables ($v, \gamma, \chi, x_g, y_g, h, m$ etc.), we must have equal number of costate variables [$p_1(t)$, $p_2(t)$, $p_3(t)$, $p_4(t)$, $p_5(t)$, $p_6(t)$, $p_7(t)$ etc.].
Thus, we have the costate vector as:

$$p(t) = \begin{vmatrix} p_1(t) \\ p_2(t) \\ \cdot \\ \cdot \\ p_7(t) \end{vmatrix} \quad (70)$$

Therefore, the Hamiltonian, defined by equation (66) can be expressed as:

$$\begin{aligned}
H &= p_0 L[x(t), u(t), t] + p_1(t)\left\{\frac{1}{m}\left(P\cos\alpha - \frac{1}{2}\rho v^2 S C_D\right) - g\sin\gamma\right\} \\
&+ p_2(t)\left\{\frac{1}{mv}\left(P\sin\alpha + \frac{1}{2}\rho v^2 S C_L\right)\cos\varphi - \frac{1}{v}\left(g - \frac{v^2}{R+h}\right)\cos\gamma\right\} \\
&+ p_3(t)\left\{\frac{\sin\varphi}{mv\cos\gamma}\left(P\sin\alpha + \frac{1}{2}\rho v^2 S C_L\right)\right\} + p_4(t) v\cos\gamma\cos\chi \\
&+ p_5(t) v\cos\gamma\sin\chi + p_6(t) v\sin\gamma - p_7(t) q
\end{aligned} \quad (71)$$

Where $p_o \leq 1$ (constant scalar and normally is zero).
The costate equations, as defined by equation (62), can be derived by taking the partial derivative of the Hamiltonian (71) with respect to each state variable as mentioned below:

$$\begin{aligned}
\dot{p}_1^{\bullet} &= -\frac{\partial H}{\partial x_1(t)} = -\frac{\partial H}{\partial v(t)}, \\
\dot{p}_2^{\bullet} &= -\frac{\partial H}{\partial x_2(t)} = -\frac{\partial H}{\partial \gamma(t)} \\
\dot{p}_7^{\bullet} &= -\frac{\partial H}{\partial x_7(t)} = -\frac{\partial H}{\partial m(t)}
\end{aligned} \quad (72)$$

7.5 Cost Functional or Objective Function

The general form of the cost functional of a system is given by the relation defined by equation (61), but the decision about the cost functional or objective function depends on the task of the system. Therefore, the decision about the cost functional or objection function depends upon the type of the aircraft and the requirement of the particular mission to be accomplished by the aircraft. In general, a cost functional can be a functional of different functions or variables. The scaling among these functions or variables may not be same due to difference in dimensions; or it may be desired to make the cost functional more sensitive to one variable as compared to others. In such case, weighing (or scaling) factors are introduced in the cost functional to implement the required scaling or desired effects of the different variables on the cost functional. For example, in case of an aircraft, considering both time and the fuel consumption as the measure of performance, the cost functional can be expressed as:

$$J(u) = \int_{t_i}^{t_f} dt + W(m_i - m_f)$$

or

$$J(u) = \int_{t_i}^{t_f} dt + W(\Delta m) \qquad (73)$$

where m_i and m_f is the mass of the aircraft at the initial time t_i and the final time t_f of the trajectory respectively, and W is the weighing factor.

7.6 Minimization of the Hamiltonian and Solution of the Differential Equations

Once all the aerodynamic and engine data is available, then the differential equations are solved to get the state and costate variables. At each time interval, the Hamiltonian is calculated for all permissible values of the control parameter and the optimal value is selected. This optimal value is used in the solution of the differential equations for that particular time. The equations are solved at each time interval, from the initial time to the terminal time of the trajectory to be optimized. The values of the costate variables evaluated at the terminal time are compared with the desired values of the costates, which are obtained using transversality conditions (65) as following:

$$p_{jf} = \frac{\partial K}{\partial x_{jf}} + \left(\frac{\partial \psi_1}{\partial x_{jf}} \quad - \quad - \quad \frac{\partial \psi_k}{\partial x_{jf}} \right) \begin{vmatrix} \mu_1 \\ - \\ - \\ \mu_k \end{vmatrix} \qquad (74)$$

(j = 1, 2, 3, - - - n)

where K is the terminal cost, ψ is k-dimension terminal conditions, and μ is also k-dimension unknown parameters analogous to Lagrange multipliers, the values of these parameters are chosen as a guess while assigning the initial guess values of the costate variables. Therefore, the desired final values are function of these initial guess values of μ. The comparison between the values of the costate variables evaluated at the terminal time and the desired values of the costates will determine whether or not the boundary conditions are satisfied. If not, the initial guess value of the costates and μ are changed using optimization subroutine Simplex and the process is repeated until the boundary conditions are met. The initial values of costate and μ, that result in meeting the terminal conditions, are used along with the other terminal conditions to evaluate the optimum trajectory.

7.7 Examples

To verify the approach, a manoeuvrable aircraft TC1.1 (an experimental aircraft) [6] was selected for the evaluation of optimal trajectories in horizontal flight and in vertical flight and the results are shown in the figures 1 through 5.

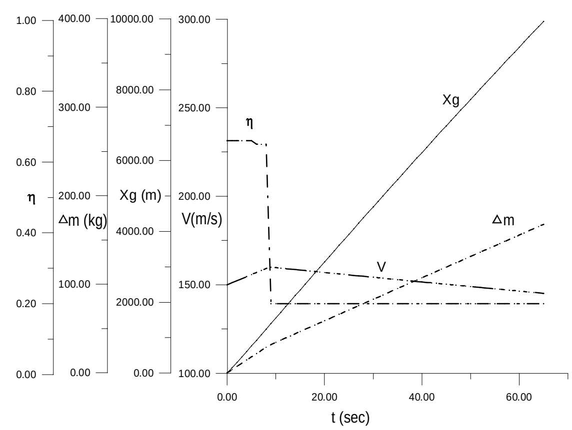

Figure 1. Minimum fuel state trajectories (Case II)

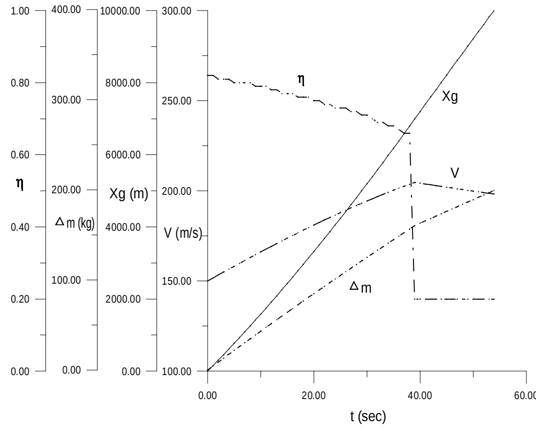

Figure 2. Minimum time and minimum fuel state trajectories (Case III)

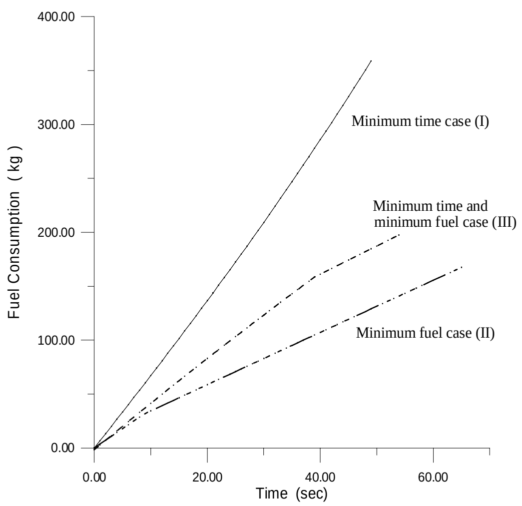

Figure 3. Comparison among the three flight cases

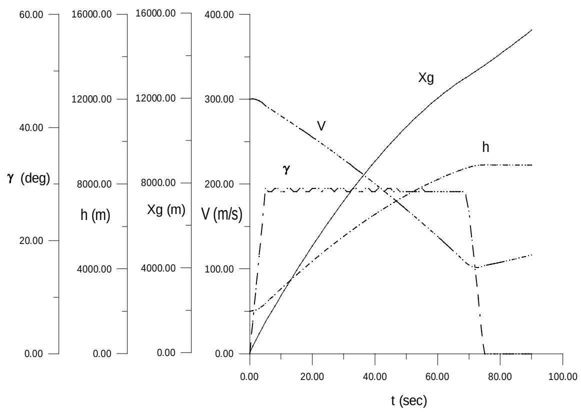

Figure 4. Optimal state trajectories of flight in vertical plane

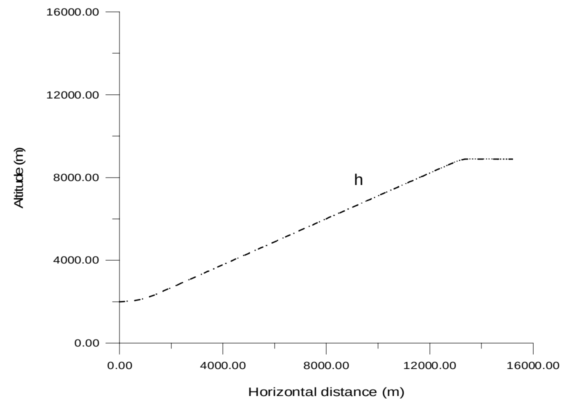

Figure 5. Flight path in vertical plane

7.8 Analysis of results

The horizontal flight trajectories of the above examples shown in figures 1, 2, and 3 reveal the desired results. The results of flight in vertical plane as shown in figure 4 and 5 reveal the desired flight path control. However, it was observed that as the terminal conditions can be set in two possible ways, for example:

$$\psi\left(x_g(t_f), t_f\right) = 0$$

$$x_{greq} - x_{gf} = 0 \qquad \text{or}$$

$$x_{gf} - x_{greq} = 0$$

Both should yield the same results, but practically it was observed the inspite of convergence in both the cases the results were different. In minimum time case I of example in the horizontal flight trajectory, for the terminal condition $x_{greq} - x_{gf} = 0$, the result is logical, i.e. it results in maximum throttle control and minimum time. While for the terminal condition $x_{gf} - x_{greq} = 0$, it results in minimum throttle setting and takes maximum time, although it meets the end conditions. This shows that the Pontryagin's minimum principle provides only the necessary conditions and not the sufficient conditions for optimality. The solution may or may not be optimal. In this case both the results yield the extremal trajectories, but only one is optimal. It was also observed that the change in boundary conditions for the state variables results in change of the optimal trajectories, however, the change in the initial guess of the costate variables does not effect the results.

8 CONCLUSION

The following conclusions are drawn from the analysis of the results:

- Flight trajectory optimization problems can be solved by any of the optimization techniques: the Calculus of Variations, the Belman's Dynamic Programming, the Pontryagin's Maximum Principle, etc. depending on the complexity and nature of the constraints, however, the Pontryagin's Maximum Principle is applied as a test case.

- It has been proved by the results that Pontryagin's maximum principle provides only the necessary conditions and not the sufficient conditions for optimality, but still it can be successfully applied for optimization of trajectories of any aircraft for any flight phase. Only it has to be ensured to analyze all the extremal trajectories and find the optimal trajectory by their comparison.

- The change in boundary conditions for the state variables results in change of the optimal trajectories; however, the change in the initial guess of the costate variables does not effect the results.

- The Pontryagin's maximum principle may be applied in similar way for the optimization of trajectories of the spacecraft, and the same atmospheric model can be used for the launch or re-entry phase of the spacecraft.

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