PRE-STRESSED SYMMETRIC COMPOSITE BEAMS IN AIRCRAFT - ANALYSIS AND DESIGN IMPLICATIONS

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Abstract

The uniform beam has a curved elastic core. It is reverse-bent prior to the attachment of the two stiffer outer-layers. When the bonds are dry and after the pre-loading bending-moment has been removed, the residual stresses modify the internal loads. From strain-compatibility at the interface, the allowable values of the associated stresses depend on the material properties and the geometry of the cross-section. Their limitations are determined by balancing the relaxation bending-moments, whilst equating the strains yields the beam curvatures used in its fabrication.

The pre-stress is relieved before reaching the maximum load, when both materials are designed to arrive at their ultimate stresses together. Were it not for core shear flexibility, the values of these stresses would be limited solely by the ratios of Young's Modulus and the material's strength. The maximum relaxation stresses in the outer-layers are made equal to the opposite load-direction ultimate strengths. Then for optimal pre-stress, the residual stress and core flexibility are determinate. This analysis provides workable limits for the geometric and stiffness ratios.

It is found that under the design loading, the static strengths of the regular and pre-stressed beams are identical, although when loaded in the opposite direction, the latter has less strength. However, the pre-stressed beam has more flexibility, with a greater capacity for energy absorption. The least structure needed for absorbing a specific amount of kinetic energy (such as in a leaf-spring landing gear or a windscreen post for resisting bird-impacts) is the pre-stressed beam. Mass savings of up to 50% are theoretically possible, compared to regular beam designs.

1 Introduction

Pre-stressed ferro-concrete floor-beams have been used in buildings for many years. They enable the civil engineers to successfully reduce the structural mass (and cost) of their designs. This concept possibly could be used to advantage in certain other engineering situations too and its application for use in flight-vehicles will be examined here. It is thought that particular kinds of aircraft structural components could be reduced in mass by significant amounts, when they employ designs having this feature.

The past use of pre-stressing in aircraft structures has been confined to a few specific applications. Examples are the cold-working of attachment holes, high squeeze riveting, shot-peening (used for shaping sheet metal and improving the resistance to fatigue), pre-tensioned bolted-joints, the inflation of tires and shock-absorbers (landing-gears) and pressured airships (blimps). Except when the inflation pressure is used to apply the pre-load, these methods do not result in a significant a degree of improvement in the structural design; not at least when compared to the achievement in civil engineering. The other aeronautical applications are more localized and, in order to make the best use of this feature, there is a need to find a situation which more closely resembles that of the floor- beams.

In this study, pre-stressing of a symmetric composite beam will be examined. The three structural elements of the beam have uniform cross-sections and behave as if they were continuous. Advantage will be taken of the static strength properties of these elements, whose two materials have different mechanical strength properties, positions and shapes.

2 Method of Construction and of Introducing Pre-Stress

The method of pre-stressing is applied to the following built-up structure, which is a composite I-beam of uniform cross-section, designed to resist a specific bending-moment. It consists of a central rectangular core and two equal outer-layers. The core is made from a comparatively low-density material having a reduced capacity to resist to the axial loads (tension and compression), but good shear strength. The outer top and bottom layers of the beam are made from a relatively dense material having high axial strength. However these facings do not provide much shear resistance. The beam is shown in side-view in Fig. 1.

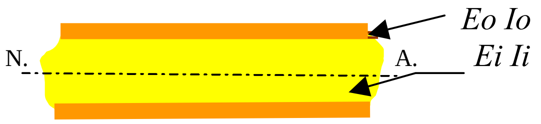

Fig. 1 The Composite Beam

Instead of the usual method of fabrication, which is to bond the three beam elements together as straight pieces, the core of the beam is first formed as a curved member, as shown in Fig. 2.

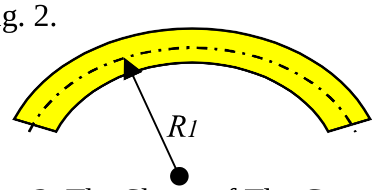

Fig. 2 The Shape of The Core

To pre-load the composite-material beam, two equal bending-moments *Mi* are applied at the ends of the core to reverse bend it. The situation is shown in Fig. 3. Here, the + sign shows the presence of the pre-tensile stress, and the - sign indicates the pre-compression, both of which depend on the nature of the imposed strains.

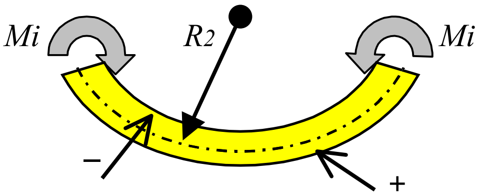

Fig. 3 Pre-Stressed Condition of The Core

The outer-layers are attached and bonded whilst the loading continues to be applied, see Fig. 4.

In cases where these layers are very thin they may be pre-formed flat, because they are easily bent to suit the upper and lower surfaces of the core. Thicker layers need to be cured *in-situ*.

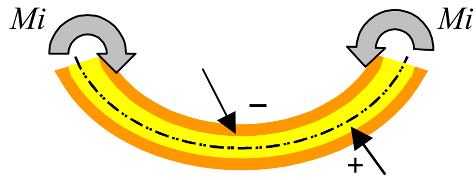

Fig. 4 The Fabrication of the Outer-Layers

When the materials have set, the bending-moments *Mi* are then released and the beam springs back (or relaxes) to a position that lays between the previous two extreme curved forms. The resulting shape may be designed for an intermediate radius, or it may be straight. The distribution of tension and compression relaxation stresses across the beam at a typical cross-section is shown in Fig. 5. This relaxed load situation is at a central lengthwise part of the beam.

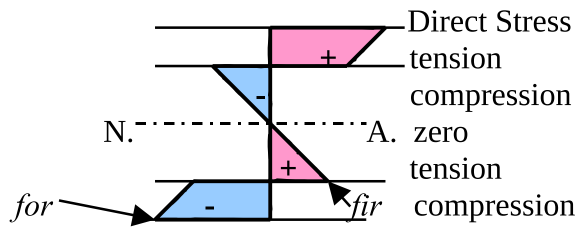

Fig. 5 The Residual Stress Distribution on The Cross-section

Due to the pre-stressing, the inner-core carries a shear stress and a corresponding strain. After relaxation, the compatibility of the longitudinal strains at the interfaces, transfers part of them from the core to the outer-layers. Across this interface, the different values of Young's Modulus produce a significant change in magnitude of the resulting residual direct-stresses in these materials. As shown, they vary linearly with distance from the neutral axis.

By this procedure the beam is internally pre-loaded. After their application, the external bending loads on the beam compress the relaxation-tension side and stretch the relaxation-compression side. The direct-stress distribution under full loading is shown in Fig. 6, the shape of which should be compared to the previous figure (*fi* > *fir* , *fo* > - *for*).

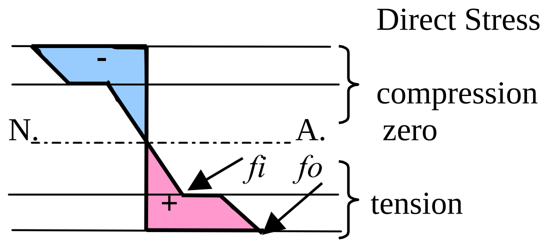

Fig. 6 The Fully-Loaded Stress Distribution

As the external loading is applied, the residual shear stresses between the core and the outer-layers are affected. These stresses return to zero before changing sign and increasing in size in the opposite direction. They are described in more detail in Section 3.7.

3 Stress and Strain Analyses

To determine in numerical terms how this process may be used to fabricate a composite pre-stressed beam, some of its basic structural properties are required as input. This implies that the procedure will yield additional structural properties as design outputs. The following theoretical algebraic analysis establishes the relationships between the significant parameters of both kinds.

3.1 General Approach and the Core Pre-Load Bending-Moment *Mi*

According to the classical Engineers' Theory of (Elastic) Bending [1], the general relationships for beam's direct-stresses and deformation are shown below:

$$M/I = f/y = E/R \qquad (1).$$

where M is the applied bending-moment and I is the second moment of inertia of the beam's cross-section, about its centroid in the plane containing the bending axis. This is the Neutral Axis (N.A.) of the cross-section, where the longitudinal direct stress changes sign. The resulting stress f is felt on a fiber that is spaced distance y from the N.A. Linear elastic properties are assumed to apply in all of the analysis. E is the value of Young's Modulus and R is the change in radius on the center-line.

The general expression for the moment of inertia I, when taken about the center-line of a rectangular cross-section of width B and height T is given by :

$$I = B\,T^3/12$$

and the greatest value of $y = T/2$ (2).

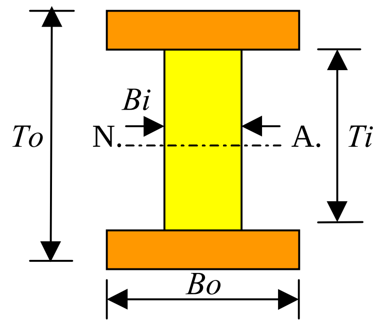

Fig. 7 Cross-Section of The Uniform Beam

The composite beam's cross-section is shown in Fig.7, having symmetry about it's N.A. The suffix i is used to indicate the uniform inner part or core and the suffix o is for the outer-layers. It is assumed that sufficient core breadth/thickness ratio Bi/Ti is provided so as to avoid compression and shear instabilities.

Using these equations, the greatest bending-moment, which the compound beam can carry:

$$M = Mo + Mi = 2\,(Io\,fo\,/To + Ii\,fi\,/Ti)$$

$$= Bo\,(To^3 - Ti^3)\,fo\,/(6\,To) + fi\,Bi\,Ti^2/6 \qquad (2a).$$

For convenience the ratio of this beam's thicknesses is written as $Ti/To = t$ and the ratio of its breadths is written as $Bi/Bo = b$. Then:

$$M = Bo\,To^2\,fo\,(1 - t^3 + b\,t^2\,fi/fo)/6 \qquad (2b).$$

The analysis is now presented for the fabrication process described above. The magnitude of the stresses $\pm\,p\,fi$ that are felt by the core, depends on the bending-moment Mi initially applied to it. The factor p is the pre-stressing proportion of the ultimate stress fi and the size of p lays between zero and one. From equations (1) and (2):

$$Mi = p\,fi\,Ii/yi = p\,fi\,Bi\,Ti^2/6 \qquad (3).$$

Due to Mi the radius of curvature changed from the original amount R_1 , to R_2 in the opposite direction, which accounts for the negative sign in the following expression, based on equation (1):

$$Mi = Ei\ Ii\ (1/R_1 - 1/R_2) \qquad (4).$$

3.2 The Relaxation Bending-Moment Effect

After the materials have cured, the applied pre-loading bending-moment *Mi* is released. The curvature that the composite beam then adopts is due solely to its internal relaxation bending-moments. Using the suffix *r* to indicate the relaxed load condition, the bending-moment *Mr* is what remains of *Mi* in the core, and an equal and opposite amount is carried by the outer-layers.

In a similar manner to the derivation of equation (3) , for the core :

$$Mr = fir\ Ii/yi = fir\ Bi\ Ti^2/6 \qquad (5).$$

We can also write for the outer-layers:

$$Mr = -\, for\ Io/yo = -\, for\ Bo\ (To^3 - Ti^3)/(6\ To) \quad (6).$$

The minus sign has been included due to the reversal of stress at the interfaces between the core and the outer-layers, *for* is of opposite sign compared to *fir*. Using equations (5) and (6), and making *- for* the subject:

$$-\, for = fir\ To\ Bi\ Ti^2/(Bo\ (To^3 - Ti^3))$$

$$= fir\ b\ t^2/(1 - t^3) \qquad (7).$$

The product of the breadth ratio and the relaxation stress in the core *(b fir)*, depends exclusively on the thickness-ratio *t* and the residual stress in the outer-layers *- for*. The value of *(b fir)* is normally chosen to be smaller than *- for*. Then it is found that the outer-layers must be relatively thin (*t* only slightly less than 1). Table 1 shows this relationship in numerical terms.

TABLE 1. RELATIONSHIP BETWEEN THE CORE THICKNESS-RATIO AND THE RELAXATION STRESS- RATIO USING EQUATION (7).

t =Ti/To	1.0	0.95	0.90	0.85	0.80	0.75	0.70
-b fir/for	0.0	0.158	0.335	0.534	0.765	1.028	1.341

The ratio *- b fir/for*, grows rapidly as the thickness-ratio *t* is progressively reduced in size. Its magnitude exceeds unity when *t* is slightly less than 0.755 . When *t* is smaller than this, the core material should be made stronger (and stiffer) than the outer layers. This concept lays outside of the original premise and it does not appear to be useful.

3.3 Compatibility of Relaxation Strains at The Interfaces

The smaller equilibrium moment *Mr* with its reaction are then carried by both kinds of elements of the composite beam. It will be seen later that the pre-stress in the core *p fi* is unable to raise the material to its ultimate value of stress, and $p < 1$ in equation (3).

Equal amounts of added strain occur on both sides of the interface, where the relaxation stresses are now developed. In general, strain ε is equal to *f/E*. Using this relationship and equations (1) and (2), the strains at the interface (laying distance *Ti* from the neutral axis) are:

for the core: $\varepsilon i = (p\,fi - fir)/Ei$

$$= (Mi - Mr)\ Ti/(2\ Ei\ Ii) \qquad (8).$$

It is noted that for the core, *Ei* may be an equivalent value of stress/strain, to include the effect of core shear flexibility. Similarly for the outer layers: $\varepsilon o = t\ for/Eo$

$$= -\, Mr\ Ti/(2\ Eo\ Io) \qquad (9).$$

But for strain *compatibility* at the interface:

$$\varepsilon i + \varepsilon o = 0 \qquad (10),$$

hence by substitution of equations (8) and (9) into this:

$$(Mi - Mr)\ Ti/(2\ Ei\ Ii) - Mr\ Ti/(2\ Eo\ Io) = 0\ .$$

For convenience the ratio of the elasticity modulii of the core and the outer layers is written as $Ei/Eo = e$, then after substitution for the moments of inertia and re-arrangement :

$$Mi/Mr - 1 = e\ b\ t^3/(1 - t^3) \qquad (11).$$

(Note that when both kinds of elements are of equal bending stiffness, $e\,b\,t^3/(1 - t^3) = 1$ and:

$t = 1/(1 + e\,b)^{1/3}$, then $Mr = Mi/2$ (11a).)

The ratio of the relaxation to initial stresses $fir/(p\,fi)$ in the core can now be determined by taking the ratio of the applied bending-moments, from equations (4) and (5):

$$fir/(p\,fi) = Mr/Mi\ ,$$

and we may substitute equation (11) into this to yield:

$$fir/(p\,fi) = 1/(e\,b\,t^3/(1 - t^3) + 1)$$
$$= (1 - t^3)/(1 - (1 - e\,b)\,t^3) \quad (12).$$

Equation (12) may also be derived from the stress-related terms in equations (8) and (9), instead of the bending-moment related ones. For practical values of e, b and t, all of which are less than 1, the residual stress in the core, fir is smaller than the factored ultimate value $p\,fi$ that was introduced by the first-applied bending-moment Mi. Also, by using equation (7) with equation (12):

$$-for/(p\,fi) = b\,t^2/(1 - (1 - e\,b)\,t^3) \quad (13),$$

which can exceed unity when t is close to it.

3.4 The Bending Radii Relationships

From the general bending equation (1), it is easy to show that $\varepsilon = y/R$. Suppose that the relaxed beam adopts a radius Rr, then at the interfaces:

for the outer-layers: $\varepsilon o = Ti\,(1/R_2 - 1/Rr)/2$

and similarly for the core:

$$\varepsilon i = Ti\,(1/R_1 - 1/Rr)/2\ .$$

From equation (10):

$$Ti\,(1/R_1 + 1/R_2 - 2/Rr)/2 = 0$$

and $Rr = 2/(1/R_1 + 1/R_2)$ (14).

This enables the value of Rr to be design-controlled. When the relaxed shape of the beam is straight, Rr becomes infinite. Then it is found that R_1 and R_2 must be of equal magnitude but opposite in sign. Even when the linear shear displacements of the core are included, this result is true.

3.5 The Direct Stresses Under Full External Loading

After the full external bending-moment is applied to the relaxed composite beam, it loads the outer-layers in opposition to their relaxation stresses (initially reducing their size), whilst in the core the direct stresses are simply added to the relaxation stresses. The pre-stressing proportion p is arranged so that after relaxation and under the full bending-moment, the stress levels in the outer layers and in the core both reach their maximum designed (ultimate) values together. This is expressed by the increments of strain in both kinds of elements being the same at their interface. The stress increment in the outer layers is $(fo - for)$, where the value of for is negative, resulting in a comparatively large difference, and the stress increment in the core is $(fi - fir)$, which is smaller. Then for equal amounts of added strain:

$$t\,(fo - for)/E = (fi - fir)/Ei\ ,$$

which compares with the relaxation equations (8) and (9).

For the core outer-fibers we can therefore write:

$$fi = fir + e\,t\,(-for + fo) \quad (15),$$

Using equation (7) to eliminate fir, the above expression then becomes:

$$fi = -for\,(1 - t^3)/(b\,t^2) + e\,t\,(-for + fo)$$

$$= -for\,(1 - (1 - e\,b)\,t^3)/(b\,t^2)) + e\,t\,fo \quad (16).$$

Now substitute for for using equation (13):

$$fi = p\,fi + e\,t\,fo \quad (17).$$

The rising value of external load causes the core to reach the pre-stress values $\pm\,p\,fi$. At this moment they act on both sides, without any load being carried by the outer layers, where the relaxation effect has just been relieved. However when the full external load applies, these elements subsequently develop the stresses $\pm fo$ and the additional strains at the interfaces are equal, as indicated above. After re-arrangement:

$$p = (1 - e\, t\, fo/fi) \qquad (18).$$

For pairs of materials having equal ductility, $e = fi/fo$ and by equation (18) $p = (1 - t)$ and only a small amount of pre-stressing is possible (if we want to arrive at fo and fi together). However in practice, many pairs of structural materials have a different ratio of their ultimate stresses compared to the ratio of their values of Young's Modulii e. These values are even more separated when an allowance for the effect of shear flexibility of the core is included. Consequently we now introduce the factor m to allow for the requisite general unequal elongation at ultimate stresses so that $e\ m = fi/fo$. Then equation (18) may be simplified to give:

$$p = (1 - t/m) \qquad (19)$$

and from equation (13) the ratio of $- for/fi$ becomes:

$$- for/fi = p\, b\, t^2/(1 - (1 - e\, b)\, t^3)) \ .$$

With equation (19) it yields:

$$- for/fi = (1 - t/m)\, b\, t^2/(1 - (1 - e\, b)\, t^3) \qquad (20).$$

Multiplying by $fi/fo = e\ m$ (from above) on the left-hand and right-hand sides respectively, this expression then provides:

$$- for/fo = (m - t)\, e\, b\, t^2/(1 - (1 - e\, b)\, t^3)) \qquad (21).$$

When $m = 1$ the stress ratios in these two equations both become small and there appears to be little structural capacity for pre-stressing. However, with the physical arrangement of parallel elements, the core shear flexibility acts in series with its interfacial strain. The effect of this spring is to raise the effective value of m , possibly to the amount corresponding to the ratio of the ultimate stresses.

3.6 The Ideal Relaxation Condition

The ideal design upper limit for $- for/fo$ is 1. This causes the bending-moment capacity of the outer-layers to double, which is the maximum amount possible for these elements. Substituting $- for/fo = 1$ into equation (21), and using the suffix $_{max}$ throughout:

$$1 = (m_{max} - t)\, e\, b\, t^2/(1 - (1 - e\, b)\, t^3)$$

or $$m_{max} = t + (1 - (1 - e\, b)\, t^3)/(e\, b\, t^2) \qquad (22).$$

and from equation (19):

$$p_{max} = (1 - t/(t + (1- (1 - e\, b)\, t^3))/(e\, b\, t^2))$$

$$= 1 - e\, b\, t^3/(1 - (1 - 2\, e\, b)\, t^3) \ ,$$

or: $$p_{max} = (1 - (1 - e\, b)\, t^3) /(1 - (1 - 2\, e\, b)\, t^3) \qquad (23).$$

These values cannot be exceeded. As previously suggested, the greatest pre-stressing p_{max} that can be used is less than unity. It depends on the structural properties of the elements, provided that the shear stiffness of the core is suitably adjusted. From equations (13) and (23):

$$- (for/fi)max = b\, t^2/(1 - (1 - 2\, e\, b)\, t^3) \qquad (24).$$

Equations (22), (23) and (24) for the 3 "$_{max}$" quantities, are expressed numerically in Table 2 below.

No value of m_{max} is less than 2, but only in extreme situations are values greater than 3.5 needed (see boundary-lines in the tables). As was anticipated the amount of p_{max} required is severely limited. It lays above 0.5 but does not usefully approach 1.0 . Similarly the amount of relaxation in the outer layers compared to the ultimate strength of the core rarely exceeds unity. This suggests that it is difficult to make good use of this compression unless the core breadth is small. For example, when using the practical values: $e\ b = 0.40$, $t = 0.90$, we find from the above table that:

$m_{max} = 2.736$, $p_{max} = 0.671$,

$- [for/(b\, fi)]_{max} = 0.948$ and using Table 1,

$- (b\, fir)/for = 0.335$.

These results are due to the ultimate strength limitations, defined by equation (15) as presented in equation (21), and the choice for the maximum value of the stresses in the relaxed core. Here the outer-layers work out to be the thinner stronger material, and the core is more flexible but of a greater cross-sectional area.

Combining equations (24) with (7):

$$(fir/fi)_{max} = (1 - t^3)/(1 - (1 - 2\,e\,b)\,t^3) \quad (25).$$

This is the resulting stress-ratio in the core after relaxation. It should be compared to the corresponding ratio p_{max} that is used for the pre-stress loading, where the ratios are $(1 - t^3)/(1 - (1 - e\,b)\,t^3)$ times greater. The values of $(fir/fi)_{max}$ are shown below in Table 3.

TABLE 2. RELATIONSHIPS BETWEEN THE CORE THICKNESS-RATIO t AND ITS RELATIVE STIFFNESS FACTORED BY THE BREADTH-RATIO $e\,b$, AND THE DERIVED MAXIMUM VALUES OF: m, p AND $-for/(b\,fi)$.

a) REQUISITE RELATIVE STRAIN AT FAILURE
$m_{max} = [fi/(e\,fo)]_{max}$ equation (22)

e b =	**0.20**	**0.40**	**0.60**	**0.80**	**1.00**	**1.20**
t **= 1.00**	2.00	2.00	2.00	2.00	2.00	2.00
t **= 0.95**	2.690	2.295	2.163	2.098	2.058	2.032
t **= 0.90**	3.473	2.736	2.458	2.218	2.135	2.079
t **= 0.85**	4.370	3.035	2.590	2.368	2.234	2.145
t **= 0.80**	5.413	3.506	2.871	2.553	2.363	2.235
t **= 0.75**	6.639	4.069	3.213	2.785	2.528	2.356
t **= 0.70**	7.404	4.752	3.634	3.076	2.741	2.517

b) PRE-STRESS RATIO p_{max} equation (23)

e b =	**0.20**	**0.40**	**0.60**	**0.80**	**1.00**	**1.20**
t = 1.00	0.50	0.50	0.50	0.50	0.50	0.50
t **= 0.95**	0.647	0.586	0.561	0.547	0.538	0.532
t **= 0.90**	0.741	0.671	0.634	0.612	0.578	0.567
t **= 0.85**	0.802	0.689	0.672	0.641	0.620	0.604
t **= 0.80**	0.852	0.772	0.721	0.687	0.661	0.642
t **= 0.75**	0.887	0.816	0.767	0.731	0.703	0.682
t **= 0.70**	0.905	0.853	0.807	0.772	0.745	0.722

c) RATIO OF RELAXED STRESS IN OUTER-LAYERS TO ULTIMATE STRESS OF THE CORE, FACTORED BY BREADTH-RATIO
$[-for/(b\,fi)]_{max}$ equation (24)

e b =	**0.20**	**0.40**	**0.60**	**0.80**	**1.00**	**1.20**
t **= 1.00**	2.500	1.250	0.833	0.625	0.500	0.417
t **= 0.95**	1.956	1.089	0.770	0.596	0.486	0.410
t **= 0.90**	1.440	0.948	0.707	0.564	0.468	0.401
t **= 0.85**	1.144	0.824	0.643	0.528	0.448	0.388
t **= 0.80**	0.924	0.713	0.581	0.490	0.423	0.373
t **= 0.75**	0.753	0.614	0.519	0.449	0.396	0.354
t **= 0.70**	0.617	0.526	0.459	0.406	0.365	0.331

TABLE 3. RELATIONSHIP BETWEEN CORE THICKNESS-RATIO *t*, ITS RELATIVE STIFFNESS TO INCLUDE THE BREADTH-RATIO *e b* AND THE GREATEST VALUE OF THE CORE RELAXATION STRESS-RATIO *(fir/fi)max*.

RATIO OF GREATEST RELAXED STRESS IN CORE TO ITS ULTIMATE STRESS *(fir/fi)max* equation (25)						
e b =	**0.20**	**0.40**	**0.60**	**0.80**	**1.00**	**1.20**
t = 1.00	0.0	0.0	0.0	0.0	0.0	0.0
t = 0.95	0.2937	0.1721	0.1217	0.0942	0.0768	0.0648
t = 0.90	0.4817	0.3173	0.2365	0.1885	0.1567	0.1341
t = 0.85	0.6110	0.4399	0.3437	0.2820	0.2391	0.2075
t = 0.80	0.7044	0.5437	0.4427	0.3733	0.3228	0.2842
t = 0.75	0.7741	0.6314	0.5331	0.4613	0.4066	0.3635
t = 0.70	0.8272	0.7054	0.6148	0.5449	0.4892	0.4439

The numerical values are considerably smaller than those of *pmax* except in the "extreme situation" region, where the greatest advantage is taken of the residual stresses.

The numerical values in Tables 2 and 3 are regarded as non-symmetrical, because a reversal of the materials in the core and the outer layers does not give the same result unless *e b* = 1. This is due to the location of the shear elasticity in the core, being in series with the stiffest element, which is loaded in parallel with the least stiff one.

3.7 The Shear Stresses and Strains

For beams having a rectangular cross-section or a symmetrical I-beam shape, the greatest shear-loads normally occur on the center-line, at the neutral axis. For a position on the cross-section that lays inside the outer fibers, the local shear is the sum of the end-loads acting outside of its location. These end-loads include those due to the residual stresses. This may be expressed as an integral over the depth of the cross-section. Then the shear-stress equals this integral divided by the local width *B* and length *L* of the beam, which is the distance between the supports or from the external load to its support position.

The integral applies even when the outer layers, consisting of material having greater stiffness and strength, after they are bonded top and bottom. However, at the junction of the two kinds of materials there is a change in the slope of the distribution (or rate of end-load development with depth co-ordinate). For a regular beam design without the pre-stress, these distributions of direct and shear stresses are illustrated in Fig. 8.

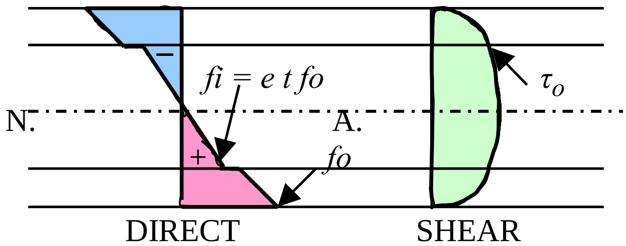

Fig. 8 Regular Beam, Distribution of Direct and Shear Stresses

For the pre-stressed beam design the distribution of the relaxation loads leads to the direct and shear distributions as shown in Fig. 9.

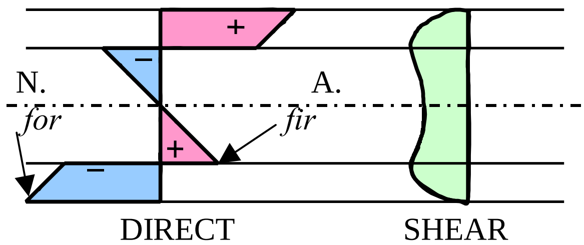

Fig. 9 Relaxation, Distribution of Direct and Shear Stresses

When the beam is fully loaded, these distributions are indicated by the modifications to this diagram that are shown in Fig. 10. It is seen that the shear-stress at the interface has changed sign but is no greater in magnitude than when the regular beam stresses were carried.

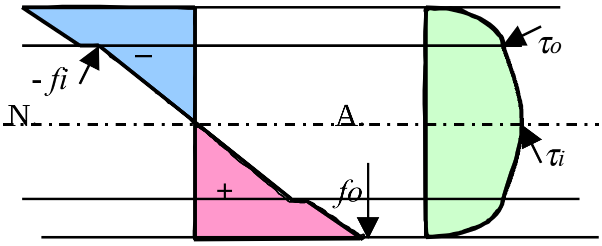

Fig. 10 Fully Loaded Beam, Distribution of Direct and Shear Stresses

In considering the bond strength it is seen that actually there is no influence in the size of the fully loaded stresses due to the use of a pre-stressed design. The only change that results from pre-stressing, is in the increment of direct stresses in the core itself. Here they increase from: $(e\ t\ fo)$ to $fi = (e\ t\ fo + p\ fi)$ see equation (17), as described in section 3.5 . An allowance for this must be made by the choice of suitable core materials.

The numerical values of the shear stress τ are determined below. From Figs. 8 and 10 the maximum shear stress occurs on the neutral axis N.A., even for the pre-stressed beam situation. In general the shear stress is found from the integral of the longitudinal stress over the depth after dividing by the width and length of the beam L .

$$\tau o = 1/(L\ Bo)\int_{Ti/2}^{To/2} y\ fo\ dy = fo\,/(L\ Bo)\Big[y^2/2\Big]_{Ti/2}^{To/2}$$

$$= fo\ To^2\ (1 - t^2)/(8\ L\ Bo) \qquad (26).$$

For the regular beam, the greatest value of shear stress, due solely the distribution of longitudinal loads in the core, may be derived in a similar manner and is:

$$\tau i - \tau o = 1/(L\ Bi)\int_{0}^{Ti/2} y\ (e\ t\ fo)\ dy$$

$$= e\ t\ fo/(L\ Bi)\Big[y^2/2\Big]_{0}^{Ti/2} = e\ t\ fo\ Ti^2/(8\ L\ Bi) \qquad (27).$$

The sum of these is the total maximum shear stress at the center-line N.A. :

$$\tau i = fo\ To^2\ (1 - t^2 + e\ t^3/b)/(8\ L\ Bo)$$

$$\tau i = fo\ To^2\ (1 - t^2\ (1 - e\ t/b))/(8\ L\ Bo) \qquad (28).$$

For the pre-stressed beam the corresponding shear stresses are determined by similar methods. At the interface the value of τo is the same as in equation (28). However due to the pre-stressing effect on fi in the core, there is a greater contribution. This is included by a modification to the relevant part of the maximum shear stress, which is found by dividing fi by $(1 - p)$. This follows from the discussion that was given below Fig. 10, and:

$$\tau i = \tau_{max}$$

$$= fo\ To^2\ (1 - t^2\ (1 - e\ t/(b\ (1 - p_{max}))))/(8\ Bo\ L)$$

Then using equation (19), this simplifies to:

$$\tau i = fo\ To^2\ (1 - t^2\ (1 - e\ m_{max}/b))/(8\ Bo\ L) \qquad (29).$$

Only for very short beams where L is comparable to To, does the shear stress approach the magnitude of the longitudinal stress.

The shear strain εis is obtained from this by dividing by the shear modulus Gi . Hence:

$$\varepsilon is = fo\ To^2\ (1 - t^2\ (1 - e\ m_{max}/b))/(8\ Gi\ Bo\ L) \qquad (30).$$

When the shear strain is included, the relative ductility $m = (fi/Ei + \tau i/Gi)/(fo/Eo + \tau o/Go)$.

In this expression the dominant term is Gi the value of which may be made small by suitably tailoring the material of the core. By this means the greatest advantage can be taken of values of m exceeding unity (see Table 2 a)), when the full pre-stress properties of the complete beam become available for exploitation.

4 Static Strength and Energy-Absorption Comparisons Between Regular and Pre-Stressed Beams

4.1 General Bending-Moment Capacity

The bending-moment BM, is carried by a beam of rectangular or I cross-section, as shown in Fig. 7. This beam has a symmetric arrangement of core and outer-layer composite materials of thickness-ratio $Ti/To = t$. In the following expressions n is used to indicate either element, thus En equals Eo for the outer-layers or Ei for the core. The size of BM is then found by the calculation of the beam's bending resistance. This can be obtained by integrating the first moment of linearly varying stress across the depth of the cross-section. For the full capacity, using the ultimate stress levels:

$$BM = 2\, I f/T = \int_{-To/2}^{+To/2} Bn\, y\, fn\, (2\, y/Tn)\, dy$$

$$= 4 \int_{0}^{+To/2} Bn\, (fn/Tn)\;\; y^2\, dy \qquad (31).$$

The symmetrical y^2 term permits the range of the integration to be halved. This variable includes the moment-arm of the elemental area $Bn\, dy$ and the linear stress distribution, which is zero on the neutral axis. The integration interval applies to the particular element. On the outer-layer it actually starts at $Ti/2$ and not at 0 .

4.2 Regular Beam

For this design of beam (without pre-stressing), the maximum stresses in the outer fibers of the outer-layers and the core are fo and fi respectively. When equation (31) is applied to the two regions :

$$BMR = 4\, [Bo\, (fo/To) \int_{Ti/2}^{To/2} y^2\, dy + Bi\, (fi/Ti) \int_{0}^{Ti/2} y^2\, dy]$$

After integration, curly brackets are used here to indicate the terms over which the limits apply:

$$BMR = 4\, (Bo/To)[fo\, \{y^3/3\}_{Ti/2}^{To/2} + fi\, (b/t)\, \{y^3/3\}_{0}^{Ti/2}]$$

$$BMR = (Bo/To)[fo\, (To^3 - Ti^3) + fi\, (b/t)\, Ti^3)]/6$$

$$= Bo\, fo\, To^2\, [(1 - t^3 + (fi/fo)\, b\, t^2]/6 \qquad (32).$$

Although it has been obtained here by integration, equation (32) in fact is the same as equation (2b).

4.3 Pre-Stressed Beam and Static Strength Comparison

For this design of beam, the same geometry is taken as before. Under the external loads, the outer layers carry the same stresses at ultimate load as in the case of the regular beam. However, the way that the stresses are developed is different. They change from the relaxed values to the ultimate ones. For the outer layers this change is from for to fo and for the core it is from fir to fi , see Figs. 9 and 10. Thus when re-writing the above analysis for the pre-stressed situation it is necessary to replace fo by $(fo - for)$ and fi by $(fi - fir)$. From equation (32):

$BMP =$

$$Bo\, To^2\, [(fo - for)\, (1 - t^3) + (fi - fir)\, b\, t^2)]/6$$

$$= Bo\, fo\, To^2\, [(1 - t^3 + (fi/fo)\, b\, t^2]/6 - Bo\, To^2\, [\, for\, (1 - t^3) + fir\, b\, t^2)]/6$$

But from equation (7): $-\, for = fir\, b\, t^2/(1 - t^3)$, which results in the second term in the square brackets becoming zero. Hence:

$$BMP = Bo\, To^2\, [(1 - t^3 + (fi/fo)\, b\, t^2]/6 = BMR \qquad (33).$$

Then the comparative static strengths of the pre-stressed and regular composite beams are identical.

4.4 Stiffness and Energy Capacity Comparisons

An outer fiber of the pre-stressed beam changes its stress from for (a relaxed stress of opposite sign) to fo when fully loaded. This compares with a change of only fo in the case of a regular beam. The two kinds of beams are assumed to have linear elastic properties. Suppose that they are made from the same materials and have the same physical dimensions. Then the ratio of their maximum strains or deflections is equal to that of these direct stresses $(-\, for + fo)/fo$. Consequently the flexibility of the pre-stressed beam is greater by the same proportion, when compared to a similar regular beam. However the maximum possible amount of pre-stress is when this ratio equals 2, see Section 3.6 . Hence in practice for the same ultimate load capacity, the pre-stressed beam can deflect by up to twice as much as a regular beam. Advantage of this feature may be taken, for the design of particular bending structures needing to absorb specific amounts of kinetic-energy.

5 Design Considerations for Energy-Absorbing Beams

5.1 General Capacity of The Beam

Before examining the beneficial effect of the pre-stress, it is useful to determine how a general beam is to function, which is designed to resist the impact of a mass by energy-absorption. Suppose that the beam is a cantilever of uniform rectangular cross-section, of breadth B , thickness T and length L . At its free end, the ratio of an applied local force to its displacement, is known to be its stiffness K .

Then: $K = 3\,E\,I/(L)^3$ see [1]. (34) .

The second moment of area (also called the moment of inertia) used in bending is:

$I = B\,T^3/12$, see equation (2) .

When this is substituted into equation (34):

$$K = B\,E\,(T/L)^3/4 \qquad (35).$$

The kinetic-energy (to be absorbed) is Q , which in terms of strain-energy is:

Q = force x deflection/2

= force²/(2 x force/deflection) = $F^2/(2\,K)$.

(Using linear elastic properties, this implies that the pre-stressed beam of equal cross-section but half the stiffness of the regular beam, develops an applied force and an associated stress level that are 0.7071 times that of the regular beam, and a displacement of 1.4142 times as much.) Then making F the subject of the formula:

$$F = \sqrt{2\,Q\,K} = \sqrt{Q\,B\,E\,(T/L)^3/2} \qquad (36).$$

However at the attachment location of the beam the greatest bending moment: $M = (F\,L)$, and from equations (1) and (2), taking this to result in the ultimate stress:

$$f = M\,(T/2)/I = 6\,F\,L\,T/(B\,T^3)$$

and using equation (36):

$$f = 6\,\sqrt{Q\,B\,E\,(T/L)^3/2}\;\;L\,T/(B\,T^3)$$

$$= 3\,\sqrt{2\,E\,Q\,/(B\,T\,L)} \qquad (37),$$

where the square root of the terms on the right-hand side, also has the dimensions of stress.

Then for a design problem where stress f , energy Q and beam length L are all held constant, the combination of "design variables" $(B\,T/E)$ is fixed. This means that for a specific material, the cross-sectional area $(B\ T)$ is constant and it is impossible to save mass by varying the proportions of its breadth to thickness. Consequently, the choice of these two quantities here, is based only on the need to load the rest of the aircraft to a limiting value which is consistent with its overall structural features.

5.2 Comparison of a Regular Beam with a Pre-Stressed Design

It has been shown in Section 4 that under a steady load a pre-stressed beam develops the same stresses at ultimate load as its regular beam equivalent. If this is the sole design criterion there is no advantage - it would be achieved by using the same cross-section. However under dynamic loading from the impact of a mass, it has also been found that the pre-stressed beam can absorb up to twice as much energy as the regular beam of equal size and shape.

In terms of the analysis given in Section 5.1 for general beams, the effective stiffness modulus E of the pre-stressed beam is reduced by a factor of up to 2. Then from equation (37), in the extreme case of design to full pre-stress, only half the cross-section is necessary to contain the same amount of kinetic-energy from the dynamic load situation.

Suppose that the breadth and width of the pre-stressed beam's cross-section were reduced in equal proportions. Then the deflection would be greater and the maximum dynamic load and stress levels smaller, than those experienced by a regular beam that performs the same role. But in practice for cases of energy-absorption by means of this structure, there are design limits on the permitted amounts of load and displacement. For example, the maximum vertical displacement of a leaf-spring landing-gear must provide the necessary ground clearance (which is associated with a specific maximum load). Consequently it is of interest to examine the way that the proportions of breadth and thickness can be modified so that these two results remain unchanged.

The stiffness ratio is directly proportional to the moment of inertia of the structure $I = B\,T^3/12$. Now consider the necessary changes to B and T for the same stiffness to be maintained. The stiffness is proportional to the product $E\ I$. Then for the old and new designs:

$$E_{old}\, I_{old} = E_{new}\, I_{new}\ ,$$

but $E_{new} = E_{old}/2$ consequently:

$$I_{new}/I_{old} = 2 = (B_{new}/B_{old})\ (T_{new}/T_{old})^3 \quad (38).$$

But the ratio of areas:

$$(B_{new}/B_{old})\ (T_{new}/T_{old}) = 1/2 \qquad (39),$$

because the capacity to absorb energy is otherwise proportional to it, and unless there is a change to the dimensions it can now absorb twice as much. Then by substitution of equation (39) into (38):

$$T_{new}/T_{old} = 2 \quad and \quad B_{new}/B_{old} = 1/4 \quad (40)\ .$$

Example of a leaf-spring main landing-gear typical of that used on light aircraft [2], the cross-section at the root was:

T_{old} = 1.25 inches and B_{old} = 4.80 inches.

Then: T_{new} = 1.25 x 2 = 2.50 inches and
B_{new} = 4.80/4 = 1.20 inches, from equation (40).

The shape of the original leaf-spring was tapered, and changes along the length of the new design should also be in similar proportions. The mass of this spring previously was 22 lb. This suggests that *with full pre-stress a reduction to 11lb. is likely.* It now becomes practical to modify the cross-section as shown in Fig. 11.

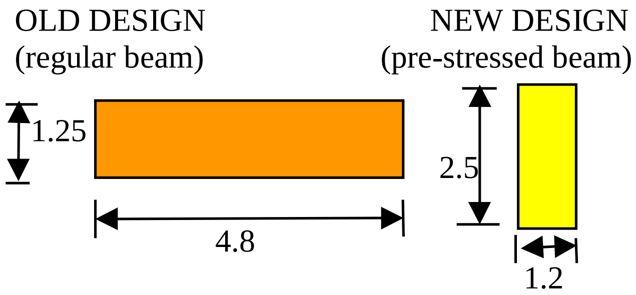

Fig. 11 Comparison of cross-sections of leaf-springs, dimensions are in inches.

In practice there are some secondary engineering considerations, such as the need for the addition of aerodynamic fairings, which will reduce the considerable weight saving. Never-the-less by pre-stressing the energy-absorbing beam, a significant mass reduction is possible compared to the regular beam design. The changes in the cross-section and the resulting mass reduction are proportional to the amount of pre-stress employed.

6 Review and Discussion

This paper presented four kinds of activities:

Section 2 - method of beam fabrication,
Section 3 - algebraic analyses including details of how to obtain the pre-stress in practice,
Section 4 - strength/energy comparison,
Section 5 - design for weight-saving.

They are summarized in more detail in Table 5.

Although the ranges of the variables m , t and $e\ b$ are limited, as was indicated in Tables 2 and 3, the practical range of data shows that it is possible to achieve a significant advantage, when pre-stressing is employed and the beam is allowed to relax before loading. In terms of design to resist the energy of an impact, this results in a saving in structure mass.

TABLE 5. STAGES IN THE ANALYSIS AND DESIGN

PARA-GRAPH	NECESSARY FEATURE	ANALYTIC RESULT	EQUATION(S)
3.1	Pre-stress ratio p is applied to the core.	Applied bending-moment Mi and the curvature $(1/R1 - 1/R2)$.	(3) and (4).
3.2	Equal relaxation bending-moments are developed in the core and outer layers.	Ratio of relaxation stresses, variation with the beam geometry.	(7).
3.3	Strains at the interfaces are made compatible.	Relation between the relaxation stresses in the core and outer layers and the geometric and stiffness properties of the beam.	(12) and (13).
3.4	Effect of equal interfacial strains.	Relationship between and values of the radii of curvature, during fabrication.	(14).
3.5	External load is applied. Relaxation stresses in outer fibers set to ultimate strength.	The magnitude of pre-stress p that is possible and the introduction of m to provide for it.	(18) and (19).
3.6	Idealization of the ratio of enlongations at failure, including the effects of shear in the core.	Relationship of the relaxation stresses in outer-layers to their ultimate stresses. Maximum shear flexibility m_{max} and the resulting pre-stress ratio p_{max}.	(22), (23), (24) and (25).
3.6	Determination of shear stresses and strains.	Confirmation of the range of the variable m , where optimum values are likely.	(29) and (30).
4.	Comparison between the static strengths of the pre-stressed and regular beams.	The maximum bending moments *BMR* and *BMP*, which are found to be equal.	(32) and (33).
5.	Study of design for the absorption impact energy.	Comparison of cross-sections of old and new designs and their associated masses.	(37) and (40)

From the practical example in Section 5, the structure mass reduction works out to be 50%, which is large enough to be very useful. Its value is dominated by the use of the effect of shear flexibility of the core, where a value of m_{max} of about 2.75 is needed. It is possible to obtain this value by careful design of the core material, particularly when using composite materials. No allowance for shear diffusion effects in the flanges has been included. This effect will reduce the ability to pre-stress the outer layers to their ideal values. It becomes significant when the value of b is less than about 0.5 and t approaches 1, and it should be determined by finite-element numerical techniques in combination with experimental methods.

7 Conclusions

The concept of pre-stressing of symmetric composite beams has been examined. It was found that by careful design of the components including the shear flexibility of the core, it is possible to modify the beam's structural properties. The comparative static-strength of two beams of the same cross-section, one having pre-stress, is the same, but the stiffness of the latter is less. This is an advantage when the structure is required to absorb the energy of impact, where a useful saving in weight of up to 50% is possible.

8 References

[1] Roark R.J. *Formulas for Stress and Strain* 4th. edition, McGraw-Hill Book Company, New York, USA, 1965.

[2] Pazmany L. *Landing Design for Light Aircraft,* Pazmany Aircraft Corporation, San Diego, California, USA, 1986.