DESIGN OF STABILIZING AND LOCALLY-ROBUST NONLINEAR CONTROLLERS FOR A CLASS OF NONLINEAR SYSTEMS

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Abstract

This paper presents a new practical technique to design stabilizing locally-robust nonlinear controllers using ideas from feedback linearization theory. The asymptotic stability of the nonlinear closed loop system is achieved by constructing outputs for the linearized plant at a given equilibrium point. A method is presented to locate transmission zeros of the linearized plant, so as to stabilize the linear closed loop system and guarantee optimal modal robustness. Outputs are then constructed to place the transmission zeros of the linearized plant at these locations. The nonlinear feedback for the nonlinear system is then constructed from these outputs.

1 Introduction

This paper presents a new technique to design stabilizing and locally robust nonlinear controllers for nonlinear systems that are affine in input. The nonlinear controller is constructed in the same manner as in the feedback linearization theory.

Feedback linearization has become a very popular technique for control system design of nonlinear systems. (See Isidori [1], Nijmeijer and Van der Schaft [2], Vidyasagar [3]). In essence, this technique utilizes state feedback and coordinate transformations to cancel nonlinearities and render the closed loop system linear in the input-output behaviour. Linear control techniques can then be applied to address various design issues. Historical developments of this technique are given in the bibliographical notes of Isidori [1], and Nijmeijer and Van der Schaft [2].

Various issues of internal stability in feedback linearization are well understood. For plants where relative degree of the nonlinear system is the same as the system order, the closed loop system can be completely linearized and later stabilized with another linear state feedback. However, for plants in which the relative degree is lower than the plant order, internal stability is governed by stability of the zero dynamics. Zero dynamics are a by-product of closing the loop with linearizing feedback. If the zero dynamics are unstable, the resultant closed loop system is internally unstable. Since the zero dynamics modes are uncontrollable from reference inputs, they cannot be stabilized by any additional feedback.

Zero dynamics are inherently nonlinear. It is possible to linearize these dynamics around the equilibrium point of interest. It has been shown in Isidori [1] that the eigenvalues of linearized zero dynamics are the same as the transmission zeros of the Jacobian linearization of the nonlinear plant at the equilibrium point. Hence, to stabilize zero dynamics, it is sufficient that the linearized plant has stable transmission zeros. The location of transmission zeros of a linear system changes when the output matrix is changed. Hence, a new technique has been proposed in this paper for constructing output matrix to place transmission zeros of the system desired locations. The same outputs are then used for constructing feedback for asymptotically stabilizing the nonlinear system. This approach is useful when stability of the closed loop system is of primary concern and not linearization obtained by virtue of the design.

The problem of locating transmission zeros by constructing the output matrix has presumably not been addressed previously. Locations of transmission zeros of a physical system are usually determined by the location of sensors and actuators on the system. As a result, none of the standard techniques like static/dynamic feedback, redistribution of inputs/outputs, etc., can change transmission zero locations. While much research has been done to understand the properties of transmission zeros, (for example, see Davison and Wang [4], Kailath [5], Chen [6], and Patel and Munro [7]) not much work has been done on the placement of transmission zeros. The only reference found that deals with a similar problem is Misra and Patel [8]. In this reference, the problem of transmission zero assignment is reduced to the problem of eigenvalue assignment using output feedback. The inverse of this feedback compensator is then used as a *feed-through* term to place transmission zeros.

The possibility of reassigning the output matrix of the system to place transmission zeros is not usually considered because this matrix cannot be changed for a physical system after the sensors have been located. In feedback linearization approach, full state feedback is used. Outputs are only notional and hence construction of a new set of outputs results in a very simple yet powerful technique.

Consider a linear system of order n, with m inputs and m outputs,

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

For such a system only n-m transmission zeros exist (Davison [4]). Therefore, a maximum of n-m zeros can be assigned by changing the matrix C. The remaining zeros are automatically assigned at infinity. The problem of transmission zero assignment has been cast in this paper as an eigenvalue assignment problem using state feedback. A state feedback is designed to place n-m eigenvalues at the desired zero locations and the remaining m eigenvalues at the origin. The matrix C is constructed from this feedback matrix.

The problem of optimally locating the transmission zeros is then addressed. Practically, it is desirable that closed loop poles of the system are robust to perturbations in system parameters. Such robustness issues have been very effectively addressed in the theory of eigenstructure assignment (see Srinathkumar [9] and Kautsky *et al.* [10]). A two-step procedure is proposed in this paper to ensure stability and maximization of local robustness of the closed loop system. In the first step, n-m

eigenvalue locations are obtained using optimization techniques to minimize the condition number of a certain eigenspace matrix. In the second step, a state feedback matrix is constructed so as to place m eigenvalues at the origin and the remaining at the optimal locations obtained from the first step. Thus, this two-step process ensures optimal robustness of the linear closed loop system. The feedback gain matrix is then used to generate the required matrix C.

This paper is divided into four sections. Section 1 is the Introduction, Section 2 formally defines the problem addressed in the paper. Section 3 presents the systematic procedure to construct outputs of a linear system so that the system has desired transmission zeros. Section 4 addresses robustness issues and describes how to choose transmission zero locations and the state feedback so that the closed loop system is stable and optimally robust. Section 5 contains an example to describe the technique presented in Sections 3 and 4. Section 6 finally presents conclusions of the paper.

2 Problem statement

Consider a multi-input nonlinear system in the affine form:

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i = f(x) + g(x)\, u \qquad (1)$$

where $x \in R^n$ are the states, $u_i \in R$, $i \in \{1, \dots, m\}$ are the m inputs, f: $R^n \to R^n$ defines the basic dynamics of the plant, g_i: $R^n \to R^n$, $i \in \{1, \dots, m\}$ are m vector fields through which the control inputs enter the system. Define $g(x) = [g_1(x)\ g_2(x)\ \dots\ g_m(x)]$ and $u = [u_1, \dots, u_m]^T$. Both f and g are assumed to be sufficiently smooth and $f(0) = 0$. Consider the following linearization of the nonlinear system (1) about the origin:

$$\dot{x} = Ax + Bu \qquad (2)$$

where $x \in R^n$, $u \in R^m$, $y \in R^m$. A: $R^n \to R^n$, B: $R^m \to R^n$. Let the outputs of the system (1) as well as (2) be defined by

$$y = Cx \qquad (3)$$

where C: $R^n \to R^m$. Let $v \in R^m$ denote external reference input vector. System (1) can be feedback linearized in some small neighborhood around the origin by the following feedback function:

$$u(x) = \left(Cg(x)\right)^{-1} v - \left(Cg(x)\right)^{-1} . Cf(x) \quad (4)$$

provided CB is invertible [1]. The linearizing feedback also stabilizes the closed loop system if the system (1), (3) has stable zero dynamics. Since the eigenvalues of zero dynamics are the same as transmission zeros of the linearized system (2), (3) [1], the following problem is solved in this paper:

Construct a matrix C such that (i) CB is invertible and (ii) the system (2),(3) has stable transmission zeros. In addition, it is desired that the *closed loop system using feedback as in (4) be locally optimally robust*, in the sense that, the perturbation of elements of A matrix should minimally affect the closed loop eigenvalue locations.

3 Assignment of transmission zeros of a linear system

This section presents a procedure to construct linear outputs to place the transmission zeros of the system at desired locations. These linear outputs can later be used to construct a nonlinear feedback that locally stabilizes the nonlinear system.

To define the transmission zeros of system (2), (3), consider a new output

$$\tilde{y} = \dot{y} = C\dot{x} = CAx + CBu\,, \qquad (5)$$

and assume that CB is an invertible matrix. The set of transmission zeros Z_T for the system (2), (3) satisfies:

$$Z_T \subset \lambda(A - B(CB)^{-1}CA). \qquad (6)$$

where, $\lambda(A - B(CB)^{-1}CA)$ denotes the spectrum of matrix $A - B(CB)^{-1}CA$. Since the new outputs $\tilde{y}$ are just the derivatives of the output y, a total of m eigenvalues of the above set would have to lie at the origin. The other n-m eigenvalues define the transmission zeros for the system (2) and (3).

The main result that shall be proved in this section is stated in the following theorem:

Theorem 1. *If the system (2) is controllable and B is a full rank matrix, then the (n-m) transmission zeros of the system (2),(3) can be arbitrarily assigned by a choice of the output C matrix.*

The proof of this theorem requires some more results, which are stated and proven first. Formally, the problem of construction of outputs of the form (3) to place n-m system transmission zeros can be stated as :Find a C matrix such that transmission zeros of (2), (3) are placed at $Z = \{z_1, z_2, z_3, \ldots, z_{n-m}\}$, where $z_1, z_2, \ldots, z_{n-m}$ are distinct non zero complex numbers and Z is a self conjugate set.

The following lemma describes a procedure that can be adopted to place the transmission zeros. Define a new set $Z' = \{z_1, z_2, \ldots, z_{n-m}, 0, \ldots, 0\}$ which is constructed by augmenting the set Z with m zeros.

Lemma 1. *Suppose there exists a gain feedback $m\times n$ matrix K such that,*

i) $\lambda(A\text{-}BK) = Z'$, and

ii) the geometric multiplicity of the eigenvalue at the origin of (A-BK) is same as its algebraic multiplicity,

then, any full rank $m\times n$ matrix C which satisfies the condition C(A-BK)=0, places the transmission zeros of system (2), (3) at Z.

The proof of this lemma requires a result on the canonical forms of closed loop matrices, which is stated in Lemma 2 below. Let the linear coordinate transformation that transforms A and B matrices of the system (2) into the controllable canonical form [12] be denoted by T: $R^n \rightarrow R^n$. Let $k_1, k_2, \ldots, k_m$ be the size of the m individual blocks of the matrix $A^*=T^{-1}AT$ corresponding to m inputs. Denote $B^* = T^{-1}B$. Let K be any feedback gain matrix, and let $K^*= KT$. Then for some constants $c_{i,j}$, $i\in\{1,\ldots,m\}$, $j\in\{1, \ldots, n\}$, the canonical form of the closed loop matrix can be written as: The canonical form A^*-B^*K^* in equation (7) is valid for all possible feedback matrices K. For the special feedback as defined in Lemma 1, this canonical form has a special structure. The following Lemma describes this special structure.

Lemma 2. *Suppose a matrix K satisfies conditions (i) and (ii) of Lemma 1, then in the canonical form of the closed loop matrix as described by Equation (7), all the entries in the column numbers 1, k_1+1, k_1+k_2+1, ... , n-k_m+1 are zero.*

Proof. The complete proof of this lemma is lengthy and shall be reported elsewhere. Basically the matrix A^*-B^*K^* is diagonalized and rearranged to shift all row vectors containing $c_{i,j}$ terms to the bottom of the matrix. Now columns 1, k_1+1, k_1+k_2+1, ... , n-k_m+1 of the right diagonalizing matrix can be easily shown to be zero. As a result the corresponding columns of A^*-B^*K^* also turn out to be zero. ■

$$
A^* - B^*K^* = \left[\begin{array}{ccccc|ccccc|cc|ccccc}
0 & 1 & 0 & \cdots & 0 & & & & & & & & & & & & \\
0 & 0 & 1 & \cdots & 0 & & & & & & & & & & & & \\
\vdots & & & \cdots & \vdots & & & 0 & & & 0 & & & & 0 & & \\
0 & 0 & 0 & \cdots & 1 & & & & & & & & & & & & \\
c_{1,1} & c_{1,2} & c_{1,3} & \cdots & c_{1,k_1} & c_{1,k_1+1} & \cdots & & \cdots & c_{1,k_1+k_2} & \cdots & \cdots & \cdots & & & \cdots & c_{1,n} \\
\hline
& & & & & 0 & 1 & 0 & \cdots & 0 & & & & & & & \\
& & & & & 0 & 0 & 1 & \cdots & 0 & & & & & & & \\
& & 0 & & & \vdots & & \vdots & \cdots & \vdots & 0 & & & & 0 & & \\
& & & & & 0 & 0 & 0 & \cdots & 1 & & & & & & & \\
c_{2,1} & c_{2,2} & c_{2,3} & \cdots & c_{2,k_1} & c_{2,k_1+1} & \cdots & & \cdots & c_{2,k_1+k_2} & \cdots & \cdots & \cdots & & & \cdots & c_{2,n} \\
\hline
& & \vdots & & & & & \vdots & & & \ddots & & & & \vdots & & \\
& & \vdots & & & & & \vdots & & & & \ddots & & & \vdots & & \\
\hline
& & & & & & & & & & & & 0 & 1 & 0 & \cdots & 0 \\
& & 0 & & & & & 0 & & & 0 & & 0 & 0 & 1 & \cdots & 0 \\
& & & & & & & & & & & & \vdots & & \vdots & & \vdots \\
& & & & & & & & & & & & 0 & 0 & 0 & \cdots & 1 \\
c_{m,1} & c_{m,2} & c_{m,3} & \cdots & c_{m,k_1} & c_{m,k_1+1} & \cdots & & \cdots & c_{m,k_1+k_2} & \cdots & \cdots & \cdots & & & \cdots & c_{m,n}
\end{array}\right] \qquad (7)
$$

We now construct the m-dimensional left null space of the matrix A^*-B^*K^*. By inspection, the following C^* matrix satisfies the condition $C^*(A^*$-$B^*K^*)=0$:

$$
C^* = \left[\begin{array}{ccccc|ccccc|c|ccccc}
-c_{1,2} & -c_{1,3} & \cdots & -c_{1,k_1} & 1 & & & 0 & & & \cdots & & & 0 & & \\
\hline
& & 0 & & & -c_{2,k_1+2} & -c_{2,k_1+3} & \cdots & -c_{2,k_1+k_2} & 1 & \cdots & & & 0 & & \\
\hline
& & \vdots & & & & & & & & \ddots & & & \vdots & & \\
\hline
& & 0 & & & & & 0 & & & 0 & -c_{m,n-k_m+2} & -c_{m,n-k_m+3} & \cdots & -c_{m,n} & 1
\end{array}\right] \qquad (8)
$$

Proof of Lemma 1.

The feedback matrix K is assumed to satisfy conditions (i) and (ii) of Lemma 1. Condition (ii) ensures that the closed loop matrix A-BK has an m-dimensional null space. In other words, there exists a full rank $m\times n$ matrix C such that

$$C(A\text{-}BK) = 0. \qquad (9)$$

For the C matrix as constructed above,

$$CA - CBK = 0,$$

$$CBK = CA,$$

and finally, the K matrix itself can be written as

$$K = (CB)^{-1}\, CA. \qquad (10)$$

provided CB is invertible.

Therefore, if CB is invertible, A-BK is effectively $A\text{-}B(CB)^{-1}CA$ for this newly defined C. By the definition of transmission zeros (eq. (6)), all the eigenvalues of the matrix A-BK other than the m eigenvalues at the origin are the transmission zeros of system (2) and (3), with the output matrix C as defined by equation (9).

Now, it only needs to be shown that, with such a choice of C, CB is invertible. For this, consider again the controllable canonical form (A.7) and C^* as defined in eq.(8). The product C^*B^* results in an identity matrix and hence C^*B^* is invertible. Define $\hat{C} = C^*T$ where T is the coordinate transform as used in Lemma 1. By transforming the coordinates back we get $\hat{C}B = C^*TT^{-1}B^*$. Hence, $\hat{C}B$ is invertible and satisfies the condition $\hat{C}(A\text{-}BK) = 0$. $\hat{C}$ is a full rank matrix and so for any full rank C that satisfies condition (9) there exists an invertible transformation M such that $C=M\hat{C}$. Therefore $CB=M\hat{C}B$ is also invertible. ■

Remark 1. Though it is assumed in the initial problem statement that the set Z contains unique non-zero complex numbers, this condition can be further relaxed. The only requirement is that K should be chosen in such a way that for each repeated eigenvalue of (A-BK), the geometric multiplicity and the algebraic multiplicity should be the same. ■

Now, it is possible to prove Theorem 1.

Proof of Theorem 1. As can be seen from the structure of the canonical form (A.7), the condition that "K should be such that, (A-BK) must have m eigenvectors corresponding to the eigenvalue at the origin", is identical to the condition that, "K should be such that each input places one eigenvalue at the origin". This is certainly possible in a system which is controllable and where B is a full rank matrix. Therefore, the theorem has been proven. ■

4 Feedback linearization as a standard linear feedback design problem

In this section it is shown that if the system outputs can be reassigned, a feedback linearization based controller can be viewed as a nonlinear extension of a linear feedback. This results in a systematic procedure to design nonlinear controllers that satisfy local design requirements.

4.1 Linear design and feedback linearization

First we show that if outputs can be freely chosen then around an equilibrium point feedback linearization becomes a standard linear feedback design with some constraints on pole locations. Consider the nonlinear affine system (1) and its linearization (2). Let the outputs be defined by (3) where the matrix C has been constructed using Lemma 1 to place transmission zeros of the linearized system. The dynamics of the closed loop system using the standard linearizing feedback as in (4) are:

$$\begin{aligned}\dot{x} = &\left(f(x) - g(x)\left(\left[Cg(x)\right]^{-1} Cf(x)\right)\right) \\ &+ g(x)\left[Cg(x)\right]^{-1} v\end{aligned}. \quad (11)$$

Note that this closed loop system is not linear because the zero dynamics of this system are n-m dimensional and nonlinear. If these closed loop dynamics are linearized (by Taylor series truncation at the equilibrium point), the linearized closed loop system would be:

$$\dot{x} = (A - B(CB)^{-1}CA)x + B(CB)^{-1}v. \quad (12)$$

Now, since C is constructed using Lemma 1, the feedback gain matrix K used to construct C satisfies condition (10). Therefore, the local dynamics of a feedback linearization based design can be rewritten as:

$$\dot{x} = (A - BK)x + B(CB)^{-1}v\,. \quad (13)$$

where K is chosen such that it satisfies conditions of Lemma 1.

Hence it can be concluded that *the choice of matrix K that is used to construct the matrix C in Lemma 1, effectively designs the linearizing feedback locally*. This result is very useful because it identifies the feedback linearization based controller design close to an equilibrium point, with a linear feedback design.

4.2 Design for linear robustness

In this section, the procedure to design a feedback for optimal robustness is presented using ideas from the eigenstructure assignment technique. The aim is to place eigenvalues of a multi-input linear system using a full state feedback in such a way that the closed loop system is robust. Here robustness is used in the sense that, the closed loop eigenvalues are optimally insensitive to perturbations of matrix A elements. The main ideas for robustness have been derived from Kautsky *et al*. [10], where the following problem is addressed:

Given the controllable system (2) with a full rank matrix B, and a self conjugate set of desired eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$, find a nonsingular matrix X of eigenvectors and a full state feedback K satisfying

$$(A\text{-}BK)X = X\Lambda, \quad (14)$$

such that some measure of robustness is optimized. Here Λ corresponds to the diagonal matrix with $\lambda_1, \ldots, \lambda_n$ as its entries. Minimization of condition number of X is chosen as the measure to improve robustness.

To solve this problem, it is first necessary to characterize which eigenvectors for a given eigenvalue λ_j, can be assigned by feedback. The space in which these eigenvectors can lie is known as the *eigenspace* of the eigenvalue and can be characterized in the following way. Let the B matrix be written as

$$B = \begin{bmatrix} U_0 & U_1 \end{bmatrix} \begin{bmatrix} Z \\ 0 \end{bmatrix},$$

with U=[U_0 U_1] an orthogonal matrix and Z a nonsingular mxm matrix. This decomposition of B can be obtained using singular value decomposition or QR decomposition of B.

Lemma 3. ([10]): *The eigenvector X_j of A-BK corresponding to the assigned eigenvalue λ_j must belong to the space*

$$\mathsf{S}_j = \mathsf{N}\left\{U_1^T (A - \lambda_j I)\right\} \quad (15)$$

where N{.} denotes the null space.

Proof. Refer [10] ■

Kautsky *et al.* [10] describe various numerical algorithms to assign poles while minimizing the condition number of the matrix of assigned eigenvectors. The lower bounds of the condition number of matrix X can also be calculated in the following way. Let S_j (λ_j) be the matrix of m orthogonal vectors lying in the space S_j. Consider the eigenspace matrix S obtained by the concatenation of matrices S_j (λ_j) for j=1, …,

n. Then the condition number $\kappa(X)$ for all possible eigenvector matrices X, is bounded by:

$$\min \kappa(X) \geq \kappa(S)/\sqrt{(n)}.$$

After testing on numerous design examples, Kautsky *et al.* have shown that their algorithms work quite well when $\kappa(S)$ is small. It has been observed that the condition numbers $\kappa(X)$ of the assigned eigenvectors X are generally close to the optimal bound $\kappa(S)/\sqrt{(n)}$.

Srinathkumar [9] generalized the ideas from [10] to assign both eigenvalues and the eigenvectors for maximizing robustness. The following two step approach is suggested.

a. Optimal location of eigenvalues is searched in a pre-specified region of left half complex plane to minimize the condition number $\kappa(S)$.
b. Algorithms in [10] are used to find a feedback, that assigns the poles at optimal locations and finds eigenvector matrix X so as to minimize $\kappa(X)$.

This approach results in a optimally robust linear state feedback and can be easily adopted to design the feedback linearization based controller. Therefore, in summary, the following procedure can be adopted to design a feedback linearization based controller to satisfy the locally robust stability requirements:

1. Construct the linear system (2) from the original nonlinear system (1).
2. Search for a self conjugate set of stable pole locations $\{z_1, z_2, \ldots, z_{n-m}, 0, \ldots, 0\}$ to minimize $\kappa(S)$.
3. Design a feedback K using methods from [10] to minimize $\kappa(X)$ and satisfy requirements of Lemma 1.
4. Find C as in Lemma 1.
5. Define the linearizing feedback as in (4) and obtain the closed loop system (11).

Though the feedback controller designed in Step 3 is linear, Step 5 results in a closed loop system with a *nonlinear feedback*. The nonlinear closed loop system would locally meet the design requirements of stability and robustness.

5 Example

The example chosen here to demonstrate the controller design technique has been taken from the thesis by El-Shaer [11]. This thesis presents a detailed model of double inverted pendulum on an inclined rail. The thesis also presents a few linear and nonlinear controllers designed to stabilize the system and simulation results. The states considered in the model are the cart position and velocity along inclined rail, angular position and velocity of lower pendulum, and angular position and velocity of upper pendulum. El. Shaer [11] considers only the force on cart as the single input. Here, torque on the lower pendulum is considered as an extra input. The equilibrium condition for this model is chosen as the origin [0 0 0 0 0 0]′. The linear model at the origin is then represented by the following A and B matrices

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -2.3237 & 0.0670 & -4.6244 & 0.0139 & -0.0047 \\ 0 & 48.3060 & -13.2765 & 13.3909 & -0.3762 & 0.1860 \\ 0 & -53.2614 & 49.2099 & -1.5488 & 0.6688 & -0.4591 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 23.0561 & -4.7655 \\ -66.7640 & 99.0676 \\ 7.7219 & -109.2304 \end{bmatrix}.$$

The eigenvalues of matrix A are [0, 8.2334, 4.5058, -9.3105, -5.6946, -3.1939] and the plant is clearly unstable.

Feedback linearization based stabilizing controller is designed for this plant by constructing the output matrix *C*. The eigenvalues of the closed system need to be located. Since the plant in this example has two inputs, it is required to place two eigenvalues at the origin to construct a C matrix. In the final closed loop, it is not a good idea to keep two eigenvalues at the origin. Hence the total feedback can be modified in the following way [1]:

$$u(x) = (Cg(x))^{-1} v - (Cg(x))^{-1} . \left[Cf(x) + \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} Cx \right] . (16)$$

This feedback would then place the two eigenvalues at $-\lambda_1$, $-\lambda_2$ instead of the origin. The procedure adopted to locate eigenvalues in this example is slightly different from the one suggested in Section 4.2. Optimization based techniques are used to locate all the six eigenvalues, two of which namely, λ_1 and λ_2 are constrained to be real and other four ($\lambda_3, \ldots, \lambda_6$) are constrained to lie in a sector of the complex plane bounded by minimum damping of 0.2. In each step of optimization the following computations are carried out.

(i) *K* matrix is constructed using *place* command in *MATLAB* to place eigenvalues at $(0, 0, \lambda_3, \ldots, \lambda_6)$.

(ii) *C* matrix is obtained by constructing null space of $(A\text{-}BK)$.

(iii) The closed loop matrix is constructed as

$$A_c = A - B(CB)^{-1} . \left[CA + \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} C \right] . (17)$$

(iv) The condition number of the eigenvector matrix of the closed loop matrix A_c is computed.

This condition number is minimized during optimization. Using this approach it is possible to locate all the system eigenvalues and design the feedback, simultaneously. For the double inverted pendulum, the final eigenvalue locations obtained are: -3.0200 ± 14.7683*i*, -0.5971 ± 2.9320*i*, -9.9345, -5.5889. The corresponding *C* matrix is given by

$$C = \begin{bmatrix} 0.3435 & -0.5740 & 0.1097 & 0.6429 & -0.0152 & 0.3562 \\ -0.7623 & -0.2555 & -0.5407 & 0.1934 & -0.0693 & 0.1379 \end{bmatrix}$$

The nonlinear controller is then given by Eq. (10). The complete system along with the controller was simulated in *MATLAB*. As mentioned in [11], the relevant domain of attraction of the closed loop system is obtained in the plane containing θ_1, θ_2 as the two axes. Using extensive simulation, all possible combinations of θ_1, θ_2 are found such that starting with these angular positions as initial conditions, the controlled double pendulum returns back to the vertically upright position. Fig. 1 shows an estimate of the domain of attraction of the stabilized inverted double pendulum using the feedback linearization based controller. For comparison, the domain of attraction obtained using a nonlinear controller designed in [11] is also plotted in the figure.

6 Conclusions

This paper has presented a simple and practical technique to design nonlinear controllers to stabilize a nonlinear system based upon feedback linearization theory. It has been shown that the freedom to choose outputs of a linear system can be very effectively used to design controllers. The problem of choice of outputs has been cast as an eigenvalue assignment problem with some constraint on choice of the eigenvalue locations. Thus many good results in literature on eigenvalue assignment have been applied to the feedback linearization theory. The natural extension of the work carried out here is to extend the linear outputs to nonlinear output functions to guarantee a large domain of attraction for the closed loop system.

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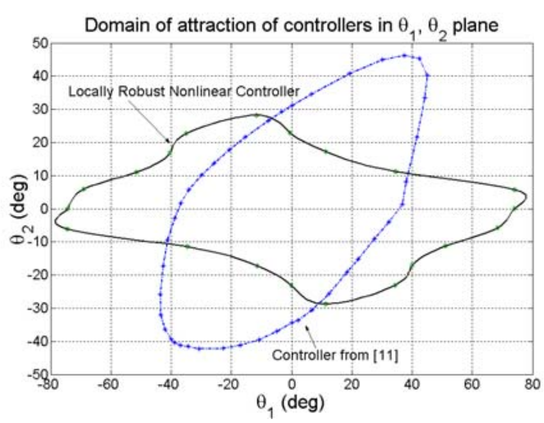

Fig.1 Domains of attraction of inverted double pendulum with two controllers.