A HIGHER-ORDER THEORY FOR FIBER-METAL LAMINATES

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Abstract

A higher-order theory is proposed for fiber-metal laminates that consist of alternating metal (isotropic) and fiber-reinforced prepreg (orthotropic) layers under generalized plane stress conditions. In order to describe interlaminar interactions, two conjugated harmonic shear- stress potentials, p *and* q*, are employed, in addition to the in-plane stress function, which exist by the requirement of interfacial displacement compatibility. The boundary-value problems under the generalized plane stress conditions are formulated in complex variables and its potential applications in dealing with problems of stress concentration in such laminates are briefly introduced.*

1 Introduction

Laminates represent a wide class of material innovation for today's industrial applications, particularly in aerospace. Examples are fiber-metal laminates (FMLs), composite patching, and multi-layer coatings, and the list may go on. One advantage of such layer-structured materials is that when one layer incurs some damage, the load transfer mechanism via interlaminar stresses will allow other layers to compensate the loss of load-bearing capacity of the damaged layer. In FML (also known as ARALL or GLARE), which consists of alternating layers of metal and fiber-reinforced prepreg, the fiber reinforcement plays dual roles of strengthening and "fiber-bridging" of fatigue/impact damages. Thus, FMLs provide a great advantage over monolithic aluminum alloys, offering superior damage tolerance properties [1,2], and they are being considered as promising materials for the next generation aircraft.

Facing with these material innovations, a great challenge to mechanists is to describe the mechanism and behavior of these materials for better design and better use of these materials on aircraft, but the theoretical development seems to be lagging behind the actual application of the material. First of all, the classical composite laminate theory, e.g. [3,4], which has been used to deal with composites for the last half century, has difficulty in describing the interlaminar interactions, because it treats the multi-layer composite as a homogeneous body, thus neglecting interlaminar stresses. Some analytical treatments have been given to the inter-laminar stresses along straight free edges [5,6], but not applicable to curvilinear edges such as circular or elliptical cutouts and cracks (with a extremely sharp curvature). A method of superposition of a hybrid and displacement approximation has also been developed for 3D stress analysis of composites [7,8], which only seeks numerical solutions based on variational principles. Therefore, higher order theories are needed to include interlaminar shear stresses into the formulation, to deal with the quasi-3D stress states in FMLs.

Recently, a higher-order theory has been developed for laminates consisting of isotropic layers under the generalized plane-stress conditions ($\sigma_z = 0$), based on the Taylor expansions of displacements up to the second order of z (z is the coordinate in the plate normal direction) [9]. It has been shown that interlaminar stresses exist by the requirement of interfacial strain compatibility in the laminates, especially in places where strong stress gradients exist. The higher-order theory employs two conjugated harmonic stress functions, p and q, in addition to the bi-harmonic stress function Ψ (Airy function), to satisfy all the 3D stress-equilibrium and strain-compatibility conditions.

The present paper extends that higher-order plane-stress theory to hybrid materials such as FMLs. In this formulation, the interlaminar stresses acting across the metal/prepreg interfaces can be derived from the harmonic potentials, p and q, which exist by the requirement of displacement compatibility across the layer. A complex-variable approach is formulated for finding the stress/displacement solution for each individual lamina of the hybrid material.

2 The Lamination Theory for FML

We consider that a typical fiber-metal laminate consists of alternating metal (isotropic) and prepreg (orthotropic) layers, as shown in Fig. 1 (only three layers are drawn without intention to lose generality) and assume that the thickness of the laminate is small compared to the planar dimensions of the plate. In addition, we assume that body forces are absent in the laminate and the stress normal to the x-y plane, σ_z, is zero.

Under the above conditions, the stress equilibrium condition in each individual lamina can be written as:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0 \qquad (1a)$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0 \qquad (1b)$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0 \qquad (1c)$$

For isotropic layers, it has been shown that the 3D stress equilibrium conditions and strain compatibility can be met by the introduction of the following stress potentials [9]:

$$\tau_{xz} = -z\frac{\partial p}{\partial x} = -z\frac{\partial q}{\partial y}, \quad \tau_{yz} = -z\frac{\partial p}{\partial y} = z\frac{\partial q}{\partial y} \qquad (2)$$

where p and q are conjugated harmonic functions ($\nabla^2 p = 0$ and $\nabla^2 q = 0$, by the Cauchy-Riemann condition); and

$$\sigma_x = \frac{\partial^2 \Psi}{\partial y^2} + p, \quad \sigma_y = \frac{\partial^2 \Psi}{\partial x^2} + p, \quad \tau_{xy} = -\frac{\partial^2 \Psi}{\partial x \partial y} \qquad (3)$$

where Ψ is a bi-harmonic function ($\nabla^4 \Psi = 0$).

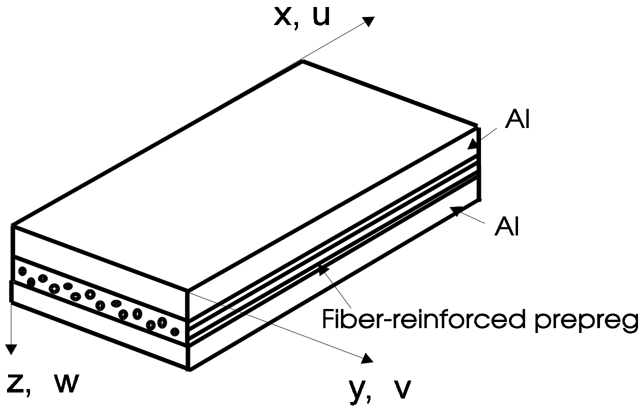

Fig. 1 Schematic of a three-layer FML.

Because FML has a symmetrical layout with metal layers as the top and bottom surface layers, the neutral plane (z = 0) of a surface layer is always at the free surface and that for an inner layer should coincide with the mid-plane of the lamina. Thus, the entire laminate can be viewed as a portion of periodical stacking of the constituent lamina, where the surface layer becomes an inner lamina but with twice of the thickness. With this view in mind, the inter-laminar shear stresses, can be generally expressed as:

$$\tau_{xz}^{i} = \mp\frac{h_i}{2}\frac{\partial p_i}{\partial x}, \qquad \tau_{yz}^{i} = \mp\frac{h_i}{2}\frac{\partial p_i}{\partial y} \tag{4}$$

where h_i is the thickness of the i-th layer (i=1,3,…) and the sign convention is observed in Fig. 1.

For a prepreg layer between metal layers (i = 2, 4, 6…), the action of the interlaminar shear stresses would produce an equivalent effect as the in-plane body forces, as defined by

$$X_i = \frac{\tau_{xz}^{i+1} - \tau_{xz}^{i-1}}{h_i} = -\frac{\partial U}{\partial x} \tag{5a}$$

$$Y_i = \frac{\tau_{yz}^{i+1} - \tau_{yz}^{i-1}}{h_i} = -\frac{\partial U}{\partial y} \tag{5b}$$

where U is the equivalent body-force potential, defined as

$$U = -\frac{1}{2h_i}\left(h_{i+1}p_{i+1} + h_{i-1}p_{i-1}\right) \tag{5c}$$

Therefore, the equilibrium conditions for the prepreg reduce to

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X_i = 0 \tag{6a}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y_i = 0 \tag{6b}$$

Then the in-plane stresses in a prepreg, by satisfying the equilibrium conditions, can be expressed as:

$$\sigma_x = \frac{\partial^2 F}{\partial y^2} + U, \quad \sigma_y = \frac{\partial^2 F}{\partial x^2} + U, \quad \tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y} \tag{7}$$

where F is the stress potential of prepreg, which should satisfy the compatibility condition for an orthotropic material [10]:

$$a_{22}\frac{\partial^4 F}{\partial x^4} + (a_{66} + 2a_{12})\frac{\partial^4 F}{\partial x^2 \partial y^2} + a_{11}\frac{\partial^4 F}{\partial y^4} = -(a_{22} + a_{12})\frac{\partial^2 U}{\partial x^2} - (a_{11} + a_{12})\frac{\partial^2 U}{\partial y^2} \tag{8}$$

where a_{ij} are the compliance coefficients of the prepreg.

Because the layout of an FML is symmetrical, the anti-plane shear stresses in the top half of the laminate do exactly counteract that in the bottom half, so that the averages of τ_{xz} and τ_{yz} over the entire thickness of the laminate are all zero. Thus, the stress-state of an FML, as defined by the stress functions Ψ, F, U, p and q, falls into the category of the generalized plane stress state.

In summary, the stresses in a metal (isotropic) layer can be obtained from the stress function Ψ and conjugated harmonic functions p and q; the stresses in a prepreg (orthotropic) layer can be obtained from the stress functions F and U. The inter-laminar stresses can be calculated using Eq. (4). These stress potentials, when satisfying the necessary compatibility conditions, should lead to a complete description of the stresses in the laminate. By the theorem of unique solution, they should represent the true stress-state of the laminate under a given generalized plane-strain condition. The problem, then, reduces to finding stress functions (or potentials) that meet the boundary-value conditions of the particular loading configuration. The mathematical approach to seek such solutions is discussed below.

3 The Complex Variable Representation

Muskhelishvili [11] Lekhnitskii [10] have developed complex-variable formulations for isotropic and anisotropic elastic bodies, which have been some powerful methods for solving problems of plane elasticity of homogeneous bodies. Now that it is shown that the anti-plane shear potentials, p and q, are needed take into account the interlaminar shear stresses in laminated materials, it is sensible to re-formulate the method to include these terms in the complex variable formulation for laminates. For the present purpose, we shall limit our discussion to the *first fundamental problem (FFP)*, i.e., to find the state of elastic equilibrium for given external stresses applied to the boundary of the body.

3.1 FFP of an isotropic layer

For an isotropic material, the stress potential Ψ is a bi-harmonic function, which can be expressed in terms of complex functions, as:

$$2\Psi = \bar{\xi}\Omega(\xi) + \omega(\xi) + \xi\overline{\Omega}(\bar{\xi}) + \bar{\omega}(\bar{\xi}) \qquad (9)$$

where $\xi = x + iy$ is the complex variable and $\bar{\xi}$ denotes its conjugate, the same meaning also applies to complex functions. We introduce a new complex potential, $\chi(\xi)$, to express the anti-plane shear stress potential p, as:

$$2p = \chi(\xi) + \bar{\chi}(\bar{\xi}) \qquad (10)$$

By manipulating these complex functions, the in-plane stresses can be obtained as:

$$\sigma_x + \sigma_y = 2\,\mathrm{Re}[2\Omega'(\xi) + \chi(\xi)] \qquad (11a)$$

$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2[\bar{\xi}\Omega''(\xi) + \omega''(\xi)] \qquad (11b)$$

and the interlaminar shear stresses can be derived as

$$\tau_{xz} + i\tau_{yz} = \mp\frac{h}{2}\,\bar{\chi}'(\bar{\xi}) \qquad (12)$$

where h is the double thickness of the surface layer and the thickness of the inner layer, the "-" sign applies to the surface normal to the positive z-direction and the "+" sign applies to the surface normal to the negative z-direction.

For the first fundamental problem, similar to that described by Muskhelishvili (1957), the boundary conditions of an isotropic laminae can be written into:

$$\Omega(\xi) + \xi\overline{\Omega}'(\bar{\xi}) + \bar{\omega}'(\bar{\xi}) = \pm i(X_n + iY_n) - \int_s p\,d\xi \qquad (13)$$

where X_n and Y_n are the resultant external forces obtained by integration of the surface traction along the boundary in the x- and y-direction respectively, the "+" sign applies when the boundary is an outer contour and the "–" sign applies when the boundary is an inner contour.

3.2 FFP of an orthotropic layer

For an orthotropic body, the general solution of the stress potential F can be found by adding a particular solution of the non-homogeneous equation, Eq. (8) to the general solution of the homogeneous system (with the removal of the U-derivatives). Following Lekhnitskii [10], the general solution of the homogeneous system can be expressed in terms of two functions, $F_1(\xi_1)$ and $F_2(\xi_2)$, of the complex variables $\xi_1 = x + \mu_1 y$ and $\xi_2 = x + \mu_2 y$, respectively, where μ_1 and μ_2 are the roots of the following algebraic (characteristic) equation:

$$\mu^4 + \frac{2a_{12} + a_{66}}{a_{11}}\mu^2 + \frac{a_{22}}{a_{11}} = 0 \qquad (14)$$

Eq. (14) has four roots, $\mu_1 = \alpha_1 + i\beta_1$ and $\mu_2 = \alpha_2 + i\beta_2$ are the two roots having positive imaginary part ($\beta_1 > 0$, $\beta_2 > 0$), the other two are their conjugates.

To obtain a particular solution, we rearrange Eq. (8) into the form of

$$D_4 D_3 D_2 D_1 F_0 = f_0(\xi) \qquad (15a)$$

where

$$2\,\mathrm{Re}[f_0(\xi)] = -(a_{22}+a_{12})\frac{\partial^2 U}{\partial x^2} - (a_{11}+a_{12})\frac{\partial^2 U}{\partial y^2} \tag{15b}$$

$\mathrm{Re}[f_0(\xi)]$ is a harmonic function, and the complex differential operator D takes the form of

$$D_k = \frac{\partial}{\partial y} - \mu_k \frac{\partial}{\partial x} \qquad (k = 1,2,3,4) \tag{15c}$$

By operating the above complex operators consecutively on $F_0(\xi)$, we obtain the fourth-order derivative of $F_0(\xi)$ as:

$$F_0^{IV} = \frac{1}{\prod_{k=1}^{4}(i-\mu_k)} f_0 \tag{16}$$

$F_0(\xi)$ can be obtained through integration of Eq. (16), once $f_0(\xi)$ is calculated from the equivalent body-force potential, U.

Adding the new function, $F_0(\xi)$, to the general solution, the complete solution of Eq. (8) can be written as:

$$F = 2\,\mathrm{Re}[F_0(\xi) + F_1(\xi_1) + F_2(\xi_2)] \tag{17}$$

Using the stress potentials F and U, the in-plane stresses in an orthotropic prepreg can be obtained from Eq. (7), as:

$$\sigma_x = 2\,\mathrm{Re}[-\Phi_0'(\xi) + \mu_1^2\Phi_1'(\xi_1) + \mu_2^2\Phi_2'(\xi_2)] + U \tag{18a}$$

$$\sigma_y = 2\,\mathrm{Re}[\Phi_0'(\xi) + \Phi_1'(\xi_1) + \Phi_2'(\xi_2)] + U \tag{18b}$$

$$\tau_{xy} = -2\,\mathrm{Re}[i\Phi_0'(\xi) + \mu_1\Phi_1'(\xi_1) + \mu_2\Phi_2'(\xi_2)] \tag{18c}$$

where $\Phi_k = F'_k(\xi_k)$ is an complex function of ξ_k for k = 0, 1, 2.

For the first fundamental problem of an orthotropic layer in FML, the boundary condition can be written as:

$$2\,\mathrm{Re}[i\Phi_0(\xi) + \mu_1\Phi_1(\xi_1) + \mu_2\Phi_2(\xi_2)] = \pm X_n - \int_s U dy \tag{19a}$$

$$2\,\mathrm{Re}[\Phi_0(\xi) + \Phi_1(\xi_1) + \Phi_2(\xi_2)] = \mp Y_n - \int_s U dx \tag{19b}$$

where the "+" sign applies when the boundary is an outer contour and the "–" sign applies when the boundary is an inner contour.

3.3 Examples

3.3.1 Patching of a circular cutout

In this section, we will present the solution for a practical problem — patching a circular cutout — as an example of the boundary-value problem formulation. We consider an infinite plate containing a circular hole of diameter 2a, which is patched with an intact plate, as shown in Fig. 2. We consider that both plates are isotropic materials (the patching plate could be a composite with cross ply of equal amount of fibers such that its overall behavior is isotropic), for simplicity, but they have different elastic modulus ($E_1 \neq E_2$) and the same Poisson's ratio ($\nu_1 = \nu_2 = \nu$). The two plates are immediately bonded (or glued) such that anti-plane shear stresses can transmit across the interface everywhere in the bonded region (r>a). At infinity, the straining of the two plates is uniform and compatible, i.e.

$$\varepsilon_x^{(1)} = \frac{S^{(1)}}{E_1} = \varepsilon_x^{(2)} = \frac{S^{(2)}}{E_2} \tag{20a}$$

$$\varepsilon_y^{(1)} = -\frac{\nu S^{(1)}}{E_1} = \varepsilon_y^{(2)} = -\frac{\nu S^{(2)}}{E_2}; \tag{20b}$$

where $S^{(1)}$ and $S^{(2)}$ are the stresses at infinity in the two plates respectively.

For the plate containing the cutout, the stress potentials are [9]:

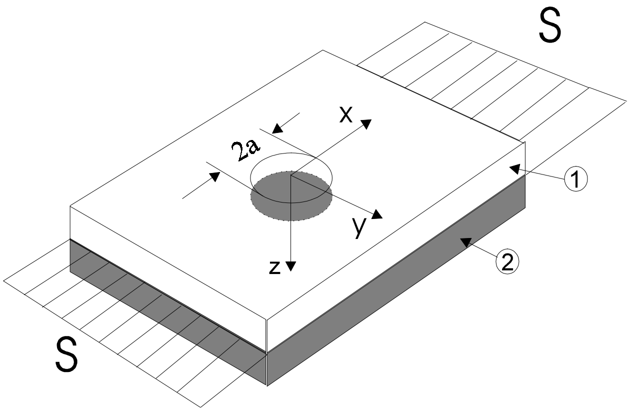

Fig. 2. A three layer FML containing a circular hole, remotely subjected to a uniform stress S.

$$\Omega_1(\xi) = \frac{S^{(1)}}{4}\xi - \sum_{m=1,3,}^{\infty} \frac{c_m}{2\xi^m} \tag{21a}$$

$$\omega_1'(\xi) = -\frac{S^{(1)}}{2}\xi - \frac{S^{(1)}a^2}{2\xi} + \frac{S^{(1)}a^4}{2\xi^3} - \sum_{m=1,3,}^{\infty} \frac{m(m+1)}{2(m+2)} \frac{c_m a^2}{\xi^{m+2}} \tag{21b}$$

$$\chi_1(\xi) = \psi_1'(\xi) = -\sum_{m=1,3,}^{\infty} \frac{mc_m}{\xi^{m+1}} \tag{21c}$$

For the patching plate, within the region $r<a$,

$$\Omega_0(\xi) = \frac{S^{(2)}}{4} + \sum_{m=1,3,}^{\infty} \frac{c_m \xi^{m+2}}{2(\kappa+1)(m+2)a^{2m+2}} \frac{h_1}{h_2} \tag{22a}$$

$$\omega_0'(\xi) = -\frac{S^{(2)}}{2} - \sum_{m=1,3,}^{\infty} \frac{(m+1)c_m \xi^m}{2(\kappa+1)a^{2m}} \frac{h_1}{h_2} \tag{22b}$$

and within the region $r>a$,

$$\Omega_2(\xi) = \frac{S^{(2)}}{4}\xi + \sum_{m=1,3,}^{\infty} \frac{\kappa c_m}{2(\kappa+1)\xi^m} \frac{h_1}{h_2} \tag{23a}$$

$$\omega_2'(\xi) = -\frac{S^{(2)}}{2}\xi + \sum_{m=1,3,}^{\infty} \frac{\kappa m(m+1)c_m a^2}{2(\kappa+1)(m+2)\xi^{m+2}} \frac{h_1}{h_2} \tag{23b}$$

$$\chi_2(\xi) = \psi_2'(\xi) = \sum_{m=1}^{\infty} \frac{mc_m}{\xi^m} \frac{h_1}{h_2} \tag{23c}$$

With the stress potentials as given in Eqs (21)-(23), the displacement and stress continuity within each plate is satisfied. Finally, the displacement continuity conditions at the interface set the equations to determine the coefficients, c_m, see reference [9] for details. A distribution of antiplane shear stress, τ_{rz} around the patched hole is shown in Fig.3. The distribution of $\tau_{\theta z}$ is at a 45^o shift relative to τ_{rz}.

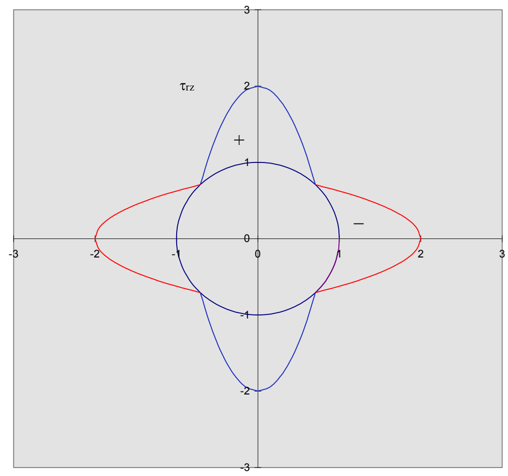

Fig. 3. Distribution of the inter laminar stress τ_{rz} around the hole of a = 1. The abscissa is parallel to the x-axis and the ordinate is parallel to the y-axis. The maximum interlaminar shear stress is normalized to be unity.

3.3.2 Cracks in a FML

In this section, we consider a GLARE-3 (3/2) panel containing a crack, *a*, which propagates in the aluminum layer from an initial through-the-thickness saw-cut, a_0 , as shown in Fig. 4. The prepreg layers are cross-plied with equal amount of fibers such that their overall behavior is isotropic within the plane.

Using the Westergaard function as the stress potential ψ for the aluminum layer containing a crack can be expressed as:

$$\psi = \mathrm{Re}\,\tilde{\tilde{Z}}(\xi) + y\,\mathrm{Im}\,\tilde{Z}(\xi) \quad (24)$$

and the shear stress potential p can be expressed as:

$$2p = \chi(\xi) + \overline{\chi}(\overline{\xi}) = \varphi'(\xi) + \overline{\varphi}'(\overline{\xi}) \quad (25)$$

Using these complex stress potentials, the stresses and displacements can be derived as:

$$\begin{aligned} \sigma_x + \sigma_y &= 2\,\mathrm{Re}\left[Z(\xi) + \chi(\xi)\right] \\ \sigma_y - \sigma_x + i2\tau_{xy} &= -i2yZ'(\xi) \\ \tau_{xz} + i\tau_{yz} &= \mp\frac{h}{2}\overline{\chi'}(\overline{\xi}) \end{aligned} \quad (26)$$

$$2G(u+iv) = \frac{2}{1+\mu}\tilde{Z}(\xi) - \mathrm{Re}\tilde{Z}(\xi) - iy\overline{Z}(\overline{\xi}) + \frac{1-\mu}{1+\mu}\varphi(\xi)$$

We choose

$$Z(\xi) = \frac{A\xi}{\sqrt{\xi^2 - a^2}} \quad \text{and} \quad \chi(\xi) = \frac{B}{\sqrt{\xi - a}} \quad (27)$$

By imposing the condition that $v^{(1)} = v^{(2)}$ at the center of the crack, x=0, we can obtain a closed-form solution for the stress and displacement distribution around the crack and the stress intensity factor at the crack tip is given by:

$$K_{eff} = \lim_{\substack{x_1 \to 0 \\ y=0}} \sqrt{2\pi x_1}\,\sigma_y =$$

$$= K_\infty^{(1)} - \frac{\sqrt{2}}{1-\mu}\,\frac{K_\infty^{(1)} - \lambda\sqrt{\dfrac{a_0}{a}}K_\infty^{(2)}}{1 + 2\lambda\dfrac{h_1}{h_2}\sqrt{\dfrac{a_0}{a}}} \quad (28)$$

where $\lambda = G_1 / G_2$,

$$K_\infty^{(1)} = A^{(1)}\sqrt{\pi a}\,,\quad K_\infty^{(2)} = A^{(2)}\sqrt{\pi a_0}$$

Fig. 5 shows the comparison of the effective stress intensity factor of the crack in the FML with that in bare aluminum. The above solution, however, yields a significant displacement discontinuity near the crack, which is inevitable. Fig. 6 shows the difference of displacements in the aluminum and the prepreg mapped on the panel plane. It is noticed that the shape of the bulged region (or the region of incompatibility) corresponds well to the shape of the delamination region observed during fatigue crack growth tests.

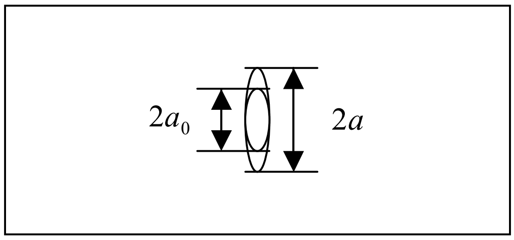

Fig. 4. A notched FML panel.

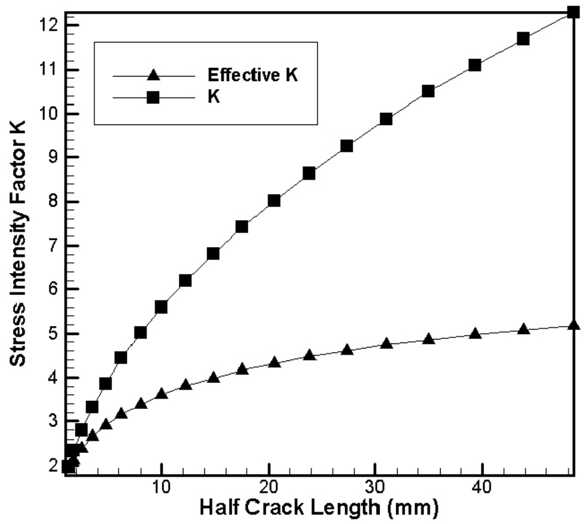

Fig. 5. Comparison of stress intensity factors in bare metal and FML.

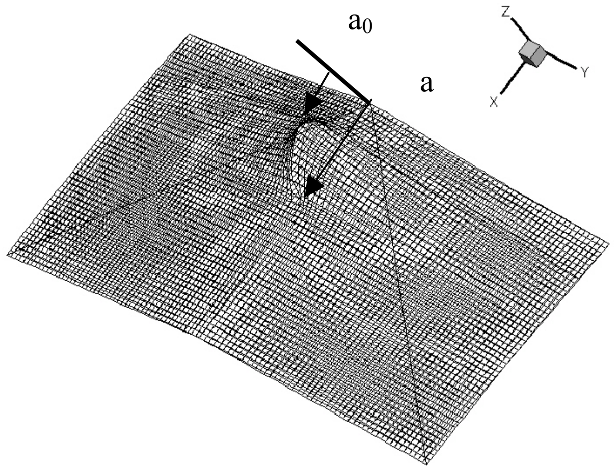

Fig. 6. The in-plane displacement discontinuity at the interface, which is believed to be the cause of delamination.

Conclusion

1. A higher-order lamination theory is developed for the elasticity of fiber-metal laminates. This theory takes into account the interlaminar stresses by employing two harmonic stress potentials, p and q. It is shown that these interlaminar interactions not only play an important role in meeting the interfacial strain compatibility condition but also affect the in-plane stress distributions in both aluminum and prepreg layers.
2. A complex variable representation of the *first fundamental problems* has been derived for FMLs in general. The theory and its use are demonstrated in the example of solving the stress problem of ARALL–3 containing a circular cutout under uniaxial tension.
3. The present stress analysis shows that the stress concentration at a circular hole in the metal layer(s) of a FML remains to be 3, rather not increased by the overall anisotropy of the laminate. This is attributed to the presence of interlaminar shear stresses. Experimental results by van Rijn (1994) seem to support this conclusion.
4. From the point of view of overall laminate stiffness, the classical theory may give a reasonable description, but it will not be very useful in descriptions for the mix-mode damage processes in the laminate, because it neglects interlaminar interactions.

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