FREE FLEXURAL VIBRATION OF RECTANGULAR PLATES HAVING SINGLE CRACKS

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Abstract

This paper presents a methodology for analyzing free flexural vibration of rectangular plates having single linear cracks. The methodology is based on the Levy method and boundary element method. Rectangular plates with simply supported boundary conditions along all four edges are used to illustrate the solution procedure. Natural frequencies obtained using the proposed methodology were compared with those obtained using ANSYS and those available in the. The procedure described in this paper methodology can be easily extended to plates with other boundary conditions with the aid of the method of superposition.

1 Introduction

The objective of this paper is to develop an accurate and efficient method for determining the natural frequencies and mode shapes of flexural vibration of a rectangular panel having a crack of arbitrary locations, orientations and lengths.

A literature review indicates that little work has been done in this area. No general comprehensive approach has been available toward resolving this challenging problem. It must be pointed out here that the finite element software package such as ANSYS failed to produce meaningful results for free flexural vibration of a rectangular plate having a general crack. Stahl and Keer [1] introduced a dual series method for analyzing vibration and buckling problems of a rectangular plate having a crack along one of the plate centerlines. Gorman [2] studied free vibration problems of a rectangular plate having three edges simply supported and the remaining edge subjected to partial boundary conditions. If the partial boundary conditions are simple support and free, the natural frequencies are those of antisymmetric modes for a plate having twice the width and a crack emanating from a plate edge.

Figure 1 shows a rectangular plate of length *a*, width *b*, and uniform thickness *h*. The plate, simply supported along all four edges, has a linear through-thickness crack characterized by its length l_c, orientation angle α_c, and center location x_c and y_c. Ranges of variations for these four crack parameters are subjected to restrictions of plate dimensions.

To obtain a solution for free vibration of a cracked plate, the entire plate is first split into two plate segments along the crack as shown in Fig. 2. An exact analytical solution satisfying the boundary conditions along the three simply supported edges may be obtained for each segment using the Levy method. Once the analytical solutions for the two segments are obtained, they may be joined together to form a solution for the cracked plate by enforcing continuity and boundary conditions along the interface. For an internal crack, there exist three distinct portions along the interface. For a plate having a crack emanating from an edge, the number of distinct portions is reduced to two.

Each segment is free along the cracked surface $\Gamma_1^{(2)}$ or $\Gamma_2^{(2)}$, and subjected to the continuity conditions along $\Gamma^{(1)}$ or $\Gamma^{(3)}$. According to the Kirchhoff classical thin plate theory [3], the free boundary conditions may be interpreted in flexural vibration as zero lateral edge reaction and zero bending moment. The continuity conditions across an interface joining the two plate segments may be interpreted as continuous lateral displacement, continuous slope taken normal to the interface, continuous bending moment, and continuous lateral edge reaction across the interface between the two plate segments.

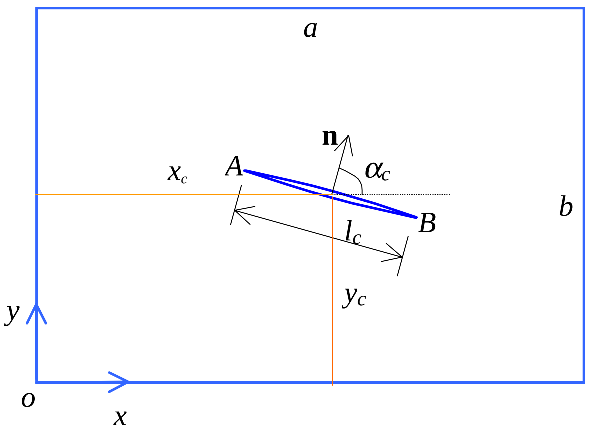


Fig. 1 A rectangular plate having a linear crack

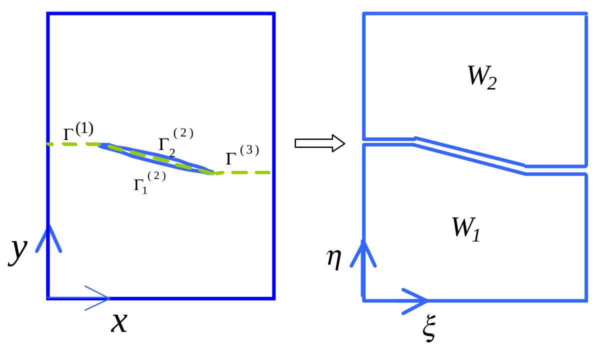


Fig. 2 Solution scheme for a cracked plate

2 Mathematical Description

Using the non-dimensional coordinates defined as $\xi = x/a$ and $\eta = y/b$, the governing differential equations for free flexural vibration of the two plate segments may be written as [3]

$$\frac{\partial^4 W_i}{\partial \eta^4} + \frac{2}{\phi^2}\frac{\partial^4 W_i}{\partial \xi^2 \eta^2} + \frac{1}{\phi^4}\frac{\partial^4 W_i}{\partial \xi^4} - \lambda^4 W_i = 0 \qquad (1)$$

$$(i = 1,2)$$

where W_i is the lateral displacement of segment i; $\lambda^2 (= \omega a^2 \sqrt{\rho h / D})$ is the non-dimensional eigenvalue parameter; ϕ $(= b/a)$ is the plate aspect ratio; ω is the natural frequency; ρ is the mass density of plate material; h is the plate thickness; $D (= Eh^3/[12(1-\nu^2)])$ is the flexural rigidity; E is Young's modulus of the plate material; ν is the Poisson's ratio of the plate material.

2.1 Analytical solutions for plate segments

For the first segment, an exact Levy type solution satisfying the simply supported boundary conditions on edges $\xi = 0$ and $\xi = 1$ may be written as

$$W_1 = \sum_{m=1}^{K} Y_m(\eta) \sin m\pi\xi \qquad (2)$$

where K is the number of terms used in the series solution. Substituting Eqn. (2) into Eqn. (1), one may obtain a series of ordinary differential equations with constant coefficients. Exact analytical solutions may be readily found for these equations. After enforcing the boundary conditions on edge $\eta = 0$, the following analytical solution is obtained

$$W_1 = \sum_{m=1}^{K} [A_m \sinh(\beta_m \eta) + B_m \mathrm{sn}(\gamma_m \eta)] \sin m\pi\xi \qquad (3)$$

where A_m and B_m are unknown constants, to be determined from the continuity and boundary conditions on the remaining edge; $\beta_m = \phi\sqrt{\lambda^2 + (m\pi)^2}$, $\gamma_m = \phi\sqrt{|\lambda^2 - (m\pi)^2|}$; $\mathrm{sn}(\gamma_m \eta) = \sin(\gamma_m \eta)$ if $\lambda^2 < (m\pi)^2$, or $\sinh(\gamma_m \eta)$ if $\lambda^2 > (m\pi)^2$.

A solution for the second segment may be obtained in a manner similar to that for the first plate segment. Upon enforcing the boundary conditions along the three edges, $\xi = 0$, $\xi = 1$ and $\eta = 1$, an analytical solution containing two additional sets of unknown constants, C_m and D_m, may be written as

$$W_2 = \sum_{m=1}^{K}[C_m \sinh(\beta_m \bar{\eta}) + D_m \mathrm{sn}(\gamma_m \bar{\eta})]\sin m\pi\xi \qquad (4)$$

where $\bar{\eta} = 1 - \eta$.

For consistency, the number of terms used in the analytical solutions for the two segments are taken to be the same.

2.2 Slope and stress resultants

Before implementing all continuity and boundary conditions, it is necessary to determine the slope of a deformed plate segment taken normal to a straight edge whose outer normal n makes an angle α with the ξ axis, the bending and lateral edge reaction along the same edge. Take the first segment as an example. The slope taken normal to the straight edge $\theta_{n,1}$, the bending moment $M_{n,1}$, and the lateral edge reaction $V_{n,1}$, according to Timoshenko and Woinowsky-Krieger [4] may be written in terms of the lateral displacement is $W_1(\xi,\eta)$ as

$$\begin{aligned}
\theta_{n,1} &= a_1 \frac{\partial W_1}{\partial \xi} + a_2 \frac{\partial W_1}{\partial \eta} \\
\frac{aM_{n,1}}{D} &= -\left(b_1 \frac{\partial^2 W_1}{\partial \xi^2} + b_2 \frac{\partial^2 W_1}{\partial \eta^2} + b_3 \frac{\partial^2 W_1}{\partial \xi \partial \eta} \right) \qquad (5) \\
\frac{a^2 V_{n,1}}{D} &= -\left(c_1 \frac{\partial^3 W_1}{\partial \xi^3} + c_2 \frac{\partial^3 W_1}{\partial \eta^3} + c_3 \frac{\partial^3 W_1}{\partial \xi^2 \eta} + c_4 \frac{\partial^3 W_1}{\partial \xi \partial \eta^2} \right)
\end{aligned}$$

where a_i, b_i and c_i are non-dimensional coefficients dependent on angle α, Poisson's ratio ν, and the plate aspect ratio ϕ. For the second plate segment, relationships similar to those in Eqn. (5) may be obtained.

2.3 Implementation of partial boundary conditions

Now that the analytical expressions for the lateral displacements, the slope and stress resultants along any straight edge are known. The two sets of analytical solutions contain $4K$ unknown constants. In principle, these constants may be determined by enforcing the appropriate boundary or continuity conditions across different portions of the interface. However, in practice, this is a challenging task to accomplish because of the non-uniformity of the boundary conditions along the interface. It is at this point that the concept of boundary element method is adopted.

Assume that each distinct portion is divided into a number of line elements. The intended boundary conditions are said to be approximately satisfied over each line element if the solution satisfies the boundary conditions at its middle point. For simplicity, the line boundary elements used in this paper have equal lengths within each distinct portion of the interface. If a reasonably large number of such line elements is used in an analysis, it is expected that the so-obtained solution is close to the exact solution.

For a plate having an internal crack, there are three partial boundary conditions along the interface between the two segments – continuity conditions for $0 \le \xi < \xi_A$, free boundary conditions for $\xi_A < \xi < \xi_B$ and continuity conditions again for $\xi_A \le \xi \le 1$, where ξ_A and ξ_B are values of the non-dimensional coordinate ξ of the two crack tips. There are four continuity conditions across each mid-point of a line boundary element in the portions identified by $0 \le \xi < \xi_A$ and $\xi_A \le \xi \le 1$, and two free boundary conditions at each point on each side of the cracked surfaces identified by $\xi_A < \xi < \xi_B$. If N_i represents the number of line boundary elements used in the i-th portion, a total of $4N$ equations may be written, where the total number of line boundary elements is $N = N_1 + N_2 + N_3$. To establish just enough equations for the unknowns, the number of line boundary elements must equal the number of terms in the analytical solutions, or $N = K$. Figure 3 illustrates a typical boundary element mesh for the two plate segments.

For a plate having a crack emanating from an edge, thee are only two partial boundary conditions along the interface. In this case, N_1 or N_3 may be set to zero.

To ensure continuity between the two

deformed plate segments across the two portions of the interface, the following conditions are enforced at the midpoint of j-th boundary element

$$\overset{j}{W}_1-\overset{j}{W}_2=0,\ \overset{j}{\theta}_{n,1}+\overset{j}{\theta}_{n,2}=0,$$
$$\overset{j}{M}_{n,1}-\overset{j}{M}_{n,2}=0,\ \overset{j}{V}_{n,1}+\overset{j}{V}_{n,2}=0 \qquad (6)$$

where $j=1,2,...,N_1$ for the first portion; $j=N_1+N_2+1, N_1+N_2+2,..., N_1+N_2+N_3$ for the third portion.

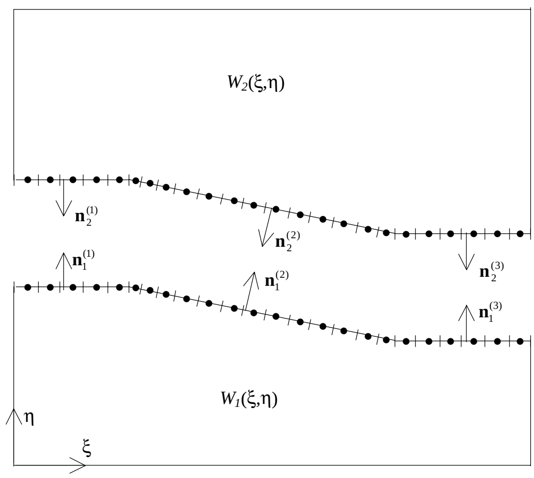


Fig. 3 An illustration of line boundary element mesh along the interface

Along the surface on either side of the linear crack, the completely free boundary conditions should be satisfied. From the classical boundary conditions in [4], this leads to the following conditions

$$\overset{j}{M}_{n,1}=0,\ \overset{j}{V}_{n,1}=0$$
$$\overset{j}{M}_{n,2}=0,\ \overset{j}{V}_{n,2}=0 \qquad (7)$$

where $j=N_1+1, N_1+2, ..., N_1+N_2$ for the third portion.

Enforcing the continuity and boundary conditions at midpoints of all line boundary elements along the interface, one obtains the following homogeneous algebraic equations

$$\left[H(\lambda^2)\right]\begin{Bmatrix}\{A\}\\ \{B\}\\ \{C\}\\ \{D\}\end{Bmatrix}=0 \qquad (8)$$

where $[H]$ is a $4K\times4K$ coefficient matrix; $\{A\}$, $\{B\}$, $\{C\}$, and $\{D\}$ are column vectors containing all unknown constants A_m, B_m, C_m, and $D_m, m=1,2,...K$.

A non-trivial solution requires that the determinant of the coefficient matrix of Eqn. (4) vanish. Eigenvalues or natural frequencies are determined by searching for the zero roots of λ, which causes the determinant to be zero. As the number of terms used in the analytical solution or the number of line boundary elements increases, the analytical solutions converge to the exact solution. In the root-finding stage, the incremental method is first used to identify the intervals where zero roots exist. Newton’s second-order method is then used to determine accurately the root in each interval.

3 Numerical Results

Numerical results for several test cases were obtained to verify the methodology presented in this paper.

The first test case involving a simply supported rectangular plate with a central crack shown in Fig. 4 was designed to examine the rate of convergence of computed natural frequencies versus the number of terms used in the series solutions. The natural frequencies obtained using different terms are presented in Table 1. To ensure the natural frequencies converge to the correct values, the ANSYS program was used to obtain a finite element solution. It is pointed out here that ANSYS cannot handle directly the flexural vibration analysis of a plate having cracks. To obtain a FEM solution, the symmetry in geometry and boundary conditions is taken into consideration. For vibration modes that are symmetric with respect to the crack interface line, the slip-shear boundary conditions [2] are imposed on the two portions of interface while free boundary conditions are imposed on the crack surface. Two families of vibration modes, symmetric or antisymmetric with respect to the plate centerline parallel to the crack, were obtained separately using ANSYS. Results shown in Table 1 indicate that the natural frequencies of the first ten modes converge rapidly to the ANSYS results. The maximum difference between the two solutions is –0.58%.

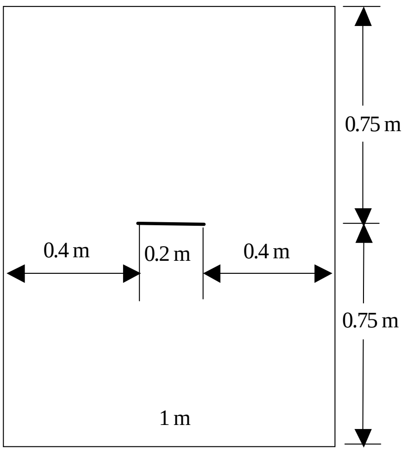


Fig. 4 A simply supported plate having an internal central crack (h = 0.05 m, E = 69 GPa, ν = 0.3)

Table 1 Convergence Test and Comparisons of Computed Natural Frequencies (Hz)

Modes	Number of Terms Used			ANSYS	% Diff.
	K = 10	K = 15	K = 20		
1	171.37	171.57	171.66	171.69	-0.02
2	331.80	332.48	332.73	331.82	0.27
3	533.85	533.84	533.84	533.45	0.07
4	580.45	582.25	583.02	586.45	-0.58
5	694.11	694.12	694.13	692.64	0.22
6	955.19	960.31	960.34	957.67	0.28
7	960.39	962.42	964.95	965.97	-0.11
8	1129.44	1130.02	1130.25	1130.40	-0.01
9	1283.00	1287.58	1289.12	1289.90	-0.06
10	1334.52	1334.57	1334.67	1329.20	0.41

For vibration modes that are antisymmetric with respect to the crack line coinciding with the horizontal plate centerline, the continuity conditions along the interface are equivalent to simple supports. For this reason, antisymmetric modes of a simply supported plate, of 1 m by 2 m, shown in Fig. 5 with a central crack emanating from an edge must be identical to a half plate (1m by 1 m) simply supported along the interface and free along the cracked surface. The latter is a mixed boundary problem, studied by Gorman [2]. The purpose of the second test case is to compare results of Gorman using the analytical method.

For crack length varying between 0 and 1 m, the eigenvalues of the first antisymmetric modes were found and compared with those of Gorman in Fig. 6. The maximum difference is about 3%.

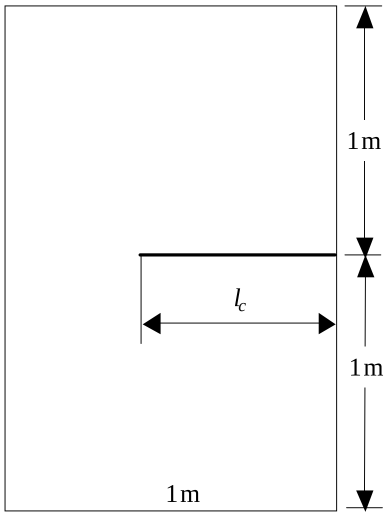


Fig. 5 A simply supported plate having a central crack emanating from an edge(h = 0.05 m, E = 69 GPa, ν = 0.3)

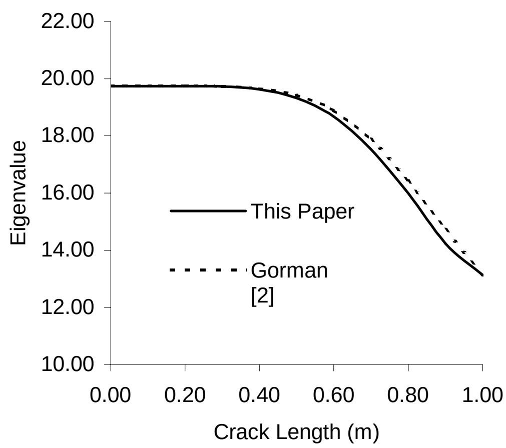
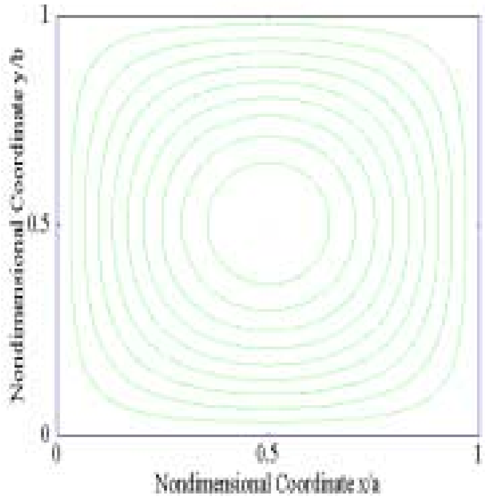
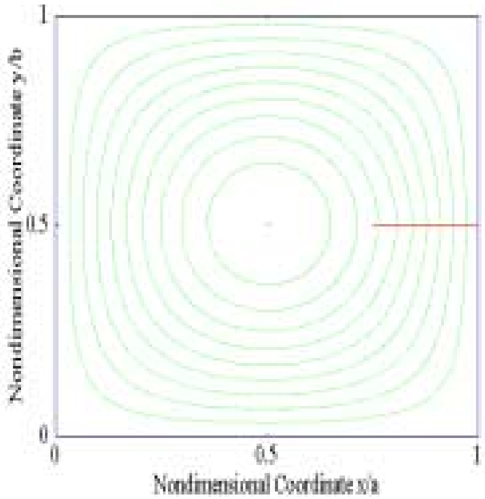


Fig. 6 Eigenvalue vs. for a plate having a central crack emanating from an edge

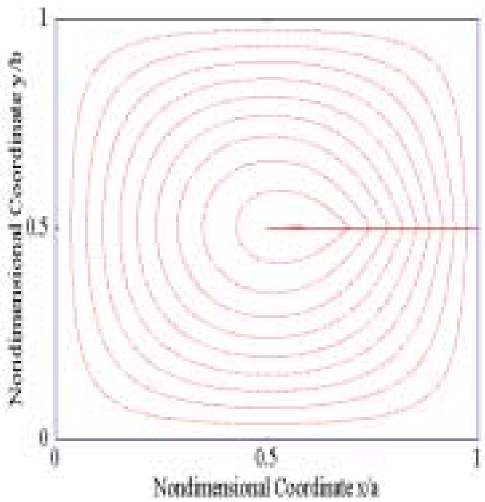
In the third case, a 1 m × 1.01 m × 0.05 m plate (ν = 0.3) having a central crack of various lengths was studied. Figs. 7-10 illustrate the mode shape contours for the first four modes. In these figures, the blue dotted lines represents the nodal lines on which the plate lateral displacement is zero; the green dotted lines represent the contour lines of equal positive displacement; the red-dotted lines represent the contour lines of equal negative displacement; and the solid red line represents the linear crack.



(a) no crack (λ^2 =19.54)

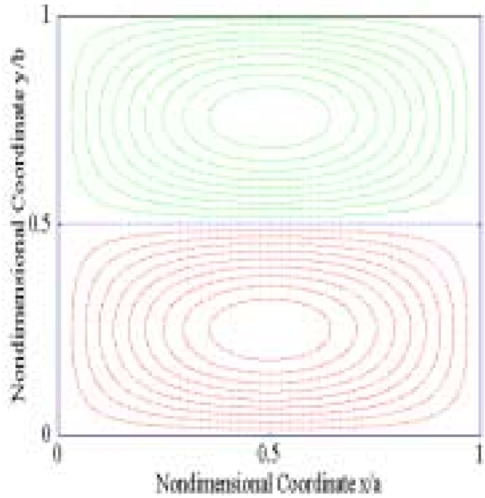


(b) a crack of 0.25 m (λ^2 =19.44)

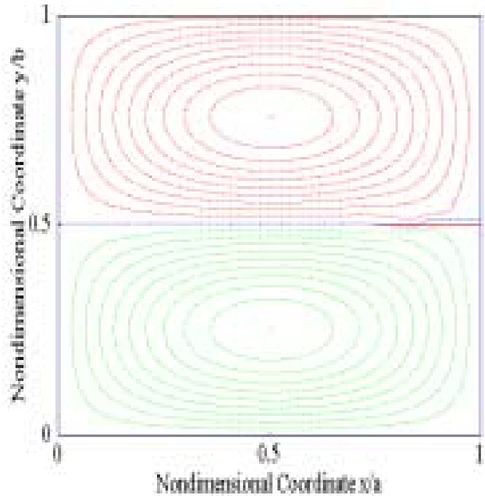


(c) a crack of 0.5 m (λ^2 =18.43)

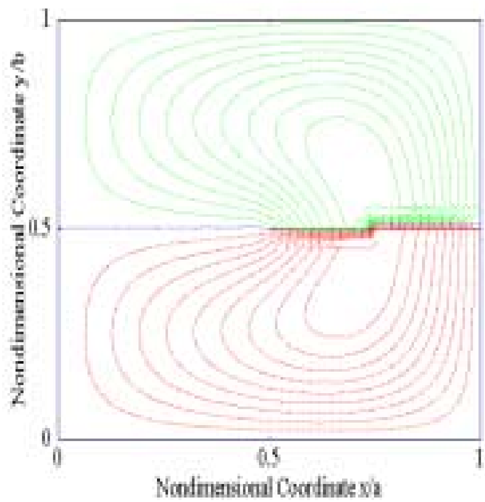
Fig. 7 Contour plot of the first mode shape of a simply supported 1 m by 1.01 m plate



(a) no crack(λ^2 =48.57)

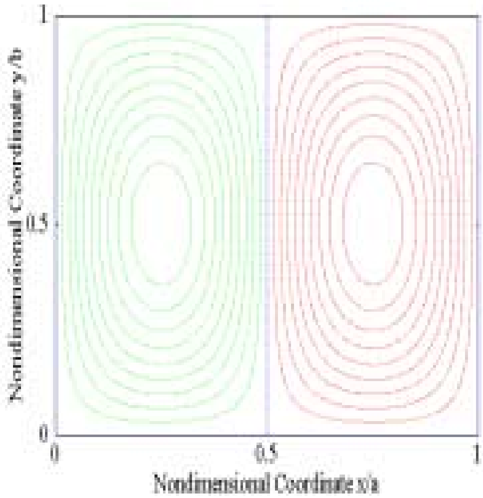


(b) a crack of 0.25 m (λ^2 =48.43)

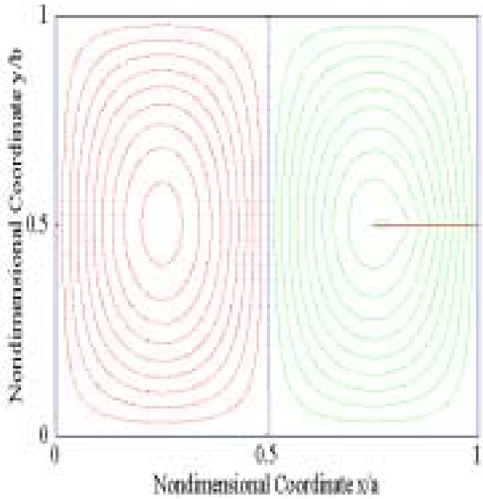


(c) a crack of 0.5 m (λ^2 =41.20)

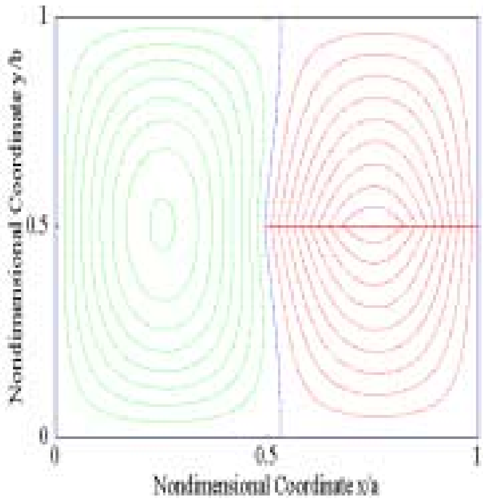
Fig. 8 Contour plot of the second mode shape of a simply supported plate having



(a) no crack (λ^2 =49.15)

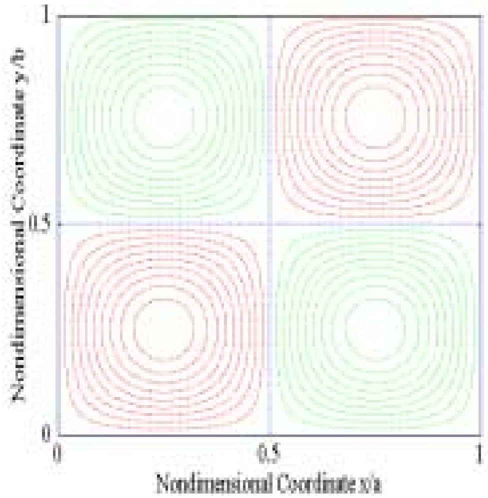


(b) a crack of 0.25 m (λ^2 =48.79)

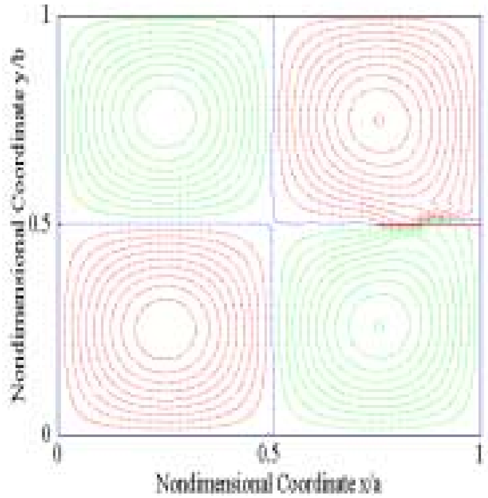


(c) a crack of 0.5 m (λ^2 =47.73)

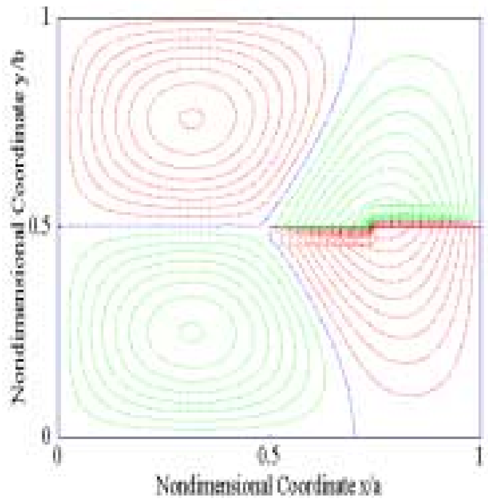
Fig. 9 Contour plot of the third mode shape of a simply supported plate



(a) no crack(λ^2 =78.18)



(b) a crack of 0.25 (λ^2 =77.12)



(c) a crack of 0.5 m (λ^2 =62.78)

Fig. 10 Contour plot of the fourth mode shape of a simply supported plate

It is obvious that the presence of cracks in plates reduces the natural frequencies of all modes and changes the mode shapes as well. However, the variations of natural frequencies and mode shapes with the crack lengths are very complicated even for a fixed crack orientation and a fixed crack location. In the case of a nearly square plate with a central crack emanating from a plate edge, there seems to exist a characteristic length or the size of the half sine wave measured in the direction of the linear crack. If the crack length varies between 0 and 50% of the characteristic length, the effects of cracks on free flexural vibration are very small. However, if the crack length exceeds 50% of the characteristic length, the effects are significant. This information may be useful when the vibration technique are employed to detect the presence of a crack in a panel. For short cracks, measurement of higher mode vibration response may be necessary while for long cracks, measurement of lower mode response should be considered.

4 Conclusions

This paper presents an accurate analytical-numerical approach for free vibration analysis of cracked rectangular plates. The approach may be extended to cracked plates with other boundary conditions with the help of the method of superposition.

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