ANALYTICAL APPROACH TO COMPOSITE BEAM ANALYSIS

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Abstract

The paper presents an analytical model, formulation and solution method for an anisotropic beam under generic distribution of loads. The most generic constitutive relations are assumed. The loads are expressed as generic functions over the cross–sections and as polynomials in the spanwise direction. The overall solution if formulated as recursive procedure. The present formulation is capable of capturing fine and detailed phenomena such as relatively small stress distributions and free edge effects.

1 Introduction

For the last three decades, the analysis of composite beams has been the focal point of many research efforts, and the rapidly growing interest in this area during the last decade testify for its potential in a vast range of engineering applications.

One of the classical approaches to the analysis of anisotropic beams has been presented by Lekhnitskii [1], who expressed the stress tensor in terms of stress functions, developed a rigorous derivation of the associated governing equations and boundary conditions, and presented analytical solutions for some specific cases. Lekhnitskii's [1] formulation for the general analysis of anisotropic beams is in general confined to cases where the stresses do not vary along the longitudinal axis of the beam. This restriction is removed in this paper.

The analytical derivation required for the present task is quite involved, and thus, parts of it were developed using symbolic computational tools.

Overall, the present formulations provides a complete and exact solution to the problem under discussion. Compared with numerical scheme, the present methodology may handle relatively simple cross–section geometries. However, its exactness may serve also as a benchmark for calibrating numerical tools that may handle more complex configurations.

2 Distributed and Tip Loading

The following analysis is derived for a general anisotropic beam, i.e., the compliance matrix contains 21 independent material characteristics

$$\mathbf{a} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ & & a_{33} & a_{34} & a_{35} & a_{36} \\ & & & a_{44} & a_{45} & a_{46} \\ & & & & a_{55} & a_{56} \\ & & & & & a_{66} \end{bmatrix}. \quad (1)$$

The beam is assumed to undergo *distributed loads* $\vec{\mathbf{F}}_s = \{X_s, Y_s, Z_s\}$ (per unit area) along the outer surface of the beam and *distributed body forces* $\vec{\mathbf{F}}_b = \{X_b, Y_b, Z_b\}$ (per unit volume) at each material point. Their components are assumed to be polynomials of degree $K \geq 0$ of z, written as

$$\vec{\mathbf{F}}_b = \sum_{k=0}^{K} \{X_b^{(k)}, Y_b^{(k)}, Z_b^{(k)}\}(x,y)z^k, \quad (2a)$$

$$\vec{\mathbf{F}}_s = \sum_{k=0}^{K} \{X_s^{(k)}, Y_s^{(k)}, Z_s^{(k)}\}(x,y)z^k. \quad (2b)$$

Hence, the case of $K=0$ stands for uniform distributed loads in the z direction, $K=1$ stands for linear distribution, etc., However, formally, the case $K=-1$ is reserved for the case where only tip loads (i.e. no surface and/or body loads) are applied.

To maintain a simple numerical indexing, we formally define the external loading *force resultant vector* $\vec{\mathbf{P}}=\{P_1,P_2,P_3\}$ and *moment resultant vector* $\vec{\mathbf{M}}=\{M_1,M_2,M_3\}$ via an integration of the stress components over a cross–section, Ω, at the spanwise location z as

$$\vec{\mathbf{P}}=\iint_\Omega\{\tau_{xz},\,\tau_{yz},\,\sigma_z\}, \tag{3a}$$

$$\vec{\mathbf{M}}=\iint_\Omega\{\sigma_z y,\,\sigma_z x,\,x\tau_{yz}-y\tau_{xz}\}. \tag{3b}$$

The above definitions enable to write the following fundamental relations between the resultant loads, the surface loads and the body forces:

$$\frac{\partial}{\partial z}\vec{\mathbf{P}}=-\oint_{\partial\Omega}\vec{\mathbf{F}}_s-\iint_\Omega\vec{\mathbf{F}}_b, \tag{4a}$$

$$\frac{\partial}{\partial z}\vec{\mathbf{M}}=-\oint_{\partial\Omega}\{yZ_s,\,xZ_s,\,yX_s-xY_s\}$$
$$-\iint_\Omega\{yZ_b,\,xZ_b,\,yX_b-xY_b\}+\{P_2,\,P_1,\,0\}. \tag{4b}$$

We may now expand P_i and M_i in a polynomial form as well, namely

$$M_i=\sum_{k=0}^{K+2}M_i^{(k)}z^k,\qquad P_i=\sum_{k=0}^{K+1}P_i^{(k)}z^k, \tag{5}$$

while the above polynomial degrees are based on considerations that will be clarified later on. The above expressions and Eqs.(2a,b, 4a,b) indicate that

$$(k+1)\vec{\mathbf{P}}^{(k+1)}=-\oint_{\partial\Omega}\vec{\mathbf{F}}_s^{(k+1)}-\iint_\Omega\vec{\mathbf{F}}_b^{(k+1)}, \tag{6a}$$

$$(k+1)\vec{\mathbf{M}}^{(k+1)}=\{P_2^{(k)},P_1^{(k)},0\}$$
$$-\oint_{\partial\Omega}\{yZ_s^{(k)},\,xZ_s^{(k)},\,yX_s^{(k)}-xY_s^{(k)}\}$$
$$-\iint_\Omega\{yZ_b^{(k)},\,xZ_b^{(k)},\,yX_b^{(k)}-xY_b^{(k)}\}. \tag{6b}$$

We shall assume that at the resultant loads are known at the beam “root” cross–section ($z=0$), and thus $P_i^{(0)}$ and $M_i^{(0)}$ are given by Equation (3a,b) as

$$\{P_1^{(0)},P_2^{(0)},P_3^{(0)},M_1^{(0)},M_2^{(0)},M_3^{(0)}\}=$$
$$\iint_\Omega\{\tau_{xz},\,\tau_{yz},\sigma_z,\sigma_z y,\sigma_z x,x\tau_{yz}-y\tau_{xz}\}_{|z=0}. \tag{7}$$

In cases where the resultant loads are known at other cross–section, it is easy to use this data and the distributed surface and body forces to determine the root resultant loads. The other components (with $k>0$) of the resultant force and moment are derived recursively from Eqs.(6a,b) while Eq.(6b) shows that

$$M_1^{(K+2)}=\frac{P_2^{(K+1)}}{K+2},\;M_2^{(K+2)}=\frac{P_1^{(K+1)}}{K+2},\;M_3^{(K+2)}=0, \tag{8}$$

which are independent of the distributed loads.

3 Presentation of Stress, Strain and Deformation

Based on the above discussion and considerations that will be clarified later on, one may assume that the five stress components $\sigma_x,\sigma_y,\tau_{yz},\tau_{xz},\tau_{xy}$ are polynomials in z of degree $K+1$, and σ_z is a polynomial in z of degree $K+2$ in the form

$$\{\sigma_x,\,\sigma_y,\,\tau_{xy},\,\tau_{xz},\,\tau_{yz}\}$$
$$=\sum_{k=0}^{K+1}\{\sigma_x^{(k)},\,\sigma_y^{(k)},\,\tau_{xy}^{(k)},\,\tau_{xz}^{(k)},\,\tau_{yz}^{(k)}\}z^k, \tag{9a}$$

$$\sigma_z=[\frac{P_1^{(K+1)}}{I_2}x+\frac{P_2^{(K+1)}}{I_1}y]\frac{z^{K+2}}{K+2}+\sum_{k=0}^{K+1}\sigma_z^{(k)}z^k, \tag{9b}$$

where $\sigma_i^{(k)}$, $\tau_{ij}^{(k)}$ are functions of x,y. To derive the integrability conditions, we first apply a series representation to the axial strain ε_z, namely

$$\varepsilon_z=\sum_{k=0}^{K+2}\varepsilon_z^{(k)}(x,y)z^k, \tag{10}$$

which is integrable with respect to z as

$$\int_0^z\varepsilon_z\,dz=\sum_{k=0}^{K+2}\varepsilon_z^{(k)}\frac{z^{k+1}}{k+1}. \tag{11}$$

In addition, Eq.(1) shows that the axial stress may be expressed as

$$\sigma_z = \frac{1}{a_{33}}(\varepsilon_z - a_{13}\sigma_x - a_{23}\sigma_y - \ldots - a_{63}\tau_{xy}). \quad (12)$$

Eqs.(9b, 12) show that $\varepsilon_z^{(K+2)} = a_{33}\sigma_z^{(K+2)}$, which may be determined independently and directly from the data as

$$\varepsilon_z^{(K+2)} = \frac{a_{33}}{K+2}[\frac{P_1^{(K+1)}}{I_2}x + \frac{P_2^{(K+1)}}{I_1}y]. \quad (13)$$

Substituting σ_z from Eq.(12) in the constitutive relations of Eq.(1) yields

$$\varepsilon_x = b_{11}\sigma_x + b_{12}\sigma_y + \ldots + b_{16}\tau_{xy} + \frac{a_{13}}{a_{33}}\varepsilon_z, \quad (14a)$$

$$\varepsilon_y = b_{12}\sigma_x + b_{22}\sigma_y + \ldots + b_{26}\tau_{xy} + \frac{a_{23}}{a_{33}}\varepsilon_z, \quad (14b)$$

$$\gamma_{yz} = b_{14}\sigma_x + b_{24}\sigma_y + \ldots + b_{46}\tau_{xy} + \frac{a_{34}}{a_{33}}\varepsilon_z, \quad (14c)$$

$$\gamma_{xz} = b_{15}\sigma_x + b_{25}\sigma_y + \ldots + b_{56}\tau_{xy} + \frac{a_{35}}{a_{33}}\varepsilon_z, \quad (14d)$$

$$\gamma_{xy} = b_{16}\sigma_x + b_{26}\sigma_y + \ldots + b_{66}\tau_{xy} + \frac{a_{36}}{a_{33}}\varepsilon_z, \quad (14e)$$

where $b_{ij} = a_{ij} - \frac{a_{i3}a_{3j}}{a_{33}}$ $(i, j = 1,2,4,5,6)$ are the *reduced elastic constants*. Analogously to the stress components, we apply a series representation to the remaining strain components (14a-e)

$$\varepsilon_x = \frac{a_{13}}{a_{33}}\varepsilon_z^{(K+2)}z^{K+2} + \sum_{k=0}^{K+1}\varepsilon_x^{(k)}(x,y)z^k, \; etc. \quad (15)$$

The above definitions enable to express the *compatibility equations*, see [2], by explicit execution of the derivatives with respect to z as

$$\varepsilon_{x,yy}^{(k)} + \varepsilon_{y,xx}^{(k)} = \gamma_{xy,xy}^{(k)}, \quad (16a)$$

$$(k+2)\varepsilon_y^{(k+2)} + \frac{1}{k+1}\varepsilon_{z,yy}^{(k)} = \gamma_{yz,y}^{(k+1)}, \quad (16b)$$

$$(k+2)\varepsilon_x^{(k+2)} + \frac{1}{k+1}\varepsilon_{z,xx}^{(k)} = \gamma_{xz,x}^{(k+1)}, \quad (16c)$$

$$\gamma_{xy,x}^{(k+1)} + \frac{1}{k+1}[\gamma_{xz,yx}^{(k)} - \gamma_{yz,xx}^{(k)}] = 2\varepsilon_{x,y}^{(k+1)}, \quad (16d)$$

$$\gamma_{xy,y}^{(k+1)} + \frac{1}{k+1}[\gamma_{yz,xy}^{(k)} - \gamma_{xz,yy}^{(k)}] = 2\varepsilon_{y,x}^{(k+1)}, \quad (16e)$$

$$\gamma_{xz,y}^{(k+1)} + \gamma_{yz,x}^{(k+1)} - (k+2)\gamma_{xy}^{(k+2)} = \frac{2}{k+1}\varepsilon_{z,xy}^{(k)}. \quad (16f)$$

At this stage one may wish to determine the displacement components by integrating the strain components $\varepsilon_z = w_{,z}$, $\gamma_{xz} = u_{,z} + w_{,x}$ and $\gamma_{yz} = v_{,z} + w_{,y}$. We first integrate ε_z, namely

$$w = \int_0^z \varepsilon_z \, dz + W_1(x,y), \quad (17)$$

where $W_1(x,y)$ is a function to be determined that include rigid body displacements and rotations. Differentiation of the above expression yields $w_{,x} = \int_0^z \varepsilon_{z,x} dz - W_{1,x}$ and $w_{,y} = \int_0^z \varepsilon_{z,y} dz - W_{1,y}$, which are subsequently substituted in Eqs.(14c,14d) and yield $u_{,z} = \gamma_{xz} + \int_0^z \varepsilon_{z,x} dz - W_{1,x}$, $v_{,z} = \gamma_{yz} + \int_0^z \varepsilon_{z,y} dz - W_{1,y}$. This enables to derive u and v via integration by z as

$$u = -\int_0^z (\int_0^z \varepsilon_{z,x} dz)\, dz + \int_0^z (\gamma_{xz} - W_{1,x})\, dz + U_1,$$
$$v = -\int_0^z (\int_0^z \varepsilon_{z,y} dz)\, dz + \int_0^z (\gamma_{yz} - W_{1,y})\, dz + V_1, \quad (18)$$

where $U_1(x,y)$ and $V_1(x,y)$ are functions to be determined that include rigid body displacements and rotations. Without lose of generality, we define the point $x=0, y=0, z=0$ as a *reference point* for all the integrations derived in what follows. The displacements at the reference points are denoted as

$$u_2^0 = U_1(0,0), \; v_2^0 = V_1(0,0), \; w_2^0 = W_1(0,0). \quad (19)$$

By using the definitions of the strain components, the rotation components and the notation of [2], the rotation components at the reference point about the x, y and z axes, resp., ω_1^0, ω_2^0 and ω_3^0, may be written as

$$\omega_1^0 = W_{1,y}(0,0) - \frac{1}{2}\gamma_{yz}^{(0)}(0,0), \quad (20a)$$

$$\omega_2^0 = -W_{1,x}(0,0) + \frac{1}{2}\gamma_{xz}^{(0)}(0,0), \quad (20b)$$

$$\omega_3^0 = \frac{1}{2}[V_{1,x}(0,0) - U_{1,y}(0,0)]. \quad (20c)$$

Based on Eqs.(10,13,1517,18), the displacements may be expand as

$$u = -\frac{a_{33}P_1^{(K+1)}z^{K+4}}{(K+2)(K+3)(K+4)I_2} + \sum_{k=0}^{K+3}u^{(k)}z^k,$$

$$v = -\frac{a_{33} P_2^{(K+1)} z^{K+4}}{(K+2)(K+3)(K+4) I_1} + \sum_{k=0}^{K+3} v^{(k)} z^k,$$
$$w = \frac{a_{33} z^{K+3}}{(K+2)(K+3)} [\frac{P_1^{(K+1)}}{I_2} x + \frac{P_2^{(K+1)}}{I_1} y] + \sum_{k=0}^{K+2} w^{(k)} z^k. \quad (21)$$

Note that free of z terms in Eqs.(17, 18) and the coefficients of z in (18) are the following:

$$u^{(0)} = U_1, \quad v^{(0)} = V_1, \quad w^{(0)} = W_1, \quad (22a)$$
$$u^{(1)} = \gamma_{xz}^{(0)} - W_{1,x}, \quad v^{(1)} = \gamma_{yz}^{(0)} - W_{1,y}. \quad (22b)$$

The strain-displacement relations show that

$$\varepsilon_x^{(k)} = u_{,x}^{(k)},\ \varepsilon_y^{(k)} = v_{,y}^{(k)},\ \gamma_{xy}^{(k)} = u_{,y}^{(k)} + v_{,x}^{(k)}, \quad (23a)$$
$$\gamma_{yz}^{(k)} = (k+1) v^{(k+1)} + w_{,y}^{(k)}, \quad (23b)$$
$$\gamma_{xz}^{(k)} = (k+1) u^{(k+1)} + w_{,x}^{(k)}. \quad (23c)$$

In a similar way, a series representation is applied to the rotation vector components

$$\omega_1 = \sum_{k=0}^{K+2} \omega_1^{(k)} z^k,\ \omega_2 = \sum_{k=0}^{K+2} \omega_2^{(k)} z^k,\ \omega_3 = \sum_{k=0}^{K+3} \omega_3^{(k)} z^k,$$

where $\omega_i^{(k)}$ are functions of x, y. Also

$$\omega_1^{(k)} = \frac{1}{2}(w_{,y}^{(k)} - (k+1) v^{(k+1)}), \quad (24a)$$
$$\omega_2^{(k)} = \frac{1}{2}((k+1) u^{(k+1)} - w_{,x}^{(k)}), \quad (24b)$$
$$\omega_3^{(k)} = \frac{1}{2}(v_{,x}^{(k)} - u_{,y}^{(k)}). \quad (24c)$$

For further purposes, we now express from Eqs.(23a-c, 24a-c) the derivatives of displacements components $u^{(k)}, v^{(k)}, w^{(k)}$ in terms of rotation and strain components

$$u_{,x}^{(k)} = \varepsilon_x^{(k)}, \quad u_{,y}^{(k)} = \frac{1}{2}\gamma_{xy}^{(k)} - \omega_3^{(k)}, \quad (25a)$$
$$v_{,x}^{(k)} = \frac{1}{2}\gamma_{xy}^{(k)} + \omega_3^{(k)}, \quad v_{,y}^{(k)} = \varepsilon_y^{(k)}, \quad (25b)$$
$$w_{,x}^{(k)} = \frac{1}{2}\gamma_{xz}^{(k)} - \omega_2^{(k)},\ w_{,y}^{(k)} = \frac{1}{2}\gamma_{yz}^{(k)} + \omega_1^{(k)}. \quad (25c)$$

From Eq.(25a,b) we obtain

$$\omega_{3,x}^{(k)} = \frac{1}{2}\gamma_{xy,x}^{(k)} - \varepsilon_{x,y}^{(k)}, \quad \omega_{3,y}^{(k)} = \varepsilon_{y,x}^{(k)} - \frac{1}{2}\gamma_{xy,y}^{(k)}. \quad (26)$$

4 The Integrability Conditions

We shall now derive the integrability conditions by specifying the conditions one should imposed on the above deformation functions in order to assure integrability.

Using Eqs.(16b,c,f) show that

$$\frac{1}{k-1}\varepsilon_{z,xx}^{(k-2)} = \gamma_{xz,x}^{(k-1)} - k\varepsilon_x^{(k)}, \quad (27a)$$
$$\frac{1}{k-1}\varepsilon_{z,yy}^{(k-2)} = \gamma_{yz,y}^{(k-1)} - k\varepsilon_y^{(k)}, \quad (27b)$$
$$\frac{2}{k-1}\varepsilon_{z,xy}^{(k-2)} = \gamma_{yz,x}^{(k-1)} + \gamma_{xz,y}^{(k-1)} - k\gamma_{xy}^{(k)}. \quad (27c)$$

We first integrate Eq.(27a) with respect to x, and Eq.(27b) with respect to y

$$\frac{\varepsilon_{z,x}^{(k-2)}}{k-1} = \frac{\kappa_1^{(k-2)}}{k-1} + k\Theta_1^{(k-1)}(y) + \gamma_{xz}^{(k-1)} - \gamma_{xz}^{(k-1)}(0,0) - k\int_0^x \varepsilon_x^{(k)}\, dx, \quad (28a)$$
$$\frac{\varepsilon_{z,y}^{(k-2)}}{k-1} = \frac{\kappa_2^{(k-2)}}{k-1} + k\Theta_2^{(k-1)}(x) + \gamma_{yz}^{(k-1)} - \gamma_{yz}^{(k-1)}(0,0) - k\int_0^y \varepsilon_y^{(k)}\, dy, \quad (28b)$$

which should be integrated under the condition

$$\varepsilon_z^{(k)}(0,0) = \varepsilon_0^{(k)}, \quad (29)$$

where according Eq.(13), $\varepsilon_0^{(K+2)} = 0$. In Eqs.(28a,b)

$$\kappa_1^{(k-2)} = \varepsilon_{z,x}^{(k-2)}(0,0),\ \kappa_2^{(k-2)} = \varepsilon_{z,y}^{(k-2)}(0,0) \quad (30)$$

are constants and $\Theta_i^{(k-1)}$, $i = 1, 2$ are arbitrary functions of one variable, the meaning of which will become clearer later on. Formally, $\kappa_i^{(k-2)}$ are required since $\Theta_1^{(k-1)}$ and $\Theta_2^{(k-1)}$ are assumed to vanish for $y = 0$, and $x = 0$, respectively. At this stage, it is useful to introduce the following definition of $\bar{D}^{(k)}$ via the identity

$$\varepsilon_z^{(k)} = \kappa_1^{(k)} x + \kappa_2^{(k)} y + \varepsilon_0^{(k)} + \bar{D}^{(k)}, \quad k \geq 0, \quad (31)$$

and subsequently, the function $\bar{D}(x, y, z)$ may be defined as

$$\bar{D} = \sum_{k=0}^{K+1} \bar{D}^{(k)} z^k. \quad (32)$$

For future purposes one should note that $\bar{D}_{,x}^{(k-2)}(0,0) = \bar{D}_{,y}^{(k-2)}(0,0) = 0$ and

$$\frac{\bar{D}_{,x}^{(k-2)}}{k-1} = k\Theta_1^{(k-1)} + \gamma_{xz}^{(k-1)}|_{(0,0)}^{(x,y)} - k\int_0^x \varepsilon_x^{(k)}\,dx,$$
$$\frac{\bar{D}_{,y}^{(k-2)}}{k-1} = k\Theta_2^{(k-1)} + \gamma_{yz}^{(k-1)}|_{(0,0)}^{(x,y)} - k\int_0^y \varepsilon_y^{(k)}\,dy, \quad (33)$$

which should be solved for $\bar{D}^{(k-2)}(0,0) = 0$.

Calculating the quantity $(\varepsilon_{z,x}^{(k-2)})_{,y} + (\varepsilon_{z,y}^{(k-2)})_{,x}$ from Eqs.(28a,b) and equating it to $2\varepsilon_{z,xy}^{(k-2)}$ of Eq.(27c), leads to

$$\frac{d}{dy}\Theta_1^{(k-1)} + \frac{d}{dx}\Theta_2^{(k-1)} = \int_0^x \varepsilon_{x,y}^{(k)}\,dx + \int_0^y \varepsilon_{y,x}^{(k)}\,dy - \gamma_{xy}^{(k)}. \quad (34)$$

The partial derivatives of Eq.(34) with respect to x and y yield the following two second–order differential equations for $\Theta_i^{(k-1)}$, $i = 1,2$

$$\frac{d^2}{dy^2}\Theta_1^{(k-1)} = B^{(k-1)}, \quad \frac{d^2}{dx^2}\Theta_2^{(k-1)} = A^{(k-1)}, \quad (35)$$

where

$$B^{(k-1)} = \int_0^x \varepsilon_{x,yy}^{(k)}\,dx + \varepsilon_{y,x}^{(k)} - \gamma_{xy,y}^{(k)}, \quad (36a)$$
$$A^{(k-1)} = \int_0^y \varepsilon_{y,xx}^{(k)}\,dy + \varepsilon_{x,y}^{(k)} - \gamma_{xy,x}^{(k)}. \quad (36b)$$

In essence, $A^{(k-1)}$ and $B^{(k-1)}$ are functions of one variable, namely x and y, respectively, which may be rewritten in an equivalent form as

$$B^{(k-1)}(y) = [\varepsilon_{y,x}^{(k)} - \gamma_{xy,y}^{(k)}]_{|x=0}, \quad (37a)$$
$$A^{(k-1)}(x) = [\varepsilon_{x,y}^{(k)} - \gamma_{xy,x}^{(k)}]_{|y=0}. \quad (37b)$$

The fact that $A^{(k-1)}$ and $B^{(k-1)}$ are functions of one variable is not always obvious when one examines the expressions of the functions. Therefore, it appears useful to substitute $y = 0$ and $x = 0$ in the expressions for $A^{(k-1)}$ and $B^{(k-1)}$, respectively, as shown in Eqs.(37a,b).

Integrating Eqs.(35) allows to express the functions $\Theta_i^{(k-1)}$ as

$$\Theta_1^{(k-1)}(y) = \frac{\theta^{(k-1)}}{k}y - \frac{1}{2}\gamma_{xy}^{(k)}(0,0)y + \int_0^y\Big(\int_0^y B^{(k-1)}\,dy\Big)\,dy, \quad (38a)$$
$$\Theta_2^{(k-1)}(x) = -\frac{\theta^{(k-1)}}{k}x + \frac{1}{2}\gamma_{xy}^{(k)}(0,0)x + \int_0^x\Big(\int_0^x A^{(k-1)}\,dx\Big)\,dx, \quad (38b)$$

which also supports the previously statement regarding that identity

$$\Theta_1^{(k-1)}(0) = \Theta_2^{(k-1)}(0) = 0.$$

In Eqs.(38a,b), $\theta^{(k-1)}$ are constants (will be derived later on). Hence,

$$\frac{d}{dy}\Theta_1^{(k-1)}(0) = \frac{1}{k}\theta^{(k-1)} - \frac{1}{2}\gamma_{xy}^{(k)}(0,0), \quad (39a)$$
$$\frac{d}{dx}\Theta_2^{(k-1)}(0) = -\frac{1}{k}\theta^{(k-1)} - \frac{1}{2}\gamma_{xy}^{(k)}(0,0), \quad (39b)$$

which ensures that the sum of the terms on the l.h.s. of the above equations equals $-\gamma_{xy}^{(k)}(0,0)$ as required by Eq.(34). Subtracting the l.h.s. terms yields

$$\frac{2}{k}\theta^{(k-1)} = \frac{d}{dy}\Theta_1^{(k-1)}(0) - \frac{d}{dx}\Theta_2^{(k-1)}(0). \quad (40)$$

We shall now express the *integrability conditions* for Eqs.(27a-c, 28a,b). Fundamentally, the following conditions make these equations integrable

$$(\varepsilon_{z,xx}^{(k-2)})_{,yy} + (\varepsilon_{z,yy}^{(k-2)})_{,xx} - 2(\varepsilon_{z,xy}^{(k-2)})_{,xy} = 0, \quad (41a)$$
$$(\varepsilon_{z,x}^{(k-2)})_{,y} - (\varepsilon_{z,y}^{(k-2)})_{,x} = 0. \quad (41b)$$

Substituting Eqs.(27a-c) in Eq.(41a) leads to Eq.(16a). From Eq.(16a) for $k \geq 0$ using Eqs.(14a-c) we obtain the integrability condition

$$(b_{11}\sigma_x^{(k)} + \ldots + b_{16}\tau_{xy}^{(k)})_{,yy} + (b_{12}\sigma_x^{(k)} + \ldots + b_{26}\tau_{xy}^{(k)})_{,xx} - (b_{16}\sigma_x^{(k)} + \ldots + b_{66}\tau_{xy}^{(k)})_{,xy} = \frac{1}{a_{33}}[a_{36}\varepsilon_{z,xy}^{(k)} - a_{13}\varepsilon_{z,yy}^{(k)} - a_{23}\varepsilon_{z,xx}^{(k)}]. \quad (42)$$

Substituting Eqs.(14c,d, 28a,b) in Eq.(41b) takes the form

$$(b_{15}\sigma_x^{(k-1)}+\ldots+b_{56}\tau_{xy}^{(k-1)})_{,y}-(b_{14}\sigma_x^{(k-1)}+\ldots$$
$$+b_{46}\tau_{xy}^{(k-1)})_{,x}=-2\theta^{(k-1)}+\frac{a_{34}}{a_{33}}\varepsilon_{z,x}^{(k-1)}-\frac{a_{35}}{a_{33}}\varepsilon_{z,y}^{(k-1)}$$
$$+k[\int_0^x A^{(k-1)}\,dx-\int_0^y B^{(k-1)}\,dy+\int_0^x \varepsilon_{x,y}^{(k)}\,dx$$
$$+\frac{a_{13}}{a_{33}}\int_0^x \varepsilon_{z,y}^{(k)}\,dx-\int_0^y \varepsilon_{y,x}^{(k)}\,dy-\frac{a_{23}}{a_{33}}\int_0^y \varepsilon_{z,x}^{(k)}dy]. \quad (43)$$

From Eq.(23a) for $k=1$, and in view of integrability condition Eq.(16a), the equations for the determination of the function W_1 become

$$W_{1,x}=\Theta_1^{(0)}+\gamma_{xz}^{(0)}-\int_0^x \varepsilon_x^{(1)}\,dx-\omega_2^0-\frac{1}{2}\gamma_{xz}^{(0)}(0,0),$$
$$W_{1,y}=\Theta_2^{(0)}+\gamma_{yz}^{(0)}-\int_0^y \varepsilon_y^{(1)}\,dy+\omega_1^0-\frac{1}{2}\gamma_{yz}^{(0)}(0,0), \quad (44)$$

which should be solve under the initial condition $W_1(0,0)=w^0$, see Eq.(19). In this case, the requirement $(W_{1,x})_{,y}=(W_{1,y})_{,x}$ serves as the integrability condition, and it it easy to see that it becomes identical to Eq.(43) for $k=1$.

The differential equations for the the functions U_1, V_1 are obtained from Eq.(23a) for $k=0$ as

$$U_{1,x}=\varepsilon_x^{(0)},\ V_{1,y}=\varepsilon_y^{(0)},\ U_{1,y}+V_{1,x}=\gamma_{xy}^{(0)}. \quad (45)$$

Using the rotation component

$$\omega_3^{(0)}=\frac{1}{2}(V_{1,x}-U_{1,y}) \quad (46)$$

from Eqs.(26) for $k=0$, with the initial condition $\omega_3^{(0)}(0,0)=\omega_3^0$, we find the functions U_1 and V_1 from the following two systems

$$U_{1,x}=\varepsilon_x^{(0)},\quad U_{1,y}=\frac{1}{2}\gamma_{xy}^{(0)}-\omega_3^{(0)} \quad (47)$$

and

$$V_{1,x}=\frac{1}{2}\gamma_{xy}^{(0)}+\omega_3^{(0)},\quad V_{1,y}=\varepsilon_y^{(0)}. \quad (48)$$

The above systems should be solved under the initial conditions $U_1(0,0)=u^0$ and $V_1(0,0)=v^0$.

It is convenient to introduce three new functions U,V,W of x,y instead of U_1,V_1,W_1 by setting

$$U_1=U-\omega_3^0 y+u^0,$$
$$V_1=V+\omega_3^0 x+v^0,$$
$$W_1=W+\omega_1^0 y-\omega_2^0 x+w^0. \quad (49)$$

5 The Deformation Measures

The constants of integration $\kappa_1^{(k)}$, $\kappa_2^{(k)}$, $\varepsilon_0^{(k)}$, $k\geq 0$, appearing in Eqs.(28a,b, 31) are components of the *curvature* and the *axial strain* functions

$$\kappa_i(z)=\sum_{k=0}^{K+2}\kappa_i^{(k)}z^k,\quad \varepsilon_0(z)=\sum_{k=0}^{K+1}\varepsilon_0^{(k)}z^k. \quad (50)$$

These displacement measures are derived using Eq.(3a,b).

We will show that the integration constants $\theta^{(k)}$ are components of the *twist* function

$$\theta(z)=\sum_{k=0}^{K+2}\theta^{(k)}z^k.$$

Below we use the auxiliary functions

$$\Theta_1(y,z)=\sum_{k=0}^{K+2}\Theta_1^{(k)}z^k,\ \Theta_2(x,z)=\sum_{k=0}^{K+2}\Theta_2^{(k)}z^k,$$
$$A(x,z)=\sum_{k=0}^{K+1}A_1^{(k)}z^k,\ B(y,z)=\sum_{k=0}^{K+1}B_2^{(k)}z^k, \quad (51)$$

and find from Eqs.(38a,b)

$$\Theta_1(y,z)=\frac{y}{z}[\int_0^z\theta(z)\,dz-\frac{1}{2}(\gamma_{xy}(0,0,z)$$
$$-\gamma_{xy}(0,0,0))]+\int_0^y(\int_0^y B(y,z)\,dy)\,dy,$$
$$\Theta_2(x,z)=-\frac{x}{z}[\int_0^z\theta(z)\,dz+\frac{1}{2}(\gamma_{xy}(0,0,z)$$
$$-\gamma_{xy}(0,0,0))]+\int_0^x(\int_0^x A(x,z)\,dx)\,dx. \quad (52)$$

In particular,

$$2\int_0^z\theta(z)\,dz=z[\Theta_{1,y}(0,z)-\Theta_{2,x}(0,z)]. \quad (53)$$

Then, using Eqs.(18, 52), we obtain the equivalent to Eqs.(17, 18) final expressions of the displacements

$$u=-\int_0^z(\int_0^z\kappa_1\,dz)\,dz-y\int_0^z\theta\,dz \quad (54a)$$
$$-\frac{y}{2}(\gamma_{xy}(0,0,z)-\gamma_{xy}(0,0,0))$$
$$-z\int_0^y(\int_0^y B\,dy)\,dy+\int_0^z\gamma_{xz}(0,0,z)\,dz$$

$$-\frac{1}{2}\gamma_{xz}(0,0,0)z+\int_0^x(\varepsilon_x-\varepsilon_x^{(0)})\,dx+U+u_r,$$

$$v=-\int_0^z(\int_0^z\kappa_2\,dz)\,dz+x\int_0^z\theta\,dz \tag{54b}$$
$$+\frac{x}{2}(\gamma_{xy}(0,0,z)-\gamma_{xy}(0,0,0))$$
$$-z\int_0^x(\int_0^x A\,dx)\,dx+\int_0^z\gamma_{yz}(0,0,z)\,dz$$
$$-\frac{1}{2}\gamma_{yz}(0,0,0)z+\int_0^y(\varepsilon_y-\varepsilon_y^{(0)})\,dy+V+v_r,$$

$$w=\int_0^z(\kappa_1x+\kappa_2y+\varepsilon_0+\bar{D})\,dz+W+w_r, \tag{54c}$$

where u_r, v_r and w_r are rigid body components

$$u_r=\omega_2^0z-\omega_3^0y+u^0,\; v_r=\omega_3^0x-\omega_1^0z+v^0,$$
$$w_r=\omega_1^0y-\omega_2^0x+w^0, \tag{55}$$

which are all determined by the geometrical boundary conditions. Therefore, there will be no need to pay any attention to the integration constants that may emerge during the integration process of the functions $U_1(x,y)$, $V_1(x,y)$ and $W_1(x,y)$.

By differentiating Eq.(54c) we obtain the following expressions of deformation measures:

$$\kappa_1(z)=w_{,xz}(0,0,z),\quad \kappa_2(z)=w_{,yz}(0,0,z),$$
$$\varepsilon_0(z)=w_{,z}(0,0,z). \tag{56}$$

Also we find the rotation vector along the axis:

$$\omega_1(0,0,z)=\int_0^z\kappa_2\,dz-\frac{1}{2}\gamma_{yz}\Big|_{(0,0,0)}^{(0,0,z)}+\omega_1^0, \tag{57a}$$
$$\omega_2(0,0,z)=-\int_0^z\kappa_1\,dz+\frac{1}{2}\gamma_{xz}\Big|_{(0,0,0)}^{(0,0,z)}+\omega_2^0, \tag{57b}$$
$$\omega_3(0,0,z)=\int_0^z\theta\,dz+\omega_3^0. \tag{57c}$$

By integration of Eq.(3a,b), the curvature and axial strain functions may be expressed as

$$\kappa_1I_2=a_{33}M_2+\frac{a_{34}}{2}M_3-\iint_\Omega\bar{D}x$$
$$+\iint_\Omega[\frac{1}{2}(a_{13}x^2-a_{23}y^2)(X_b+\tau_{xz,z})$$
$$+(a_{23}xy+\frac{a_{36}}{2}x^2)(Y_b+\tau_{yz,z})$$
$$+\frac{1}{2}(a_{34}xy+a_{35}x^2)(Z_b+\sigma_{z,z})]$$
$$+\oint_{\partial\Omega}[\frac{1}{2}(a_{13}x^2-a_{23}y^2)X_s+(a_{23}xy$$
$$+\frac{a_{36}}{2}x^2)Y_s+\frac{1}{2}(a_{34}xy+a_{35}x^2)Z_s], \tag{58a}$$

$$\kappa_2I_1=a_{33}M_1-\frac{a_{35}}{2}M_3-\iint_\Omega\bar{D}y$$
$$+\iint_{\partial\Omega}[(a_{13}xy+\frac{a_{36}}{2}y^2)(X_b+\tau_{xz,z})$$
$$+\frac{1}{2}(a_{23}y^2-a_{13}x^2)(Y_b+\tau_{yz,z})$$
$$+\frac{1}{2}(a_{35}xy+a_{34}y^2)(Z_b+\sigma_{z,z})]$$
$$+\oint_{\partial\Omega}[(a_{13}xy+\frac{a_{36}}{2}y^2)X_s+\frac{1}{2}(a_{23}y^2$$
$$-a_{13}x^2)Y_s+\frac{1}{2}(a_{35}xy+a_{34}y^2)Z_s], \tag{58b}$$

$$\varepsilon_0\,S(\Omega)=a_{33}P_3+\oint_{\partial\Omega}[(a_{13}x+a_{36}y)X_s$$
$$+a_{23}yY_s+(a_{35}x+a_{34}y)Z_s]-\iint_\Omega\bar{D}$$
$$+\iint_\Omega[(a_{13}x+a_{36}y)(X_b+\tau_{xz,z})+a_{23}y(Y_b$$
$$+\tau_{yz,z})+(a_{35}x+a_{34}y)(Z_b+\sigma_{z,z})], \tag{58c}$$

where an area and moments of inertia of Ω are

$$\{S(\Omega),\,I_2,\,I_1,\,0\}=\iint_\Omega\{1,\,x^2,y^2,\,xy\}\,dx\,dy. \tag{59}$$

6 The Stress Functions

One may write the *equilibrium equations* for the kth order stage, $(K+1\geq k\geq 0)$ as

$$\sigma_{x,x}^{(k)}+\tau_{xy,y}^{(k)}+(k+1)\tau_{xz}^{(k+1)}+X_b^{(k)}=0, \tag{60a}$$
$$\tau_{xy,x}^{(k)}+\sigma_{y,y}^{(k)}+(k+1)\tau_{yz}^{(k+1)}+Y_b^{(k)}=0, \tag{60b}$$
$$\tau_{xz,x}^{(k)}+\tau_{yz,y}^{(k)}+(k+1)\sigma_z^{(k+1)}+Z_b^{(k)}=0, \tag{60c}$$

while for the $K+2$ level, only $\sigma_z^{(k+2)}$ exists.

We shall now define the *body forces potential functions* $\bar{U}_i=\sum_{k=0}^{K}\bar{U}_i^{(k)}z^k$, $(i=1,2)$ and $\bar{U}_i=\sum_{k=0}^{K+1}\bar{U}_i^{(k)}z^k$, $(i=3,4)$. For $k\leq K$

$$\bar{U}_1^{(k)}=-\int_0^x[(k+1)\tau_{xz}^{(k+1)}+X_b^{(k)}]\,dx, \tag{61a}$$
$$\bar{U}_2^{(k)}=-\int_0^y[(k+1)\tau_{yz}^{(k+1)}+Y_b^{(k)}]\,dy, \tag{61b}$$
$$\bar{U}_{4,y}^{(k)}+\bar{U}_{3,x}^{(k)}=-(k+1)\sigma_z^{(k+1)}-Z_b^{(k)}, \tag{61c}$$

while for $k = K+1$,

$$\bar{U}_{4,y}^{(K+1)} + \bar{U}_{3,x}^{(K+1)} = -\frac{P_1^{(K+1)}}{I_2}x - \frac{P_2^{(K+1)}}{I_1}y. \quad (62)$$

With the above definitions, Eq.(60a-c) become

$$(\sigma_x^{(k)} - \bar{U}_1^{(k)})_{,x} + \tau_{xy,y}^{(k)} = 0, \quad (63a)$$
$$\tau_{xy,x}^{(k)} + (\sigma_y^{(k)} - \bar{U}_2^{(k)})_{,y} = 0, \quad (63b)$$
$$(\tau_{xz}^{(k)} - \bar{U}_3^{(k)})_{,x} + (\tau_{yz}^{(k)} - \bar{U}_4^{(k)})_{,y} = 0. \quad (63c)$$

At this stage we adopt the definition of the stress functions $\Phi = \sum_{k=0}^{K+1}\Phi^{(k)}(x,y)z^k$ and $\psi = \sum_{k=0}^{K+1}\psi^{(k)}(x,y)z^k$, namely

$$\sigma_x^{(k)} = \Phi_{,yy}^{(k)} + \underline{\bar{U}_1^{(k)}}, \ \sigma_y^{(k)} = \Phi_{,xx}^{(k)} + \underline{\bar{U}_2^{(k)}}, \ \tau_{xy}^{(k)} = -\Phi_{,xy}^{(k)},$$
$$\tau_{xz}^{(k)} = \psi_{,y}^{(k)} + \bar{U}_3^{(k)}, \quad \tau_{yz}^{(k)} = -\psi_{,x}^{(k)} + \bar{U}_4^{(k)}, \quad (64)$$

where the underlined terms do not exist for $k = K+1$. With the above stress functions definitions, the equilibrium equations Eqs.(60a-c) are satisfied identically. Once the above stress functions are determined, the stress components may also be calculated, and subsequently, Eq.(12) shows that the components of σ_z may be extracted, where ε_z is derived by integrating of Eq.(28a,b).

Using the definitions of Eq.(64) and the differential operators definitions of [1],

$$\nabla_2^{(2)} = b_{44}\frac{\partial^2}{\partial x^2} - 2b_{45}\frac{\partial^2}{\partial x \partial y} + b_{55}\frac{\partial^2}{\partial y^2}, \quad (65a)$$

$$\nabla_1^{(3)} = -a_{24}\frac{\partial^3}{\partial x^3} + (a_{25}+a_{46})\frac{\partial^3}{\partial x^2 \partial y} - (a_{14}+a_{56})\frac{\partial^3}{\partial x \partial y^3} + a_{15}\frac{\partial^3}{\partial y^3}, \quad (65b)$$

$$\nabla_2^{(4)} = b_{22}\frac{\partial^4}{\partial x^4} - 2b_{26}\frac{\partial^4}{\partial x^3 \partial y} + (2b_{12}+b_{66})\frac{\partial^4}{\partial x^2 \partial y^2} - 2b_{16}\frac{\partial^4}{\partial x \partial y^3} + b_{11}\frac{\partial^4}{\partial y^4}, \quad (65c)$$

one may write the integrability conditions Eqs.(42, 43) for $k \le K+1$ in a compact form as

$$\nabla_2^{(4)}\Phi^{(k)} + \nabla_2^{(3)}\psi^{(k)} = g^{(k)}(x,y), \quad (66a)$$
$$\nabla_2^{(3)}\Phi^{(k)} + \nabla_2^{(2)}\psi^{(k)} = f_0^{(k)} + f^{(k)}(x,y), \quad (66b)$$

where for convenience we assume $\Phi^{(k)}(0,0) = \Phi_{,x}^{(k)}(0,0) = \Phi_{,y}^{(k)}(0,0) = \psi^{(k)}(0,0) = 0$. The boundary conditions on $\partial\Omega$ are the following:

$$\frac{d}{ds}\Phi_x^{(k)} = -Y^{(k)} + \bar{U}_2^{(k)}\cos(\bar{n},y), \quad (67)$$
$$\frac{d}{ds}\Phi_y^{(k)} = X^{(k)} - \bar{U}_1^{(k)}\cos(\bar{n},x),$$
$$\frac{d}{ds}\psi^{(k)} = Z^{(k)} - \bar{U}_3^{(k)}\cos(\bar{n},x) - \bar{U}_4^{(k)}\cos(\bar{n},y),$$

where $\bar{n}$ is the outward normal to the contour of the cross-section. The third condition of Eq.(67) may be simplified since in case of simply connected region Ω the potentials $\bar{U}_3^{(k)}, \bar{U}_4^{(k)}$ can be chosen so as to make the expression $\bar{U}_3^{(k)}\cos(\bar{n},x) + \bar{U}_4^{(k)}\cos(\bar{n},y)$ zero on the contour.

7 The Solution Scheme

The stress functions are derived from $K+1$ recursively related boundary value problems (BVPs).

The *scheme of a solution* may be described by the following steps:

1) From initial data we calculate the tip constants $M_i^{(k)}$, $P_i^{(k)}$.

We now execute the $k-$th step using results of the previous $(k+1)-$th step for the following steps:

2) Calculating $\bar{U}_i^{(k)}$, $\Phi^{(k)}$, $\psi^{(k)}$, $f^{(k)}$, $g^{(k)}$,

3) Calculating $A^{(k)}$, $B^{(k)}$, the functions $\Theta_i^{(k)}$, and the functions $\bar{D}^{(k)}$.

4) Calculating the parameters $\kappa_i^{(k)}, \varepsilon_0^{(k)}$.

5) Solving $k-$th BVP and determining the stress functions $\psi^{(k)}, \Phi^{(k)}$, the stresses $\sigma_x^{(k)}, \sigma_y^{(k)}, \sigma_z^{(k)}, \tau_{yz}^{(k)}, \tau_{xz}^{(k)}, \tau_{xy}^{(k)}$ and strains as a linear functions of the parameters $\theta^{(k)}$.

6) Calculating $\theta_0^{(k)}$ from Eq.(3a):

$$M_3^{(k)} = \iint_\Omega (x\tau_{yz}^{(k)} - y\tau_{xz}^{(k)}). \quad (68)$$

If $k > 0$ the procedure returns to step 2) with $k-1$.

7) The *warping functions* U, V, W (when $k = 0$) are determined from the systems (44), (47), (48).

8) Finally the displacement components u, v, w are derived from Eqs.(54a-c).

8 Example: Homogeneous Beam Under Pure Torsion

In this section we present some results (by Maple program) of stress distribution in a rectangular Graphite/Epoxy cross–section under pure torsion (i.e. $M_3 \neq 0$). Fig.1 presents the τ_{xz} shear stress while Fig.2 presents the τ_{yz} component. In this case, the solution of Eqs.(66a,b) is derived in terms of Fourier series expansion in x, y and shows good convergence properties. In simpler cases (e.g. elliptical cross–section), exact polynomial solutions are applied.

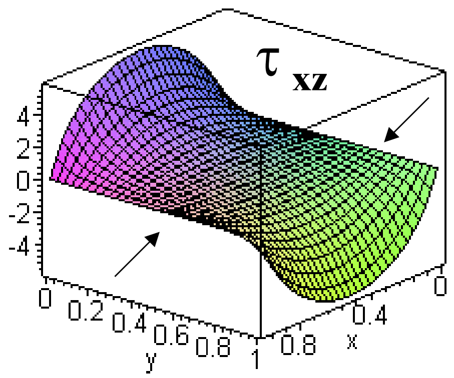

Fig. 1 The stress component τ_{xz}.

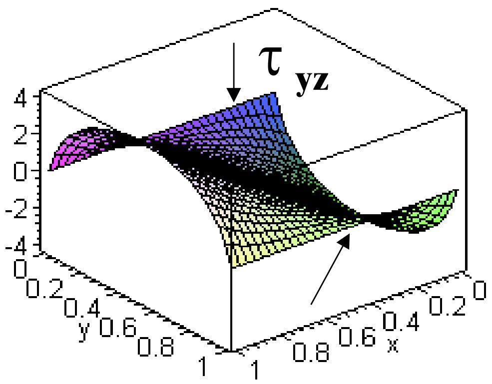

Fig. 2 The stress component τ_{yz}.

In both cases, the arrows show the edges where the boundary condition requires zero stress, namely $\tau_{xz}(0,y) = \tau_{xz}(d,y) = 0$ and $\tau_{yz}(0,x) = \tau_{yz}(x,h) = 0$. To demonstrate the

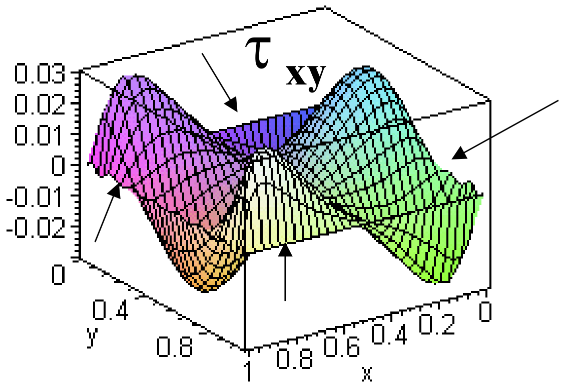

Fig. 3 The stress component τ_{xy}.

method ability to capture fine effects, the stress component τ_{xy} is shown in Fig.3.

9 Concluding

A new analytic formulation and solution for the elastic behavior of generally anisotropic beams has been presented. The present formulation is analytically exact and is capable of capturing fine and detailed effects.

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