STOCHASTIC MODELING OF LAMINAR-TURBULENT TRANSITION

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Abstract

Stochastic versions of stability equations are developed in order to develop integrated models of transition and turbulence and to understand the effects of uncertain initial conditions on disturbance growth. Stochastic forms of the resonant triad equations, a high Reynolds number asymptotic theory, and the parabolized stability equations are developed.

1 Introduction

Integrated modeling of transition and turbulence is critical to accurate CFD predictions for many aerodynamic applications such as high-lift airfoil configurations and the Mars flyer proposed by NASA. Whereas most transition-sensitized turbulence models can predict such flows provided the transition location is specified in advance, no single model can predict this location in all cases.

The practical and theoretical importance of this problem justify an attempt to formulate and solve it from a fundamental theoretical viewpoint. Our approach is to develop stochastic versions of existing deterministic transition models. The resulting statistical transition theories both naturally link transition theory to turbulence modeling and facilitate analysis of the effects of uncertain initial conditions on disturbance evolution.

A statistical transition theory can be based on transition theories with different levels of sophistication and accuracy and with different ranges of applicability. This paper presents stochastic versions of the resonant triad model, a representative high Reynolds number asymptotic theory, and the parabolized stability equations [PSE].

The analysis begins by considering the stochastic aspects of nonlinear disturbance evolution in a laminar boundary layer. There are only a few excited modes; the stochastic aspect of the problem arises from the dependence of the disturbance evolution on randomness in the initial conditions. As an example, the resonant triad model [3] is treated as a problem with random initial conditions in Sects 2–3.

Although a complete statistical solution of this problem can be given by deriving an evolution equation for the joint probability density function of the modal amplitudes [18], the calculations required are excessive. Methods like the 'particle method' [17] offer some advantage, but remain computation intensive. An alternative is offered by the Wiener-Hermite expansion, or 'polynomial chaos' [4]. Wiener-Hermite expansions are applied to random triad evolution in Sect 4. This analysis also offers a closure scheme in problems in which the joint probability density function is determined by an infinite hierarchy of equations (closure problem).

The possibility of treating transition as a 'weak turbulence' [20] of Tollmien-Schlichting waves is raised in Sect 5. Phase incoherence due to multi-mode coupling is proposed as a mechanism to control disturbance amplitude growth. This line of investigation leads to a multi-mode generalization of a typical high Reynolds number asymptotic theory.

A complete statistical transition theory would

provide a single set of equations describing the entire transition process, from initial disturbance growth, through laminar breakdown, to fully developed turbulence. This application requires treating transitional flows, like turbulent flows, as multi-mode nonlinear stochastic systems. The only practical transition model with a sufficiently wide scope for this application is the parabolized stability equations [PSE] [5].

In order to formulate the PSE model statistically, they are first embedded in a compatible turbulence closure. Kraichnan [9] had already proposed that transitional flows be analyzed by the direct interaction approximation [DIA]; this suggestion is adopted here but in the context of a parabolized reformulation of DIA consistent with PSE.

The result is a 'PSE Langevin model' in which the usual PSE equations are supplemented by a generalized damping and a random forcing. This general model is formulated, and it is shown that it could be reduced, for example, to a Smagorinsky-like subgrid model. Thus, even if the Langevin model is too complex for practical use, simplifications can be introduced, but with the advantage that the sources of error are known in advance, and corrections can be based on the more general theory.

2 Resonant triad model

The simplest model of nonlinear disturbance interactions is Craik's resonant triad model [3], which describes nonlinear interactions through a viscous critical layer. A more detailed discussion of this problem appears in our report [18].

The resonant triad model describes the weakly nonlinear evolution of the amplitudes of a resonant triad of Tollmien-Schlichting waves consisting of a two-dimensional primary wave and a pair of oblique subharmonics. Because this system is not conservative, all three modes can grow nonlinearly; the resonance condition promotes especially effective nonlinear transfer to the subharmonics.

Craik's model [3] was modified to allow for the effects of the slow growth of the mean bound-

ary layer and detuning (compare [21]). Defining amplitude-phase variables by $b_i = r_i e^{i\theta_i}$ where $1 \leq i \leq 3$, the amplitude evolution equations depend only on the single phase variable

$$\theta = \theta_1 + \theta_2 - \theta_3 \qquad (1)$$

which satisfies an equation of the form

$$\begin{aligned}
\dot{\theta} = {} & R_{3,12} \sin(\phi_3(x) - \theta) \\
& - R_{2,31} \sin(\theta + \phi_2(x)) \\
& - R_{1,23} \sin(\theta + \phi_1(x)) \qquad (2)
\end{aligned}$$

where the coefficients $R_{i,jk}$ are determined by the theory.

A resonant triad typically evolves as follows. First, the subharmonics grow through parametric excitation by the primary; the amplitudes of the subharmonics tend to equalize and a condition of phase-locking ($\theta \approx 0$) develops. After the onset of phase-locking, all three amplitudes undergo explosive nonlinear growth, leading to a finite-time singularity of the solution [19]. Amplification can occur if the subharmonics are linearly stable and even if the primary is linearly stable [13].

Although these phases of triad evolution are generic, the details of the evolution can depend on the initial conditions, especially during the early to moderate time evolution. To understand this dependence, a weighted trajectory analysis suggested by the particle method [17] of computational combustion theory was applied: a family of triads was computed with uniformly distributed initial phase. The result is a family of trajectories, each of which can be considered equally likely. If desired, an approximate amplitude evolution pdf could be obtained from this information [18].

The results are shown in Fig. 1. The top graph shows the phase evolution as a function of Reynolds number and the bottom graph shows the amplitudes of the primary and subharmonics as functions of Reynolds number for various initial values of θ. The bottom figure can be compared to the amplitude evolution shown in Fig. 2 of Ref. [21].

Two features are noticeable from Fig. 1: first, the point where the subharmonics overtake the primary depends on the initial phase, albeit rather weakly in this case, and second, the location of this point appears closely linked to the development of phase coherence.

The connection just noted between subharmonic amplification and initial phase has been verified experimentally. Fig. 2 shows the experimentally measured subharmonic amplitude (top graph) at a fixed downstream location as a function of the initial phase difference. The bottom graph shows the result of a calculation using the resonant triad model with similar initial conditions.
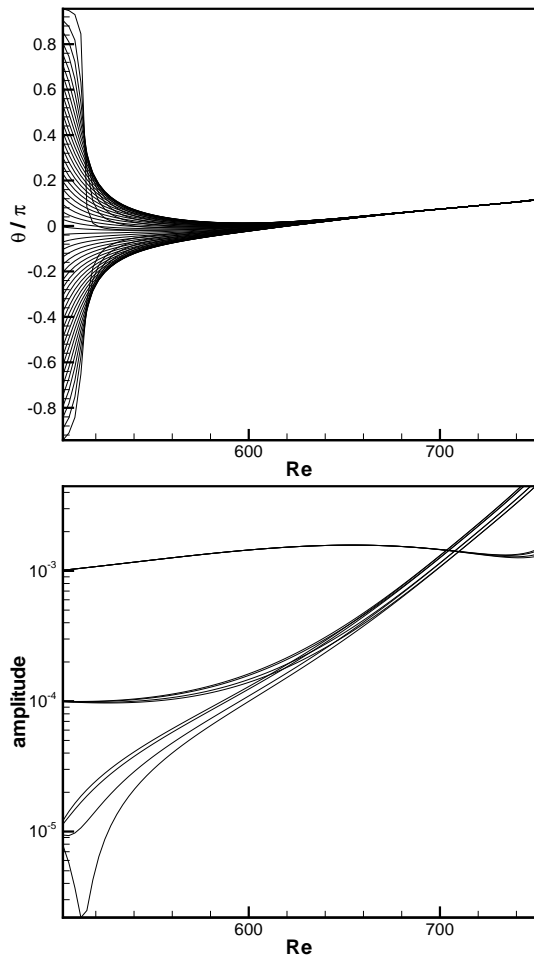

Fig. 1 Phase locking and subharmonic amplitude equalization during triad evolution in the problem analyzed in Ref. [21].

To explore the connection between initial phase and subharmonic amplification in more de-

tail, the initial conditions analyzed in [21] were modified slightly: initial subharmonic amplitudes were Gaussian with mean one-half the initial primary amplitude and standard deviations ten percent of the mean. The result is shown in Fig. 3. The effect of initial phase difference on subharmonic growth is considerably enhanced by this change.
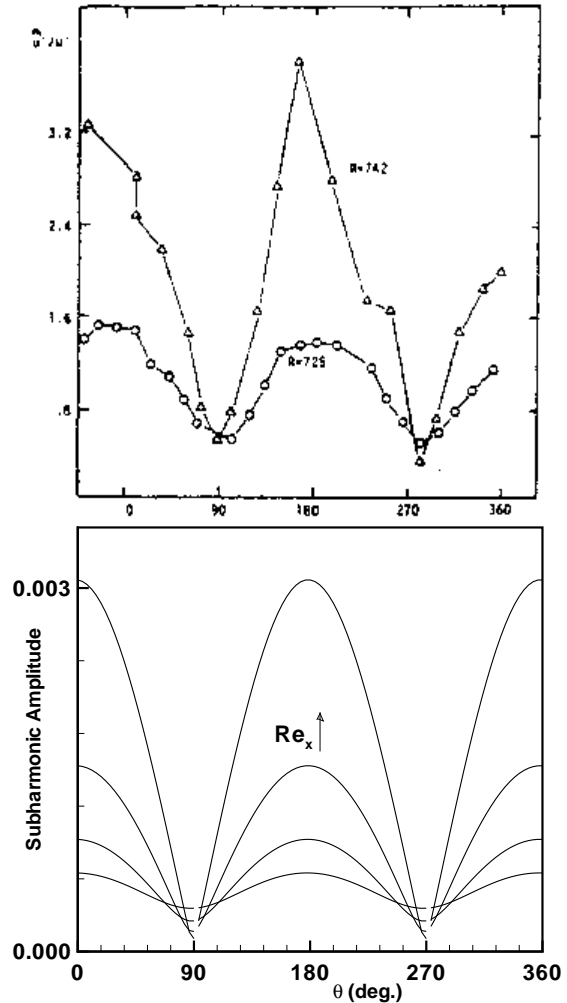

Fig. 2 Effect of initial phase on subharmonic growth: experimental comparison

These results suggest using subharmonic amplification as the basis of a heuristic transition criterion: the transition location is set at the point where the subharmonic amplitudes equal the primary. The results of Fig. 3 correspond to the probability density of transition onset location in Fig. 4. This result is only intended to illustrate how a statistical transition theory might be ap-

plied to predict an uncertain transition point. The crossover location is not the same as the transition onset location, which could not be predicted by the resonant triad interaction model in any case; however, the amplitude crossover does harbinger the onset of stronger nonlinear interactions and hence the approach of transition.
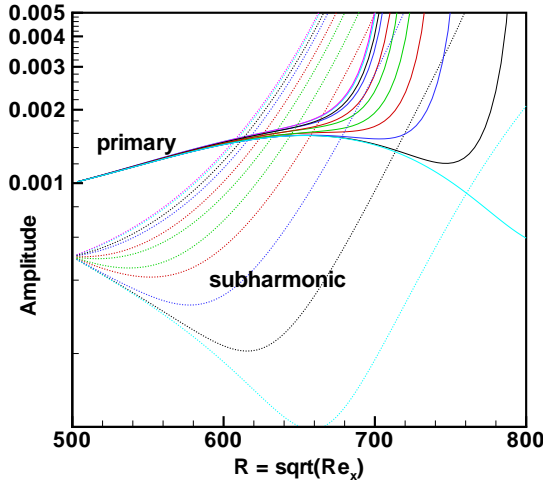

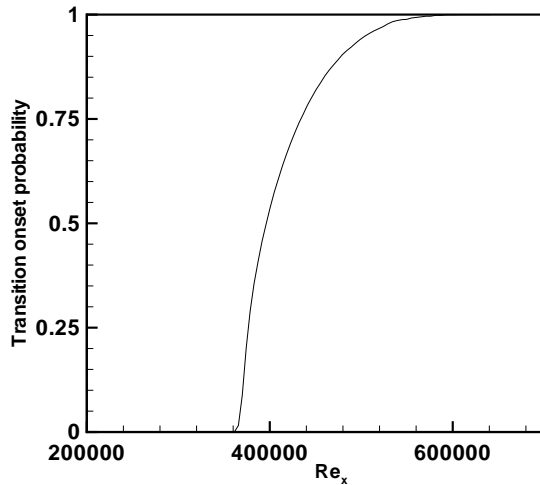**Fig. 3** Effect of initial phase on triad evolution.

Fig. 4 Probability density of transition onset location

3 Phase correlations and amplitude growth

To investigate the connection between phase coherence and subharmonic growth more closely, the computation was repeated with the phase evolution in Eq. (2) modified by adding white noise of various amplitudes to its right hand side. Specifically, Eq. (2) was replaced by

$$\dot{\theta} = R_{3,12}\sin(\phi_3(x) - \theta) \\ -R_{2,31}\sin(\theta + \phi_2(x)) \\ -R_{1,23}\sin(\theta + \phi_1(x)) + Aw(x) \quad (3)$$

where $w(x)$ is a white noise process and the amplitude A varied between 0.010 and 0.200.

At the smallest noise amplitude $A = 0.010$, the phase locking is only weakly perturbed: the phase fluctuates by about 0.1π about the deterministic value. Corresponding to this weak disturbance of the growth of phase correlations is a very weak perturbation of the nonlinear amplitude growth (not shown). Fig. 5 shows the amplitude evolution corresponding to the highest noise level, $A = 0.200$. In this case, the phase evolution is almost entirely dominated by the random forcing. Correspondingly, subharmonic growth is almost entirely suppressed. At an intermediate level of phase randomization, controlled growth can occur.

It is crucial that in Fig. 5, the suppression of energy growth in the subharmonics has been accomplished entirely through phase randomization; energy is not removed from the disturbances directly. The calculation demonstrates that the phase-locked attractor with explosive subharmonic growth can be destroyed by stochasticity.

Although the triad theory provides a plausible explanation for the development of three-dimensional disturbances in transition, the prediction of a finite-time singularity obviously limits its applicability to relatively early phases of disturbance growth. A multi-mode triad theory [21] would be a link between transition and turbulence if phase randomization could be shown to occur as a consequence of multi-mode coupling. That is, as one set of subharmonics grows, parametric excitation of higher-order subharmonics might occur by the same mechanism. Could nonlinear coupling of all of these modes induce phase randomization? If so, an ensemble of interacting Tollmien-Schlichting waves could provide a transition scenario with realistic amplitude growth instead of a finite-time singularity.
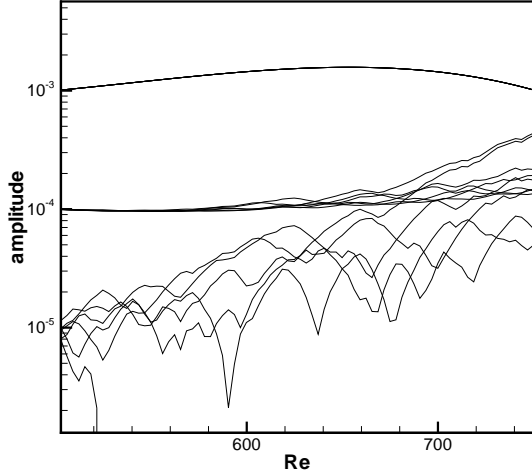
Fig. 5 Effect of random noise perturbation of phase evolution equation on subharmonic amplification during resonant triad evolution.

This scenario would amount to a 'weak turbulence' [20] description of transitional flow. In this hypothetical theory, the modal frequencies would be determined by the dispersion relation of a linear theory, the weakly non-parallel Orr-Sommerfeld equation in this case, but weak nonlinear interactions would be allowed and would even be crucial to a realistic of disturbance amplitude growth through phase randomization.

Unfortunately, this picture encounters difficulties. Consideration of nonlinear critical layers suggests that the nonlinear coupling among resonant triads is itself highly constrained (X Wu, private communication) and would not lead directly to multi-mode coupling. Moreover, the idea of weak turbulence of linearly non-neutral modes may be questionable (P Hall, private communication).

4 Critical layer theories

The algebraic nonlinearity in Craik's triad equations does not completely represent the disturbance dynamics within the critical layer, where nonlinear effects predominate. More careful consideration of critical layer dynamics lead to integro-differential equations for disturbance growth: an example is the theory of Mankbadi et al [12]. This theory offers a deterministic

mechanism by which subharmonic growth is reduced due to oscillations of the subharmonic amplitudes. An *ad hoc* generalization of this theory to multiple interacting triads was formulated:

$$\dot{A}_i^0 = \sigma_i^0(x)A_i^0(x) + S_i^{0\pm}(x)A_i^+(x)A_i^-(x)$$
$$+i\lambda A_i^0(x)\int_0^x dy \sum_p \left\{ |A_p^0(y)|^2 \right.$$
$$+|A_p^+(y)|^2 + |A_p^-(y)|^2 \right\}$$
$$\dot{A}_i^+ = \sigma_i^+(x)A_i^+(x) + S_i^\pm(x)A_i^0(x)A_i^-(x)^*$$
$$+i\lambda A_i^+(x)\int_0^x dy \sum_p \left\{ |A_p^0(y)|^2 \right.$$
$$+|A_p^+(y)|^2 + |A_p^-(y)|^2 \right\}$$
$$\dot{A}_i^- = \sigma_i^-(x)A_i^-(x) + S_i^\mp(x)A_i^0(x)A_i^+(x)^*$$
$$+i\lambda A_i^-(x)\int_0^x dy \sum_p \left\{ |A_p^0(y)|^2 \right.$$
$$+|A_p^+(y)|^2 + |A_p^-(y)|^2 \right\} \qquad (4)$$

A ten-mode system was solved with initial conditions with random initial phases. The results are shown in Fig. 6 The amplitudes are normalized by the same initial amplitude, so that the effect of initial randomization can be seen in the graphs. The results are not much different than when the initial amplitudes are deterministic. The mode amplitudes appear to evolve roughly independently.

There appears to be no tendency to phase randomization due to multi-mode coupling in this system. Preliminary consideration of multimode coupling through cubic mean flow interactions (X Wu, private communication) did not appear to lead to phase randomization, but this possibility may deserve further investigation.

5 Initial condition dependence modeling by polynomial chaos

A complete statistical description of triad evolution is given by the joint probability density function [pdf] of the complex modal amplitudes. Because triad evolution is described by ordinary differential equations, derivation of the evolution equation of this joint pdf is straightforward. The large dimensionality of the resulting equation (a
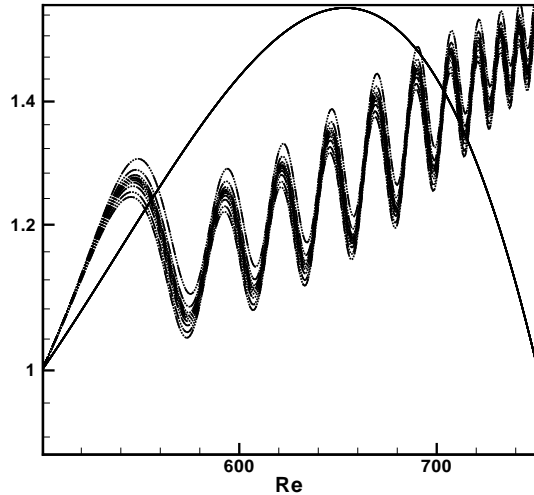
Fig. 6 Primary (solid) and subharmonic (dashed) amplitude evolution in ten-mode system with random initial phases.

triad is determined by three complex amplitudes, hence the joint pdf evolves in a 6-dimensional space) makes numerical solution less straightforward.

Although the computational difficulties can be mitigated by alternate approaches like the 'particle method' [17], even the possibility of deriving a closed pdf evolution equation is lost if the modal evolution equation is history-dependent, as in the high Reynolds number asymptotic theories, or if higher order spatial derivatives occur, as in the PSE or higher order triad theories [21]. The coupling of information from different points immediately generates a closure problem, since the single-point pdf depends on the two-point pdf, and so on.

The Wiener-Hermite expansion has recently been revisited, generalized, and re-named 'polynomial chaos' [4]. If a nonlinear problem can be characterized by a finite number of stochastic variables, then Wiener-Hermite expansion yields a systematic convergent expansion of the solution in polynomials in the stochastic variables. The lowest order response is purely Gaussian, but the higher order terms give non-Gaussian corrections.

This is an attractive approach to characterizing initial-condition sensitivity of transition,

since the initial conditions can be chosen as the stochastic variables. Moreover, truncating a Wiener-Hermite expansion also leads to a closure for problems which generate infinite unclosed hierarchies.

However, an important criticism of truncated Wiener-Hermite expansions for multi-mode nonlinear stochastic systems was made by Orszag [15], who observed that in truncated Wiener-Hermite expansions, nonlinearity implies non-Gaussianity, because only in a linear system does the expansion terminate with the first term. But the inviscid thermal equilibrium ensembles of statistical turbulence theory are both nonlinear and Gaussian. The truncated Wiener-Hermite expansion is inconsistent with this possibility. This inconsistency was confirmed by showing that the Wiener-Hermite expansion for a simple problem of conservative three-wave interaction truncated at lowest nontrivial order cannot maintain the statistics of the thermal equilibrium solution even if they are given as initial conditions.

This result raises the question whether higher-order truncation might lead to more satisfactory results. A calculation using ninth order Wiener-Hermite expansion is shown in Fig. 7: the solution maintains the thermal equilibrium solution much longer than the lowest order truncation, but ultimately exhibits windows of non-convergence. Despite these difficulties, we believe that polynomial chaos could be helpful in understanding the initial condition sensitivity of transitional flows and are currently applying this approach to study the problem of the evolution of Görtler vortices.

6   Parabolized stability equations

Parabolized stability equations [PSE] ([5], [2]) are a general, heuristic approach to transition calculations. It is perhaps the most successful current theory, since it has been used to predict the sudden increase in skin friction following transition onset [2]. Nevertheless, the success is only partial, since the calculations cannot be continued into the fully turbulent region. The difficulty could originate either in the rapid broadening of
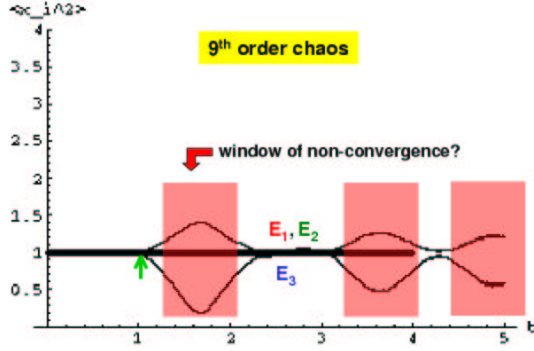
Fig. 7 Three mode system solved using ninth order polynomial chaos.

the spectrum or in limitations of the parabolizing assumptions.

The problem of spectral broadening can be attacked by developing PSE 'subgrid' models. A conventional Smagorinksy model, modified so that the eddy viscosity approaches zero in the laminar region, could be applied.

This section will outline a more theoretical approach based on embedding PSE in a two-point turbulence closure. Such a marriage of a two-point closure with a transition theory is natural because like transition theories, two-point closures are formulated in terms of modal amplitudes. Moreover, two-point closures can be formulated for arbitrary separation of large-scale and fluctuating motion, not just for Reynolds averaging. This means that subgrid models and 'phase averaged' models are derived without any change of method.

Two-point turbulence closures make no assumptions about Reynolds number, the form of correlation functions, stationarity, homogeneity, isotropy, or wall proximity. In particular, the correlation function can be Kolmogorov, self-similar, or not as the dynamics requires. This feature is critical for computing transitional flows, in which the stochastic aspects can be inhomogeneous and non-universal.

The closure equations include both *mean flow-wave* interactions and *wave-wave* interactions. Therefore, these models can reduce to quasi-linear theories [11], [14] if the dynamics requires it. But since nonlinearity is not enitrely suppressed, closure models admit the *stochas-*

tic relaxation which is absent from quasi-linear models like [11] and [14] and which can consequently be overly sensitive to initial conditions [16].

The derivation consists of the following steps:

(1) We apply the direct interaction approximation [DIA] closure [8]. DIA can be understood as a generalized stress transport model for the correlation function (which integrates to the Reynolds stress) and the modal time-scales. Salient points about this closure are: first, unlike the ε equation, the time scale (response) equation is rationally formulated and second, both linear and nonlinear time-scales are possible. The dynamics decides between them, not the modeler.

(2) The physical-space formulation of DIA [9] is combined with the spectral formulation [10] in a form appropriate to the semi-spectral formulation of PSE.

(3) Parabolizing approximations like those of PSE are introduced so that the downstream evolution variable becomes causal. The result is a semi-parabolic model in which downstream history integrals appear.

(4) The downstream evolution is Markovianized, again consistent with PSE. The result is a fully parabolic model with respect to downstream evolution. The projection-independence of the formulation is used to introduced the parabolized turbulence closure as a model for unresolved PSE modes.

For applications to two-dimensional mean flow, the fluctuations are considered stationary in time and spanwise homogeneous; accordingly the partial Fourier representation

$$u_m(\mathbf{x},t) = \int d\zeta d\omega\, u_m(x,y,\zeta,\omega)e^{i(\zeta z-\omega t)} \quad (5)$$

is appropriate.

Eliminate the pressure using the solenoidal projection operator

$$\mathsf{P} \cdot \mathbf{u} = \nabla^{-2}\nabla \times \nabla \times \mathbf{u} \quad (6)$$

so that in index notation $(\mathsf{P} \cdot \mathbf{u})_i = P_{ij}u_j$ and define

$$P_{imn}(\nabla) = \nabla_m P_{in}(\nabla_\perp, i\zeta) + \nabla_n P_{im}(\nabla_\perp, i\zeta) \quad (7)$$

Define $U_{mn}(\mathbf{x},t;\mathbf{x}',t') = \langle u_m(\mathbf{x},t)u_n(\mathbf{x}',t')\rangle$. The result of applying steps $(1) - (4)$ above is the PSE Langevin model

$$-i\omega u_i(x,y,\zeta,\omega) + P_{in}(\partial/\partial y, i\zeta) \times$$
$$\left\{ U_x(x,y)\frac{\partial u_n}{\partial x}(x,y,\zeta,\omega) + U_y(x,y)\frac{\partial u_n}{\partial y} \right.$$
$$\left. +\frac{\partial U_n}{\partial x}(x,y)u_x(x,y,\zeta,\omega) + \frac{\partial U_n}{\partial y}(x,y)u_y \right\}$$
$$+P_{in}(\partial/\partial y, i\zeta)i\zeta_m U_m(x,y)u_n(x,y,\zeta,\omega)$$
$$= -P_{imn}(\partial/\partial y, i\zeta)\int d\zeta' d\omega' \times$$
$$u_m(x,y,\zeta-\zeta',\omega-\omega')u_n(x,y,\zeta',\omega')$$
$$+\nu(\partial^2/\partial y^2 - \zeta^2)u_i(x,y,\zeta,\omega)$$
$$-P_{ir}(\partial/\partial y, i\zeta)\exp\left[i\int_0^x dx'\, \alpha(x',y,\zeta,\omega)\right] \times$$
$$u_r(x,y,\zeta,\omega)$$
$$-\frac{1}{2}\int dy''\eta_{ir}(x,y,y'',\zeta,\omega)u_r(x,y'',\zeta,\omega)$$
$$+f_i(x,y,\zeta,\omega) \tag{8}$$

These are simply the PSE equations with two added terms: the damping factor in η, and the random force f. The damping factor acts in this model as a subgrid viscosity. It is defined by

$$\eta_{is}(x,y,\zeta,\omega) = P_{imn}(\partial/\partial y, i\zeta)\int d\zeta' d\omega' \times$$
$$P_{mrs}(\partial/\partial y, i\zeta - i\zeta') \times$$
$$\theta(x,y,\zeta,\zeta-\zeta',\zeta',\omega,\omega-\omega',\omega') \times$$
$$U_{nr}(x,y,\zeta',\omega') \tag{9}$$

where the time-scale θ models the downstream history dependence, somewhat like the quantity α in PSE. It will be defined later.

The random force is defined by

$$f_i(x,y,\zeta,\omega) = -iP_{imn}(\partial/\partial y, i\zeta)w(x) \times$$
$$\int d\zeta' d\omega' \int dy'' \times$$
$$\theta^{1/2}(x,y,y'',\zeta,\zeta-\zeta',\zeta',\omega,\omega-\omega',\omega') \times$$
$$\xi_m(x,y,y'',\zeta-\zeta',\omega-\omega') \times$$
$$\xi_n(x,y,y'',\zeta',\omega') \tag{10}$$

where ξ is a random Gaussian field with the same correlation function as the velocity. In effect, the

ξ are the velocity field but with random phases. In Eq. (10), $w(x)$ denotes spatial white noise with unit variance.

The correlation function U satisfies the evolution equation,

$$-i\omega U_{ij}(x,y,y',\zeta,\omega) + P_{in}(\partial/\partial y, i\zeta) \times$$
$$\left\{ U_x(x,y)\frac{\partial U_{nj}}{\partial x}(x,y,y',\zeta,\omega) + U_y(x,y)\frac{\partial U_{nj}}{\partial y} \right.$$
$$\left. \frac{\partial U_n}{\partial x}(x,y)U_{xj}(x,y,y',\zeta,\omega) + \frac{\partial U_n}{\partial y}(x,y)U_{yj} \right\}$$
$$+P_{in}(\partial/\partial y, i\zeta)i\zeta_m U_m(x,y)U_{nj}(x,y,y',\zeta,\omega)$$
$$= \nu(\partial^2/\partial y^2 - \zeta^2)U_{ij}(x,y,y',\zeta,\omega)$$
$$-\frac{1}{2}\int dy''\eta_{ir}(x,y,y'',\zeta,\omega)U_{rj}(x,y'',y',\zeta,\omega)$$
$$+P_{i'mn}(\partial/\partial y, i\zeta)P_{jrs}(\partial/\partial y, i\zeta)\int d\zeta' d\omega' dy''$$
$$\theta_{ii'rr'ss'}(x,y,y'',\zeta,\zeta-\zeta',\zeta',\omega,\omega-\omega',\omega')$$
$$U_{mr'}(x,y',y'',\zeta-\zeta',\omega-\omega') \times$$
$$U_{ns'}(x,y',y'',\zeta',\omega') \tag{11}$$

The simplified θ equation has the general structure

$$U_x(x,y)\frac{\partial\theta}{\partial x}(x,y) + U_y(x,y)\frac{\partial\theta}{\partial y}(x,y) + S\theta$$
$$= \nu(\partial^2/\partial y^2)\theta(x,y)$$
$$-\frac{1}{2}\int d\zeta d\omega\eta(x,y,\zeta,\omega)\theta(x,y,\zeta,\omega) \tag{12}$$

where S is the total strain rate.

This model, which combines the effects of large-scale motion through PSE with energy drain and decorrelation due to small-scale motion perhaps realizes the statement of Bayly, Orszag and Herbert [1] that 'The most promising tools at the moment are some kind of decomposition of the flow into coherent structures plus pseudo-stochastic noise in such a way as to merge purely deterministic instability theory with statistical turbulence theory in a consistent and sensible way.'

In order to make a connection with standard LES modeling, it is convenient to integrate the modal evolution equation Eq. (8) with respect to ζ and ω: since the two-point dependence in y is ignored, this integration reduces to the case

of single-point quantities considered in standard LES.

To begin, ignoring wall effects, set

$$P_{imn}(\partial/\partial y, i\zeta) \approx \partial/\partial y \qquad (13)$$

Then approximating

$$\int d\zeta' d\omega' \theta(x, y, \zeta, \zeta - \zeta', \zeta', \omega, \omega - \omega', \omega') \times$$
$$U_{nr}(x, y, \zeta', \omega')$$
$$\approx \theta K \qquad (14)$$

the integrated subgrid term ηu can be approximated as

$$\int_{\zeta \leq \zeta_c, \omega \leq \omega_c} d\zeta d\omega \eta_{ir}(x, y, \zeta, \omega) u_r(x, y, \zeta, \omega)$$
$$\approx \frac{\partial}{\partial y} K\theta \frac{\partial}{\partial y} u_i(x, y) \qquad (15)$$

where K is the subgrid kinetic energy and θ is a time scale. Since this is an eddy viscosity representation of the subgrid stress in terms of the resolved velocity field, this is a Smagorinsky-type subgrid eddy viscosity model with eddy viscosity $\nu_t = K\theta$ in which transport equations are provided for the quantities K and θ.

Standard LES models have been applied successfully to transitional flows, despite the lack of any apparent justification for such success. It is interesting that the present calculation suggests the somewhat novel approach of applying standard subgrid models to the PSE rather than to the Navier-Stokes equations. The application is non-standard, because PSE solves for wave amplitudes rather than the velocity field itself.

The advantage of the present derivation is that having a more comprehensive theory, it will be possible to address shortcomings of these simple models in a systematic fashion.

These transport equations can also be approximated to reveal their connection with common heuristics. Namely, simplifying Eq. (11) for U_{ij}, the correlation function for subgrid modes, we can write

$$U_x(x, y) \frac{\partial K}{\partial x}(x, y) + U_y(x, y) \frac{\partial K}{\partial y}(x, y)$$

$$+ \frac{\partial U_n}{\partial x}(x, y) \langle u_x(x, y) u_j(x, y) \rangle$$
$$+ \frac{\partial U_n}{\partial y}(x, y) \langle u_y(x, y) u_j(x, y) \rangle$$
$$= \nu(\partial^2/\partial y^2) K(x, y)$$
$$- \frac{1}{2} \int d\zeta d\omega \eta(x, y, \zeta, \omega) U(x, y, \zeta, \omega)$$
$$+ \frac{1}{2} \int d\zeta d\omega F(x, y, \zeta, \omega) \qquad (16)$$

On the left side are the standard convection and mean-flow production terms. The first term on the right side is the viscous dissipation. For the interpretation of the following terms, we refer to the usual understanding of the DIA closure, in which the term in η represents a loss of energy and the term in F represents a gain. In fact, these terms are the energy transfer to small scales ε and the production of subgrid motion by the resolved scales.

The simplified θ equation should be understood as a substitute for the dissipation rate equation.

7   Conclusions

Stochastic versions of transition models have been applied to the problems of initial condition dependence in disturbance growth and the integration of transition and turbulence modeling.

The possibility of using uncertainty in initial conditions to predict a probability density of transition location was illustrated using the resonant triad theory. The application of Wiener-Hermite expansion methods to transition has been discussed. Although fundamental objections to truncated Wiener-Hermite expansions as descriptions of multi-mode nonlinear systems exist, their application to transition under uncertain initial conditions may be sound and warrants further investigation.

Several models leading to integrated modeling of transition and turbulence have been investigated: the resonant triad model with random forcing of the phase equation, a multi-mode critical layer theory, and a stochastic form of the parabolized stability equations. The randomly

forced resonant triad model was suggested as a motivation for a description of transition as a weak turbulence of Tollmien-Schlichting waves. Although attractive in some respects, further development of this picture appears problematical.

A stochastic form of the PSE, a PSE Langevin model with random forcing and generalized damping determined by turbulence closure, appears to be a promising approach to integrated turbulence-transition modeling. Reduction of this model to a Smagorinsky model under strong simplifying assumptions has been demonstrated.

8 Acknowledgments

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