NONLINEAR ELASTIC CONSTRAINED MULTIBODY SYSTEMS – AN ADAPTIVE TIME–STEPPING ALGORITHM BASED ON THE TIME–DISCONTINUOUS GALERKIN SCHEME

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Abstract

This work is related to the time integration of the equations of motion of nonlinear constrained multibody systems discretized using the finite element method. The resulting system of nonlinear differential-algebraic equations is prone to high frequency oscillations and instabilities of a purely numerical origin. A time-discontinuous Galerkin scheme is used to obtain the transient responses of the system. The scheme is cast in a predictor-multicorrector form, which provides unconditional stability, third order accuracy and high frequency numerical dissipation. The total energy of the system, which monitors the stability characteristics of the response, is used as error measure and a time step adaptation algorithm is employed. The scheme is not sensitive to configuration singularities that may occur during the motion of the structural system. Some of the results presented, compared to their analytical counterparts, showed excellent agreement. Kinematic constraints are satisfied at the order of machine accuracy. The algorithm may be fully implemented on existing structural dynamic computer codes requiring minor modifications on the routines. Already implemented nonlinear structural finite element libraries do not need to be rewritten. However, the finite element formulation of the mechanical joints has to be done accordingly and then included into the code's library of elements.

1 Introduction

This work is related to the use of numerical algorithms employed on the design and analysis of complex structural systems that can be modelled by the finite element method. More specifically, it addresses the study of direct time integration schemes used on the solution of the equations of motion of multibody systems. In today's industrial reality great emphasis has been placed on designing high-speed, lightweight precision systems which turns evident the extreme importance of the coupling between rigid and nonlinear elastic displacements as well as the vibrations of each structural member of the system. These are very important details that have to be taken into account when developing an algorithm to solve the equations of motion of such systems.

Multibody systems usually present general and complex topologies. In order to accurately model such systems one has to carefully address factors such as the type of coordinate systems to be used; the formulation of elastic members; the

modelling of kinematic constraints, which represent the joints used to connect each body to one another; and the parametrization of finite rotations, since large displacements and large rotations will develop during the motion of the structure. Thus, different methodologies result from the way such factors are addressed.

This work concerns mainly the development of numerical algorithms to be used in general purpose computer codes. It aims the modelling and analysis of nonlinear constrained multibody systems discretized using the finite element method. Due to its generality, low degree of nonlinearity and independence on the topology of the system, the formulation employs Cartesian coordinates to represent the position of each body with respect to an inertial frame. Clearly, the proper parametrization of the finite rotations is fundamental to this approach. An augmented Lagrange multiplier technique is used to enforce the kinematic constraints among the various bodies. Although this approach does not involve the minimum set of coordinates [1], it allows a modular development of finite elements to represent a variety of kinematic constraints, so that general multibody configurations can effectively be modeled. The resulting nonlinear differential-algebraic systems of equations are stiff due to the presence of high frequencies associated with the nonlinear motion of the elastic members. Such a condition is worsen by the fact that kinematic constraints introduce "infinite frequencies" into the system, since no mass is associated with the Lagrange multipliers resulting in algebraic equations coupled to the differential equations of the system. These high frequency contents result in numerical oscillations and instabilities that may forbid the time integration procedure when a direct time integration scheme is used to solve the equations of motion of the structural system. Geradin [2] shows the destabilizing effect of the kinematic constraints on the time integration. Finite difference techniques are extremely popular and the well known Newmark method [3] is probably the most widely used time integration scheme for structural dynamic problems. It is known that its weak instability [2, 4] makes the Newmark scheme not good a choice for time integrating the equations of motion of constrained multibody systems. In order to circumvent such instability problems it is common practice to rely on artificially added *numerical dissipation* [5]. Although the added numerical dissipation minimizes the high frequency related numerical oscillations it also causes a reduction in the scheme's accuracy. Some direct time integration schemes present such large an amount of numerical dissipation that even real physical instabilities may not be detected during the time integration process.

The study herein presented describes the use of a time integration formulation that generates time integration schemes that intrinsically present numerical dissipation, *i.e.*, there is no need to artificially add numerical damping in order to control numerical instabilities. Moreover, the resulting scheme is unconditionally stable, presents third order accuracy and very simple modifications must be performed on existing software architecture in order to have the algorithm fully implemented.

2 Direct Time Integration on Nonlinear Structural Dynamics

In structural dynamic analyses of nonlinear constrained multibodies, the problem of solving for the transient response of the systems deals with matrix dynamic equilibrium equations written in the form

$$M\underline{\ddot{u}}(x,t) + N[\underline{u}(x,t), \underline{\dot{u}}(x,t)] \;=\; P[\underline{u}(x,t), t] \tag{1}$$

associated with kinematic constraints

$$Q[\underline{u}(x,t)] \;=\; 0\,, \tag{2}$$

and initial conditions $\underline{u}(x,t_0) = \underline{u}_0$ and $\underline{\dot{u}}(x,t_0)] = \underline{v}_0$. M is the consistent finite element integrated mass matrix, N is the nonlinear internally resisting forces in the structure, which may depend on displacements and velocities, and P is the externally applied forces that vary in time, generally, but which may also depend on the displacements.

The quantities $\underline{u}(x,t)$, $\dot{\underline{u}}(x,t)]$ and $\ddot{\underline{u}}(x,t)]$ represent the displacements, velocities, and accelerations vectors, respectively. The mechanical joints that connect each body to one another are represented by constraints, as in Eq. (2), where Q is a set of holonomic, or more specifically, scleronomic or stationary, equality constraints.

A Newton-type iteration, standard or modified, along with a time integration scheme based on a time-discontinuous Galerkin formulation, is used on the solution of the final system of equations associated with Eqs. (1,2). The scheme is unconditionally stable, third order accurate, and inherently presents high frequency numerical dissipation.

3 A Time-Discontinuous Galerkin Integration Scheme

Using the natural framework of the second-order hyperbolic equations of elastodynamics, instead of relying on converting the equations to a first-order symmetric hyperbolic form, which is actually not always possible, Hulbert [8] developed time and space-time discontinuous Galerkin finite element methods to solve the equations of motion associated with structural dynamics and elastodynamics problems.

Hulbert [8] has developed schemes for single- and two-field formulations. A single-field formulation that uses quadratic functions to interpolate the displacements in time generates a time integration scheme which is third order accurate, unconditionally stable, but do not present high frequency numerical dissipation. However, as previously explained, the resulting system of equations will be solved in a *predictor-multicorrector* form, which then generates a scheme presenting high frequency numerical damping. Therefore, with the use of a single-field quadratic formulation, along with a predictor-multicorrector form, a scheme with excellent algorithmic damping and frequency error characteristics is obtained. The presence of high frequency numerical dissipation is inherent to the formulation, and the resulting systems of equations are smaller than the ones obtained with a two-field formulation.

The statement [8] of the time-discontinuous Galerkin finite element method for the single-field formulation, applied to the ordinary differential equations associated with the semidiscrete form of linear elastodynamics is:

Find $\underline{u}^h \in S^h$ such that for all $\underline{w}^h \in W^h$

$$\begin{aligned}\int_{t_{n-1}^+}^{t_n^-} [\dot{\underline{w}}^h \cdot (M\ddot{\underline{u}}^h + C\dot{\underline{u}}^h + K\underline{u}^h - \underline{F})]dt \quad + \\ \dot{\underline{w}}^h(t_{n-1}^+) \cdot M[\dot{\underline{u}}^h(t_{n-1}^+) - \dot{\underline{u}}^h(t_{n-1}^-)] \quad + \quad (3) \\ \underline{w}^h(t_{n-1}^+) \cdot K[\underline{u}^h(t_{n-1}^+) - \underline{u}^h(t_{n-1}^-)] \quad = \quad 0.\end{aligned}$$

In Eq. (3), $n = 1, 2, \cdots, N$, where N is the number of time intervals. Variables $\underline{u}^h$ and $\underline{w}^h$ are, respectively, displacements and weighting functions, $\dot{\underline{u}}^h$ and $\ddot{\underline{u}}^h$ are, respectively, velocities and accelerations. The last two terms on the left-hand side weakly enforce the initial conditions for each time interval. These jump terms are stabilizing operators that have the effect of up-winding information with respect to time [8]. Also, M, C and K are the mass, damping and stiffness matrices, respectively, and F is the force vector. Since the displacements are interpolated as quadratic functions, the resulting system of equations is three times larger than the ones solved by commonly used semidiscrete methods.

4 The Adaptive Time-Stepping Procedure

The response of constrained multibody systems often rapidly varies in time, indicating the need for an automated time step size adaptation procedure. Moreover, in modern structural dynamic analysis, in general, it is convenient that a time integration scheme allows automatic time step size control.

An adaptive time-stepping procedure, based on a time-discontinuous Galerkin scheme, for selecting the proper time step size is presented in [9], which uses a two-field formulation, namely the P1-P1 formulation [8], to interpolate displacements and velocities as piecewise linear functions.

In the study herein developed a single-field formulation is used with displacements approxi-

mated as a quadratic function. The resulting systems of equations are smaller than the ones resulting from a two-field formulation with the advantage of improved accuracy characteristics [8].

The time adaptive algorithm presented in [9] is herein applied to constrained nonlinear multibody systems. For the automatic time step size control the relative error at a time t_n is defined as

$$\varepsilon_n = \left| \frac{E(t_n) - E(\bar{t})}{E(\bar{t})} \right| \tag{4}$$

where E is the total energy, $E(\cdot) = K(\cdot) + V(\cdot)$, *i.e.*, the sum of kinetic and potential energies, and $E(\bar{t})$ is a reference energy of the system. It is expected that the relative errors satisfy the condition

$$\varepsilon_n \leq \varepsilon^{tol} \quad , \tag{5}$$

where ε^{tol} is a specified error tolerance. If requirement in Eq. (5) is not satisfied, a time step refinement is performed. The corresponding solution is rejected and given that the convergence rate for the algorithm is $O(\Delta t^3)$, the new time step size that will satisfy the error tolerance criterion is calculated, [9, 10], as

$$\Delta t_n^{tol} = \left(\frac{\theta_t \, \varepsilon^{tol}}{\varepsilon_n} \right)^{1/3} \Delta t_n \quad , \tag{6}$$

where $\theta_t \leq 1.0$ is a reducing factor used to avoid the new predicted time step size being rejected. On the contrary, if the calculated error is much smaller than the tolerance ε^{tol}, *i.e.*,

$$\varepsilon_n < \gamma \, \varepsilon^{tol} \tag{7}$$

the solution is accepted but the time step size may be increased according to Eq. (6) when the criterion in Eq. (7) is satisfied for a certain successive number of time steps. In Eq. (7) γ is a number much smaller than 1.0.

5 Numerical Examples

5.1 Simple Pendulum

Consider a simple pendulum problem modeled as a point mass m and a massless rigid link of length ℓ. Two degrees of freedom u_x and u_y, respectively, vertical and horizontal displacements, describe the position of the mass m. Only gravity acts upon the system. The constraint $Q = u_x^2 + u_y^2 - \ell^2 = 0$, that guarantees the constant length ℓ of the pendulum is added to the system through an augmented Lagrange multiplier technique. Such a constraint introduces an infinite frequency into the system of equations, which then becomes prone to numerical instabilities and oscillations of a purely numerical origin. Cardona & Geradin [4] have shown the impossibility of solving this problem with the Newmark time integration scheme. Nevertheless, the time-discontinuous Galerkin scheme efficiently solves this nonlinear constrained problem as shown by Damilano [6].

Despite the reduced number of degrees of freedom and its rigid body nature, this problem is studied aiming the possibility of applying the technique herein described to more representative multibody systems, e.g., problems with large number of degrees of freedom, several kinematic constraints and nonlinear elastic members. For the present study $m = 1.0$ kg, $\ell = 0.5$ m, $g = 9.81$ m/s^2, and the initial conditions are $u_x = 0.5$ m, $u_y = \dot{u}_x = 0.0$, $\dot{u}_y = -1.695$ m/s. The solution is calculated for 50 seconds and the numerical results are in excellent agreement with their analytical counterparts. A conventional analysis of the scheme based on the characteristics of the amplification matrix [7], for linear systems, results the period elongation $\Delta T/T = \omega^4 \Delta t^4/270 + O(\omega^6 \Delta t^6)$, and the algorithmic damping $\zeta = \omega^3 \Delta t^3/72 + O(\omega^5 \Delta t^5)$, where $\omega^2 = k/m$. Thus, the scheme is third-order accurate. However, there is no guarantee the same accuracy will be observed with nonlinear constrained systems. To assess the order of accuracy of the scheme applied to the nonlinear pendulum, the results at time $t = 10$ seconds were used to calculate the errors of the solution as functions of the time step size. The results indicated that, in fact, the scheme is third-order accurate. The time adaptive algorithm predicts the solution for two different error tolerances, $1.0\text{x}10^{-05}$ and $1.0\text{x}10^{-04}$, with $\theta_t = 0.95$ and $\gamma = 0.6$ for both cases. Figure (1)

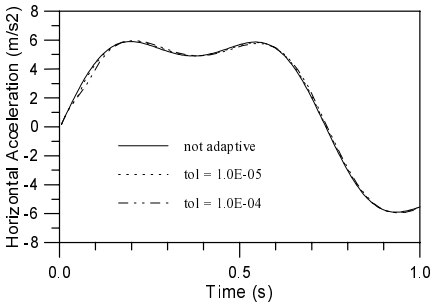

Fig. 1 Horizontal accelerations for the simple pendulum.

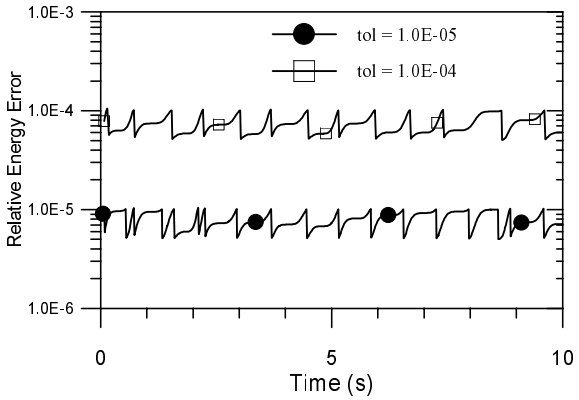

Fig. 2 Error in energy for the simple pendulum.

shows the smooth results for horizontal accelerations at the first second of calculations. There are no high frequency numerical oscillations present in the response. The solution obtained without time adaptivity falls on top of its analytical counterpart. The estimated errors in energy for the initial 10 seconds of computation are presented in Fig. (2). It is important to remember that the numerical dissipation inherently present in the scheme results a total energy decay. Evidently, it turns out to be impossible for the algorithm to use the same reference energy $E(\bar{t})$ in Eq. (4), throughout the entire computation of the solution. Thus, the calculation of the response starts using $E(\bar{t})$ equal to the initial total energy of the system. Then, if at a given time step the convergence criterion in Eq. (5) is not satisfied the time step size will be reduced. At each time step that Δt has to be reduced, this process goes on up to a maxi-

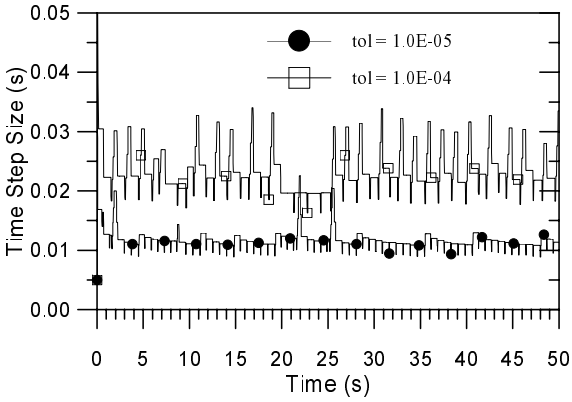

Fig. 3 Time step size variation for the simple pendulum.

mum number of repetitions, or until Δt reaches a specified minimum value. In either case the solution is accepted but the new energy of reference is the average between the reference energy at the previous time step and the energy computed at the present time step. Results for both analyses show the energy errors bounded by their respective limits. In both cases, the maximum value allowed for the time step size is $\Delta t = 5.0\text{x}10^{-02}$. A spectral analysis of the solution for constant Δt showed that $\Delta t = 3.7\text{x}10^{-02}$ is the largest time step size that could accurately integrate the equations of motion. Figure (3) shows that the algorithm keeps the time step size within the necessary limit for accuracy. During the entire period of the computations the kinematic constraint, which guarantees the constant length of the pendulum, is satisfied at the order of machine accuracy, *i.e.*, $1.0\text{x}10^{-15}$, as shown in Fig. (4). The solution without adaptivity runs with 10000 time steps. The adaptive algorithm with the error tolerance $1.0\text{x}10^{-05}$ used 57% of the computation time required to obtain the solution with a constant Δt and reduced the number of time steps to 4341. Relaxing the error tolerance to $1.0\text{x}10^{-04}$ results a reduction in time of computation to 36% of the computation time for a constant Δt, further reducing the number of time steps to 2195.

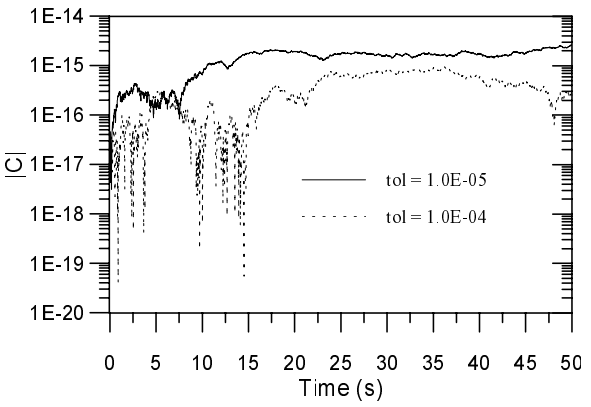

Fig. 4 The kinematic constraint for the simple pendulum.

5.2 Slider Crank Mechanism

In order to assess the capability of the algorithm to overcome problems with singularities, a slider crank mechanism is analyzed. Figure (5) shows the mechanism, which consists of two equal masses $m_1 = m_2 = 1.0$ kg, connected by means of two massless rods of equal length $\ell_1 = \ell_2 = \ell = 1.0$ m. Mass m_2 is constrained to frictionlessly move on a horizontal line such that the distance d remains equal to zero, which is the kinematic constraint imposed onto the system. The only force acting upon the system is gravity $g = 9.81$ m/s^2. The kinematic constraint is expressed as $d = \ell[\cos\theta_1 + \cos(\theta_1 + \theta_2)]$. An augmented Lagrange multiplier technique is used to enforce the constraint upon the system.

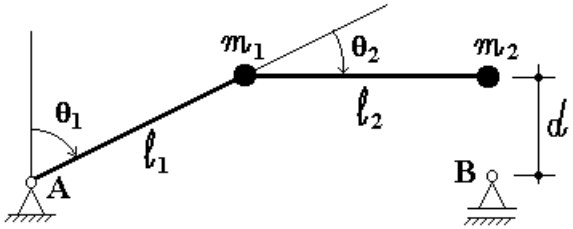

Fig. 5 Slider crank mechanism.

For specific geometric configurations, and depending on the solution scheme to be used, singularities may rule out the numerical calculations. For instance, if the method of coordinate partitioning is to be used, coordinates θ_1 and θ_2 are related as

$$\tan\theta_1 = \frac{1}{\tan\frac{\theta_2}{2}} \,. \tag{8}$$

Evidently, for specific positions of the masses, such as when $\theta_1 = 180$ deg. and $\theta_2 = -180$ deg., and $\theta_1 = 0$ deg. and $\theta_2 = 180$ deg., the solution will break down since Eq. (8) cannot be solved for such configurations. With initial conditions $\theta_1 = 45$ deg., $\theta_2 = 90$ deg., and $\dot{\theta_1} = \dot{\theta_2} = 0$, the numerical results are not sensitive to the singularities and do not present any sort of high frequency vibration content, falling on top of their analytical counterparts. Once again, the relative energy error shows an energy decaying in the response. This energy decaying characteristic of the time-discontinuous Galerkin scheme results from its capability of dissipating any spurious high frequencies that may artificially be introduced into the system. As shown in the literature [4] accelerations of constrained systems are the responses most sensitive to spurious high frequencies oscillations and instabilities. Accelerations for masses m_1 and m_2 coincide with the analytical solutions and, as in the previous example, the infinite frequencies introduced into the system by the kinematic constraint, are completely dissipated. The error tolerance in energy for the analysis is $\varepsilon^{tol} = 1.0\text{x}10^{-05}$. Since the limit tolerances are $1.1\,\varepsilon^{tol}$ and $0.6\,\varepsilon^{tol}$, respectively for the maximum and minimum tolerances, Fig. (6) shows the relative energy error oscillating between the bounds defined by such limits. Finally, the variation in time step size along the time integration process is presented in Fig. (7). The maximum size allowed for Δt is $2.0\text{x}10^{-01}$. It is clear, from Fig. (7), that the adaptive algorithm takes that value for Δt, after the first few steps of calculation. However, it rapidly comes back to a much smaller value and oscillates within the limits for accurately obtaining the response for the system. It can still be seen that the smallest values for Δt coincide, in time, with the regions where the accelerations curves present the steepest gradients. The variation in time step size emphasizes the accuracy characteristics of the algorithm, in the sense that it reduces the time step size where needed and

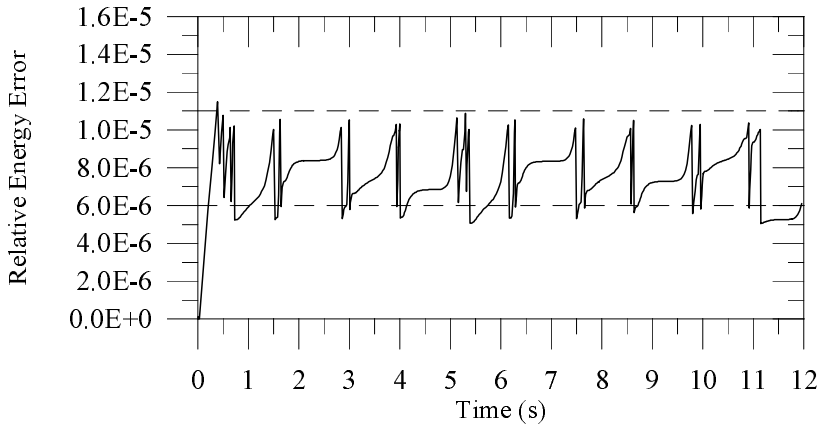

Fig. 6 Relative energy error for the slider crank mechanism with $\varepsilon^{tol} = 1.0\text{x}10^{-05}$.

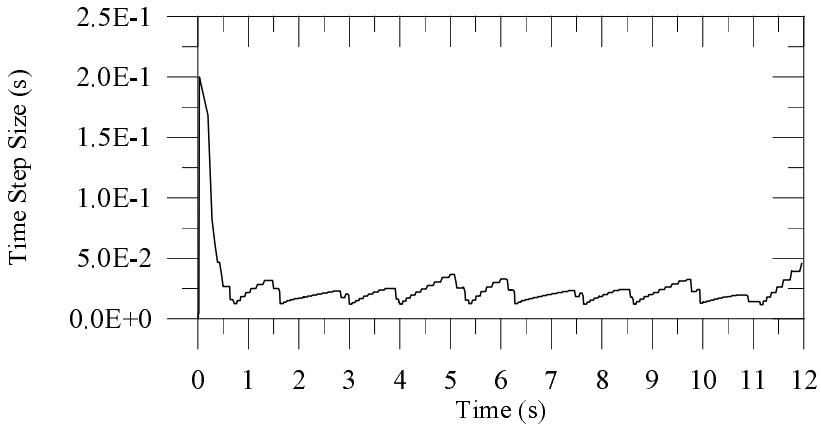

Fig. 7 Temporal time step size for the slider crank mechanism.

enlarges it otherwise.

5.3 Cantilever Elastic Beam under a Tip Load

The last example deals with a cantilever beam under a tip load $F(t)$. The beam is 2.40 m long modeled with 4 4–noded beam elements. The physical characteristics of the beam are: axial stiffness $EA = 146.2$ MN, bending stiffnesses $EI_{yy} = 736.6$ kN.m^2, $EI_{zz} = 211.7$ kN.m^2, torsional stiffness $GJ = 6.0$ kN.m^2, and mass per unit span $m = 5.41$ kg/m. Axis x is considered along the length of the beam, whereas axes y and z lay on the cross section of the beam. Gravity loads are neglected. The load $F(t)$, applied at the tip of the beam in the positive direction of the axis y, has the following time history

$$\begin{aligned} F(t) &= \bar{F} * \left(\frac{t_1}{T}\right) \\ &= \bar{F} * \left(2.0 - \frac{t_2}{T}\right) \end{aligned} \tag{9}$$

where $\bar{F} = 5.0$ x 10^{+04} N, $\bar{T} = 0.25$ s, $0 \leq t_1 \leq 0.25$ s, $0.25 < t_2 \leq 0.50$ s. Also, for $t > 0.50$ s the load $F(t) = 0$. Such a load is large enough to bring the structure to a nonlinear state of deformation, as the displacements at the tip of the beam are larger than 1/3 of its length, as depicted in Fig. 8.

During the period in which the load is being applied the time step size is kept constant, even when the time adaptive algorithm is on. In this initial analysis the time adaptive algorithm is off. As can be observed in Fig. 8 after $t = 0.5$ s, since there are no loads acting on the beam, it vibrates

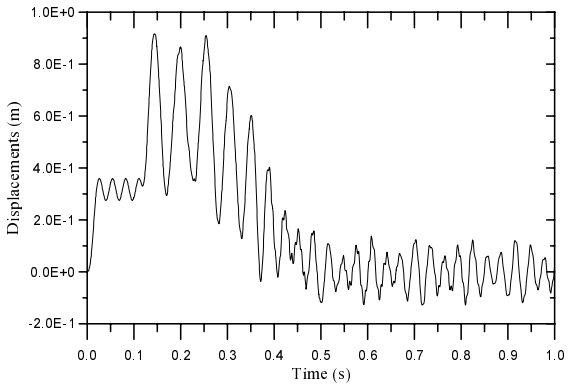

Fig. 8 Time history of transverse displacements at the beam tip.

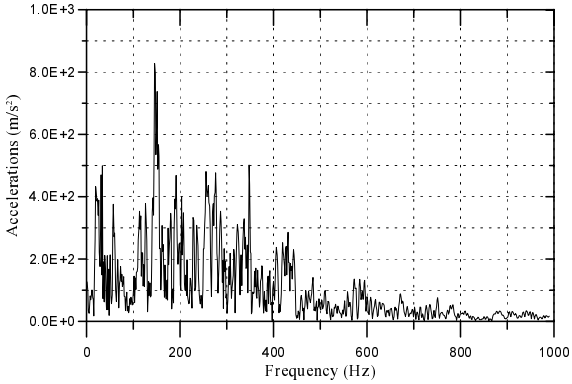

Fig. 10 Spectra of transverse accelerations of the beam tip.

around the equilibrium configuration at zero displacements. It is clear from Fig. 8 that the tip displacements oscillate between ± 0.15 m. Accelerations in the direction of the applied load are represented in Fig. 9. Although of small amplitudes, a high content of vibrations occur in the beam since there are considerable excitations on

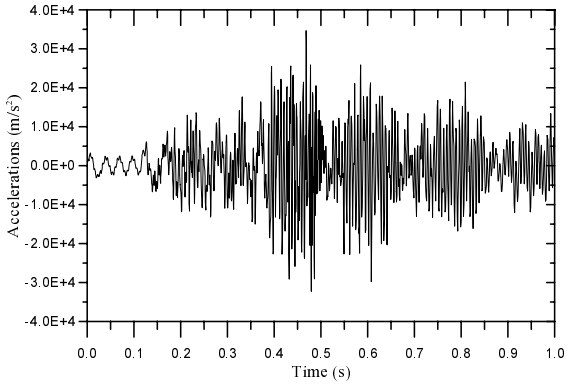

Fig. 9 Time history of transverse accelerations at the beam tip.

practically all frequencies up to 600 Hz, as can observed in Fig. 10. Also, from Fig. 10 it is clear that a small time step size should be used in the simulation in order to accurately obtain the responses of the structure. If one assumes that up to 400 Hz there are excitations with significant amplitudes, the corresponding period of $2.5 \text{ x } 10^{-03}$ should be divided in 20 points, at least, to run the simulation with accuracy. Instead, for reasons beyond the scope of this work, a constant $\Delta t = 5.0 \text{ x } 10^{-04}$ was used in the analysis. Since the scheme is unconditionally stable, the simulation was not prevent from running. However, as would be expected, a loss in accuracy of the responses did occur, as can be observed in Fig. 11, which shows a total energy dissipation of around 20 % during the last 0.5 seconds of the simulation when there are no loads acting on the beam. Evidently, for an exact solution no energy dissipation should take place. During these calculations the time step size was kept constant and the nonlinear equations of motions were solved with an error tolerance of $1.0 \text{ x } 10^{-06}$. Following, the time adaptive algorithm was tested. In order to compare the results with those of the simulation already described, it was assumed that a maximum total energy dissipation of 20 % (as

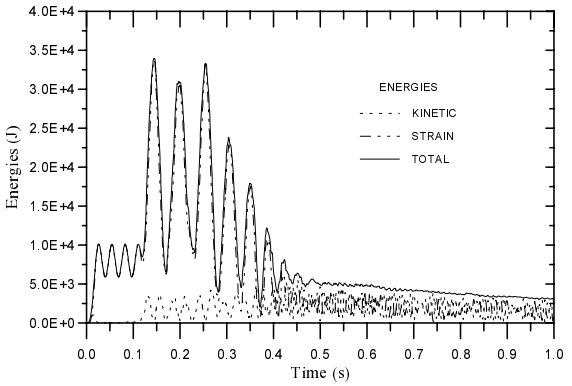

Fig. 11 Time history of energies.

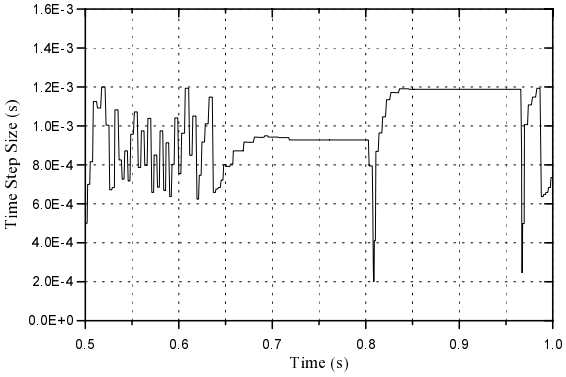

Fig. 12 Time history of the time step size.

occurred in the previous case) could take place. Time adaptivity was allowed only when there was no load on the structure. The initial time step size $\Delta t = 5.0 \text{ x } 10^{-04}$ was maintained and minimum and maximum limits were set as $5.0 \text{ x } 10^{-06}$ s and $1.2 \text{ x } 10^{-03}$ s, respectively. The time history of the time step size, during the period in which it occurs, is depicted in Fig. 12. Obviously, the large error allowed in energy dissipation does not require time step sizes smaller than $2.0 \text{ x } 10^{-04}$ as shown in the figure, whereas the maximum limit of $1.2 \text{ x } 10^{-03}$ is sufficiently small to satisfy such an accuracy at certain periods.

6 Conclusions

A time integration scheme based on the time-discontinuous Galerkin formulation is used on the solution of nonlinear constrained multibody systems. The system of differential-algebraic equations resulting from the modelling of nonlinear constrained multibody systems are prone to high frequency oscillations and instabilities of a purely numerical origin. The inherent high frequency numerical dissipation of the scheme obtained with a time-discontinuous Galerkin formulation completely eliminates these undesired instabilities. A single-field formulation of the time-discontinuous Galerkin finite element scheme using quadratic interpolation functions produces coupled systems of equations that are three times larger than the original ones. The resulting systems are solved in a predictor-multicorrector form, which alleviates the high computational cost of solving the fully coupled systems, and improves the scheme's characteristics of accuracy and high frequencies numerical dissipation. The total energy of the system, which monitors the stability characteristics of the response, is used as error measure and a time step adaptation routine is employed. The scheme is not sensitive to configuration singularities that may occur during the motion of the structural system. Kinematic constraints are satisfied at the order of machine accuracy, which confirms the high order of precision of the scheme and the efficiency of the algorithm as well. The algorithm may be fully implemented on existing structural dynamic computer codes requiring minor modifications on the routines. Already implemented nonlinear structural finite element libraries do not need to be rewritten. The finite element formulation of the mechanical joints has to be done accordingly and then included into the code's library of elements.

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