AN AGILE AIRCRAFT NON-LINEAR DYNAMICS BY CONTINUATION METHODS AND BIFURCATION THEORY

Krzysztof Sibilski
Military University of Technology, Warsaw, Poland

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Abstract

The paper provides a brief description of the modern methods for investigation of non-linear problems in flight dynamics. A study is presented of aircraft high angles of attack dynamic. Results from dynamical systems theory are used to predict the nature of the instabilities caused by bifurcations and the response of the aircraft after a bifurcation is studied. A non-linear dynamic model is considered which enables to determine the aircraft's motion. The aerodynamic model used includes nonlinearities, hysteresis of aerodynamic coefficients (unsteady aerodynamics), and dynamic stall effect. Aerodynamic model includes also a region of higher angles-of-attack including deep stall phenomena. In the paper continuation method is used to determine the steady states of the MiG-29 fighter aircraft as functions of the elevator deflections, and bifurcations of these steady states are encountered. Bifurcations of the steady states are used to predict the onset of wing rock, and "Cobra" manoeuvres. Dynamics of spatial motion of a supersonic combat aircraft with a straked wing is considered for post stall manoeuvres, and observations of chaotic motion in post stall manoeuvres are guided.

1 Introduction

An aircraft is the inherently non-linear and time varying system. Non-linear dynamics is central to several important aircraft motions, including roll-coupling and stall/spin phenomena. Linearized equations of motion can not be used to analyse these phenomena. Indeed, roll-coupling instabilities were first discovered in flight, often with fatal results, because the linearized equations of motion used for analysis at that time did not contain the instability [1].

There are many problems associated with flight dynamics for modern and advanced aircraft, which are not solved (or solved rather unsatisfactory) with traditional tools. A list of such problems includes among others flight control for agile and post-stall aircraft. The post-stall manoeuvrability has become one of the important aspects of military aircraft development. Such manoeuvres are jointed with a number of singularities, including "unexpected" aircraft motion. As the result of them, there is dangerous of faulty pilot's actions. Therefore, it is need to investigate aircraft flight phenomena at high- and very-high angles of attack.

The appearance of a highly augmented aircraft required a study of its high angle of attack dynamics The primary aim of the paper is to discuss capabilities of dynamical system theory methods as the tools for to analyse such phenomena.

Dynamical system theory has provided a powerful tool for analysis of non-linear phenomena of aircraft behaviour. In the application of this theory, numerical continuation methods and bifurcation theory have been used to study roll-coupling instabilities and stall/spin phenomena of a number of aircraft models. Results of great interest have been reported in several papers (it can be mentioned papers by Jahnke and Culick [1], Carroll and Mehra [2], Guicheteau [3], or Avanzini and de Matteis [4]). Continuation methods are numerical techniques for calculating the steady states of systems of ordinary differential equations and can be used to study roll coupling instabilities and high-angle of attack instabilities.

Carroll and Mehra [2] were the first to use a continuation technique to calculate the steady states of aircraft. They determined the steady states of a variable sweep aircraft and the F-4 fighter aircraft. By studying the steady states of these two aircraft they explained that wing rock appears near the stall angle of attack due a Hopf bifurcation of the trim steady state. They also calculated the steady spin modes for the aircraft and predicted the control surface deflections at which the aircraft would undergo stall/spin divergence. They have calculated the resulting steady state of the aircraft and have developed recovery techniques using their knowledge of the steady spin modes for the aircraft. Guicheteau [3] has used continuation methods and bifurcation theory in analysis of the non-linear dynamics of aircraft model that includes unsteady aerodynamic coefficients. He has analysed the effects of a lateral offset of the c.g. and the influence of gyroscopic momentum of rotating engine's masses on spin entry recovery. Jahnke and Culick [1] have recalled the theoretical background of dynamical system theory and bifurcation technique and they have presented profound review of the relevant investigations in this field. They have studied the dynamic of the F-14 fighter aircraft by determining the steady states of the equations of motion and seeking bifurcations. And have shown that continuation method is very useful in analysis of the wing rock instability, spiral divergence instability and spin dynamics. Avanzini and de Matteis [4] have analysed the dynamics of a relaxed stability aircraft by dynamical system theory. The principal objective of their study was to assess the practical worth of dynamical system theory in simulations where the dynamics of aircraft are tailored by the full-authority control system according to different mission tasks. They have remarked that the use of manoeuvre demand control in high-performance aircraft somehow limits the capability of dynamical system theory to provide global stability information in as much as transient motions cannot be predicted or quantified by bifurcation theory (compare with results obtained by Carol and Mehra [2]). Avanzini and de Matteis [4] have proved, that certain additional problems have been faced with the modelling of the stability and control augmentation system with the related high number of additional states and the handling of the so-called breakpoint nonlinearities coming from control system elements.

The present paper is continuation of the previous works of the author [5], [6], [7]. After a brief description of the methodology and associated procedures, Cobra maneuver and associated wing-rock oscillations are studied by means of checking the stability characteristics related to unstable equilibria. Numerical simulations are used to verify the predictions. High angle of attack maneuvers were studied to observe chaos phenomenon in post stall motion. Unsteady aerodynamics for prediction of airfoil loads is included, and the ONERA type stall model is used [8], [9], [10].

2 Theoretical background

2.1 Dynamical systems theory

In this paper we will study equations of the following form

$$\dot{\mathbf{x}} = f(\mathbf{x}, t; \boldsymbol{\mu}) \qquad (1)$$

and

$$\mathbf{x} \mapsto g(\mathbf{x}; \boldsymbol{\mu}) \qquad (2)$$

with $\mathbf{x} \in U \subset \mathbf{R}^n, t \in \mathbf{R}^1, and\ \boldsymbol{\mu} \in V \subset \mathbf{R}^p$, where U and V are open sets in $\mathbf{R}^n$ and $\mathbf{R}^p$, respectively. We view the variables μ as parameters. We refer to (1) as a *vector field* or ordinary differential equation and to (2) as a *map* or *difference equation*. Both will be termed *dynamical systems*.

By a solution of Eq. (1) we mean a map, $\mathbf{x}$, from some interval $\mathrm{I} \subset \mathbf{R}^1$ into R^n, which we represent as follows

$$\begin{aligned} &\mathbf{x} : \mathrm{I} \rightarrow \mathbf{R}^1, \\ &t \mapsto \mathbf{x}(t) \end{aligned} \qquad (3)$$

such that $\mathbf{x}(t)$ satisfies (1), i.e.,

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), t; \boldsymbol{\mu}) \qquad (4)$$

Dynamical systems theory (DST) provides a methodology for studying systems of ordinary differential equations. The most important ideas of DST used in the paper will be introduced in the following sections. More information on DST can be found in the book of Wiggins [11].

The first step in the DST approach is to calculate the steady states of the system and their stability. Steady states can be found by setting all time derivatives equal to zero and solving the resulting set of algebraic equations. The Hartman-Grobman theorem proves that the local stability of a steady state can be determined by linearizing the equations of motion about the steady state and calculating the eigenvalues [12], [13], [14], [15].

The implicit function theorem (Ioos and Joseph [16]) proves that the steady states of a system are continuous function of the parameters of the system at all steady states where the linearized system is non-singular. Thus, the steady states of the equations of motion for an aircraft are continuous functions of the control surface deflections. Stability changes can occur as the parameters of the system are varied in such a way that the real parts of one or more eigenvalues of the linearized system change sign. Changes in the stability of a steady state lead to qualitatively different responses for the system and are called bifurcations. Stability boundaries can be determined by searching for steady states, which have one or more eigenvalues with zero real parts. There are many types of bifurcations and each has different effects on the aircraft response. Qualitative changes in the response of the aircraft can be predicted by determining how many and what types of eigenvalues have zero real parts at the bifurcations point. Bifurcations for which one real eigenvalue is zero lead to the creation or destruction of two or more steady states. Bifurcations for which one pair of complex eigenvalues has zero real parts can lead to the creation or destruction of periodic motion. Bifurcations for which more than one real eigenvalue or more than one pair of complex eigenvalues has zero real parts lead to very complicated behaviour

2.2 Bifurcation Theory

For steady states of aircraft motion, very interesting phenomena appear when even if one negative real eigenvalue crosses the imaginary axis when control vector varies. Two cases can be considered [16].

- The steady state is regular, i.e. when the implicit function theorem works and the equilibrium curve goes through a limit point. It should be noted that a limit point is structurally stable under uncertainties of the differential system studied.
- The steady state is singular. Several equilibrium curves cross a pitchfork bifurcation point, and bifurcation point is structurally unstable.

If a pair of complex eigenvalues cross the imaginary axis, when control vector varies, Hopf bifurcation appears [13], [14], [16], [17], [18]. Hopf bifurcation is another interesting bifurcation point. After crossing this point, a periodic orbit appears. Depending of the nature of nonlinearities, this bifurcation may be subcritical or supercritical. In the first case, the stable periodic orbit appears (even for large changes of the control vector). In the second case the amplitude of the orbit grows in portion to the changes of the control vector.

2.3 Continuation technique and methodology scheme

Continuation methods are a direct result of the implicit function theorem, which proves that the steady states of a system are continuous functions of the parameters of the system al all steady states except for steady states at which the linearized system is singular. The general technique is to fix all parameters except one and trace the steady states of system as a function of this parameter. If one steady state of the system is known, a new steady state can be approximated by linear extrapolation from the known steady state [12], [14], [15], [19]. The slope of the curve at the steady state can be determined by taking the derivative of the equation given by setting all time derivatives equal to zero. If two steady states are known, a new steady state can be approximated by linear extrapolation through the two known steady states. The stability of each steady state can be determined by calculating the eigenvalues of the linearized system. Any changes in stability from one steady state to the next will signify a bifurcation.

Taking into account experience of many researches, one can formulate the following tree-step methodology scheme (being based on bifurcation analysis and continuation technique) for the investigation of non-linear aircraft behaviour [7], [15], [19]:

- During the first step it is supposed that all parameters are fixed. The main aim is to search for all possible equilibria and closed orbits, and to analyze their local stability. This study should be as thorough as possible. The global structure of the state space (or *phase portrait*) can be revealed after determining the asymptotic stability regions for all discovered attractors (stable equilibria and closed orbits). An approximate graphic representation plays an important role in the treating of the calculated results.
- During the second step the system behaviour is predicted using the information about the evolution of the portrait with the parameters variations. The knowledge about the type of encountered bifurcation and current position with respect to the stability regions of other steady motions are helpful for the prediction of further motion of the aircraft. The rates of parameters variations are also important for such a forecast. The faster the parameter change, the more the difference between steady state solution and transient motion can be observed.
- Last, the numerical simulation is used for checking the obtained predictions and obtaining transient characteristics of system dynamics for large amplitude state variable disturbances and parameter variations.

3. Mathematical model of aircraft motion

Non-linear equations of motion of the aeroplane and the kinematic relations will be expressed by using moving co-ordinate systems, the common origin of which is located at the centre of mass of the aeroplane (Figs.1 and 2). It is used [10]:

– a system of co-ordinates *Oxyz* attached to the aircraft (the *Oxz* plane coinciding with the symmetry plane of the aircraft);

– a system of co-ordinates attached to the air flow $Ox_ay_az_a$ in which the Ox_a axis is directed along the flight velocity vector **V** and the Oz_a axis lies in the symmetry plane of the aircraft and is directed downwards.

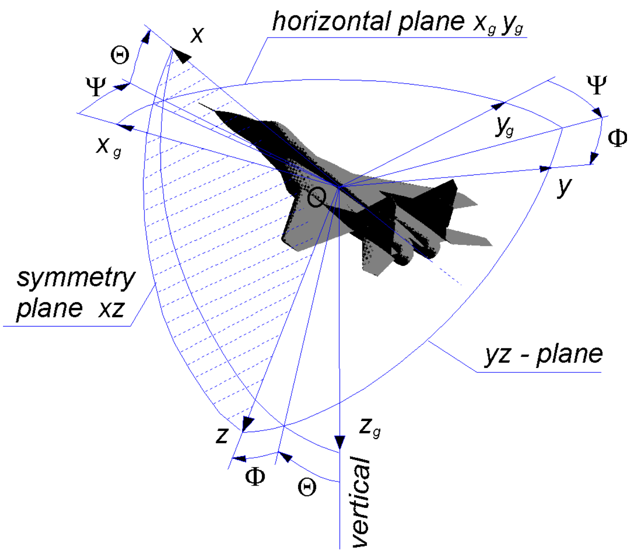

Fig. 1 System of co-ordinates attached to aircraft

The relative position of the vertical system $Ox_gy_gz_g$ and the system *Oxyz*, attached to the aircraft is described by Euler angles Θ, Φ and Ψ (Fig. 1), while the relative position of the system *Oxyz* and the system $Ox_ay_az_a$ attached to the airflow - by the angle of attack α and slip angle β (Fig.2).

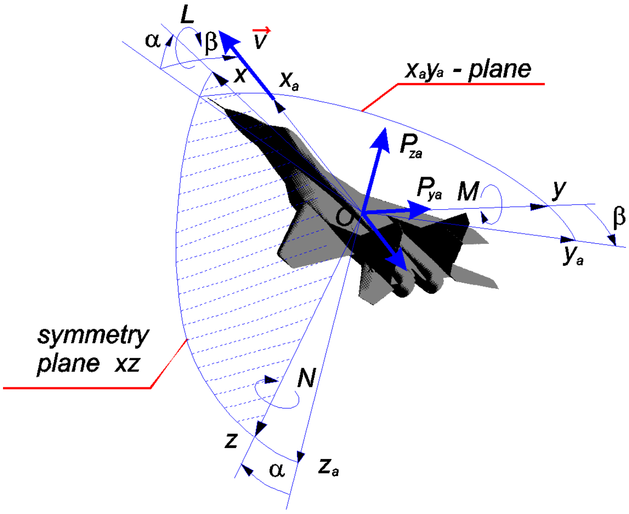

Fig. 2. System of co-ordinates attached to airflow

Usually aircraft is considered as a rigid body with moving elements of control surfaces. Gyroscopic moment of rotating masses of the engines is included. Total system of equations should be completed with the following expres-

sions: kinematic relations, kinematics of an arbitrary control system and the control laws.
The mathematical model of aircraft can be formulated in the following form [10], [20], [21]:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)), \qquad \mathbf{x}(0) = \mathbf{x}_0 \qquad (5)$$

where:

- state vector

$$\mathbf{x} = [V, \alpha, \beta, p, q, r, \Phi, \Theta, \Psi, x_g, y_g, z_g]^T \qquad (6)$$

- control vector

$$\mathbf{u} = [\alpha_{zH}, \delta_H, \delta_A, \delta_V, \delta_F]^T \qquad (7)$$

and:

$$f_1 = \frac{1}{m}\left[F\cos(\alpha+\varphi_s) - P_{X_a}\right] - g\left[\cos\Theta\sin\Phi\sin\beta - (\sin\Theta\cos\alpha - \cos\Theta\cos\Phi\sin\alpha)\cos\beta\right]$$

$$f_2 = q - (p\cos\alpha + r\sin\alpha)tan\beta - \frac{1}{mV\cos\beta}\left[F\sin(\alpha+\varphi_s) + P_{Z_a} + -mg(\sin\Theta\sin\alpha + \cos\Theta\cos\Phi\cos\alpha) + P_{Z_a}\right]$$

$$f_3 = p\sin\alpha - r\cos\alpha - \frac{1}{mV}\left\{\left[F\cos(\alpha+\varphi_s) + -mg(\sin\Theta\cos\alpha - \cos\Theta\cos\Phi\sin\alpha)\right]\sin\beta - mg\cos\Theta\cos\Phi\cos\beta - P_{Y_a}\right\}$$

$$\begin{bmatrix} f_4 \\ f_5 \\ f_6 \end{bmatrix} = \mathbf{J}^{-1}\left((\mathbf{M}_a + \mathbf{M}_F) - \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix}\mathbf{J}\right)$$

$$\begin{bmatrix} f_7 \\ f_8 \\ f_9 \end{bmatrix} = \begin{bmatrix} 1 & \sin\Phi\,\mathrm{tg}\,\Theta & \cos\Phi\,\mathrm{tg}\,\Theta \\ 0 & \cos\Phi & -\sin\Phi \\ 0 & \sin\Phi\sec\Theta & \cos\Phi\sec\Theta \end{bmatrix}\begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$$\begin{bmatrix} f_{10} \\ f_{11} \\ f_{12} \end{bmatrix} = \mathbf{A}^T\mathbf{A}_a\begin{bmatrix} V \\ 0 \\ 0 \end{bmatrix}$$

where:

P_{Xa}, P_{Ya}, P_{Za} - aerodynamic forces,
A and $\mathbf{A}_a$ - matrixes of transformations. Elements of those matrixes can be found for example in [10];
J - matrix of inertia;
$\mathbf{M}_a$=$[L.M,N]^T$ – vector of aerodynamic moment;
$\mathbf{M}_F$ - vector of momentum of aircraft engine:

$$\mathbf{M}_F = \sum_i \mathbf{M}_i^F = \sum_i \left(\mathbf{r}_i \times \mathbf{F}_i + \mathbf{J}_{F_i}\varpi_{F_i} \times \mathbf{O}\right) \qquad (8)$$

where:

$\mathbf{r}_i$ - vectors determining the distance of the engines thrust from the pole of the system of co-ordinates;
$\mathbf{F}_i$ - vector of the thrust;
$\mathbf{J}_{F_i}$ - the polar moments of inertia of the engine rotor;
$\vec{\omega}_{F_i}$ - vector of angular velocity of a rotor;
$\mathbf{O}$ =$[p,q,r]^T$- vector of angular velocity of aircraft.

Equations of motion of aircraft should be completed with equations of engine dynamics [10]:

- equation of engine rotation.

$$\tau_1\tau_2\ddot{n} + (\tau_1+\tau_2)\dot{n} = K\left[Q_p(t-\tau_0) - Q_{p0}\right]\cdot n \qquad (9)$$

where:

n - angular velocity of a rotor,
τ_0, τ_1, τ_2 - time-constants,
K - amplification factor,
Q_p - discharge of fuel for actual engine's angular velocity and for actual aircraft's altitude and airspeed,
Q_{p0} - discharge of fuel, calculated for sea level conditions and when V=0,

Time-constants are non-linear functions of engine's angular velocity, aircraft's altitude, air density, pressure and temperature on fight altitude. Discharge of fuel is following function:

$$Q_p(t) = \mathrm{f}_1(\delta_T(t)) \qquad (10)$$

where:

δ_T - displacement of cockpit power lever

- equation of thrust

$$T = T_0(n)\left(\frac{\rho}{\rho_0}\right)^{0.7}\left(K_0 + K_1 Ma + K_2 Ma^2\right) \qquad (11)$$

The engine model should be adapted from the code, data, and flow charts provided by the engine manufacturer.

3.1 Modelling of aerodynamic loads

The adequacy of mathematical modelling of high AoA (Angle of Attack) dynamics is strictly dependent on the adequacy of the aerodynamic model at these regions. There is nontrivial problem due to the very complicated nature of the separated and vortex flow in unsteady regime [22]. Precise describing of aerodynamic forces and moments found in equations of motion is fundamental source of difficulties. In each phase of flight dynamics and aerodynamics influence each other, which disturbs the precise mathematical description of those processes. The requirements for method on aerodynamic load calculations stem both from flow environment and from algorithms used in analysis of helicopter flight. The airframe model consists of the fuselage, horizontal tail, vertical tail, and wing. The fuselage model is based on wind tunnel test data (as function of angle of attack α and slip angle β). The horizontal tail and vertical tail are treated as aerodynamic lifting surfaces with lift and drag coefficients computed from data tables as functions of angle of attack α and slip angle β.

For linear extent of lifting force, an aeroplane's aerodynamic loads can be defined on the basis of algorithms, relations, diagrams and formulas shown, for example, in *DATA SHEETS* or in *The USAF Stability and Control DATCOM* [23]. However, there is no efficient method to calculate aerodynamic loads for high angles of attack. Results of investigating aeroplane's models are not always available and complete. Usually, there are no reliable aeroplane's aerodynamic characteristics, obtained by identification method, on the basis of measurements during a flight. Panel methods are appropriate to define aerodynamic loads for small and moderate angles of attack. Numerical study of spin and other manoeuvres performed on overcritical angles of attack, as well as simulation of acrobatic figures characteristic for so-called supermanoeuvre planes (for instance, Cobra or Kublit manoeuvres) requires data concerning aerodynamic characteristics for angles of attack practically from -180^{o} to $+180^{o}$. Therefore, to define aerodynamic loads in the possibly broadest extent of angles of attack, an attempt to broaden the strip theory has been made. In modification of this theory, presented below, following assumptions have been made:

– in given section of a force, aerodynamic moments depend on a local angle of attack

– the area of a flowfield is disturbed by adding to a vector of speed appropriate components resulting from plane's rotation with angular speeds *p, q, r.*

– mutual relation between flows of neighbouring strips was taken into account by adding the speed induced by flowing down whirlpools

– dynamics of whirlpool structures, including whirlpools' break-up, was taken into account

– unsteadiness of a flowfield (aerodynamic hysteresis) was taken into account, phenomenon of deep aerodynamic stall was modelled using algorithm worked out in ONERA [8], [9].

The algorithm of calculations allows defining loads of wings of any shape. In case of modern fighter aeroplanes, with strongly coupled aerodynamic configuration, it was assumed that lifting fuselage of these planes is a centre wing section, that is - in algorithms, forces and aerodynamic moments generated by lifting parts of fuselage were taken into account. The modified strip theory is interesting also because it is relatively easy to consider a phenomenon of non-symmetrical break-up of whirlpools in its algorithms (using, for example method proposed in work [10]). A wing of a plane is divided with planes parallel to the fuselage's plane of symmetry to a number of elements (strips). For each strip we define a local angle of attack and a value of total vector of speed. Then, from aerodynamic characteristics of the profile, we define aerodynamic coefficients of: lifting force, resistance force and pitching moment. Integration defined in this way forces and moments along wing span allows defining the aerodynamic loads of a plane. For purpose of numerical analysis, functions $C_z(\alpha)$ and $C_x(\alpha)$ were approximated with trigonometric polynomials:

$$
\begin{aligned}
C_z(\alpha) &= \sum_{k=0}^{n} [a_k \cos(k\alpha) + b_k \sin(k\alpha)] \\
C_x(\alpha) &= \sum_{k=0}^{n} [c_k \cos(k\alpha) + d_k \sin(k\alpha)]
\end{aligned}
\tag{12}
$$

Where coefficients $a_{k,}$ $b_{k,}$ $c_{k,}$ and d_k were calculated from Runge's scheme. Values of these coefficients are shown in work [10].

The angle of attack of - elementary strip of a wing depends on: the aeroplane's angle of attack, an angle of attack induced by whirlpools flowing down a wing and an angle of attack caused by appearance of angular speeds (pitching, rolling, and yawing). The induced angle of attack can be calculated from the relation:

$$\alpha_i = \arctan\left(V_i / V_0 \right) \tag{13}$$

The induced speed can be calculated from Biot-Savart's law:

$$\begin{aligned} V_i(y) = &-\frac{\Gamma(y)}{4\pi r_1}(\cos\varphi_1 + \cos\varphi_2) - \\ &-\frac{\Gamma(y)}{4\pi r_2}(\cos\varphi_3 + \cos\varphi_4) \end{aligned} \tag{14}$$

Where r_1 and r_2 – correspondingly, a distance from left and right bound vortex from point A (in which induced speed is calculated).

Distribution of circulation along wing span is given with following differential-integral equation:

$$\begin{aligned} \Gamma(y) = &\frac{V_0}{2} \frac{\frac{\partial C_z}{\partial \alpha} \cos\chi c_A(y)}{k_\Pi(y)} \times \\ &\times [\alpha_0(y) - \frac{1}{4\pi V_0} \int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{d\Gamma(\xi)}{d\xi} \frac{d\xi}{(z-\xi)}] \end{aligned} \tag{15}$$

Equation (15) can be solved with approximate methods (for instance, approximation of trigonometric series). Distribution of circulation along wing span can also be calculated with engineer methods (for example, classic Multhopp's method) or evaluated with help of known (for example from examining a plane in aerodynamic tunnel) distribution of pressures along wing span. On the basis of known distribution of circulation we can define distribution of induced angles of attack along wing span (and therefore for each wing's section).

The bifurcation approach is very fruitful when the sources and nature of aerodynamic phenomena are considered. Special techniques were proposed to represent the aerodynamic characteristics taking into account the dynamics effects of the separated flow [10].

Now, it must be pointed up that dynamic system theory provides a methodology for studying of such complicated ordinary differential equations (ODE). In addition, recent development in the area of numerical analysis of non-linear equations created a class of computer algorithms known as continuation method [15], [19] The set of ordinary differential equations can be solved using the continuation and bifurcation software AUTO97 available at address: *ftp://ftp.cs.concordia.ca/pub/doedel/auto*.

This very useful freeware gives all desired bifurcation points for different values of control vector components.

3 Results

Figs. 3-8 show the steady states of the MiG-29 fighter aircraft for longitudinal manoeuvres, which are at middle and high angles of attack. Steady states represented in those figures show longitudinal trim conditions and spirally divergent motions. These figures show that practically for all elevator deflections the aircraft can achieve stable or unstable trim conditions. The trim conditions for given elevator deflection can be determined by drawing the vertical line representing the desired elevator deflection on each plot; each intersection of this line with the curve of steady states gives a possible steady state of the aircraft. For example, a vertical line representing 4^o of the elevator deflection intersects three steady states. Two of them are stable one is unstable, so the aircraft could exhibit any of these three steady states. One stable steady state at 4^o elevator deflection represents the horizontal flight trim configuration (p=q=r=0). The other two stable steady states represent pull-up, or spiral trim conditions.

Continuation methods require a known steady state as a starting point for the continuation procedure. It is usually easy to determinate steady states that are at low and moderate values of angels of attack. Determining the steady states at high angles of attack is more difficult task and it is usually not possible to be certain all the steady states for a particular aircraft have been determined. The approach used to find the modes in this work was to guess an initial high-angle-of attack mode as a starting point for the continuation method algorithm, then let the algorithm run until either a true steady state was determined or the algorithm ran into numerical problems.

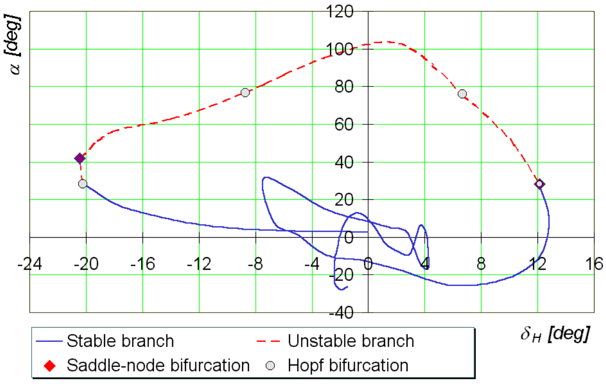

Fig. 3. Steady states for longitudinal manoeuvres

The segment of unstable steady states contains the trim conditions between the elevator deflections at -20.5^{o} and 12.1^{o} because of six Hopf or saddle-node bifurcations. Those bifurcations occur at -20.5^{o} (saddle-node bifurcation), -20.1^{o} (Hopf bifurcation), -8.8^{o} (Hopf bifurcation); 7.1^{o} (Hopf bifurcation), 12.1^{o} (saddle-node bifurcation), and 12.1^{o} (Hopf bifurcation).

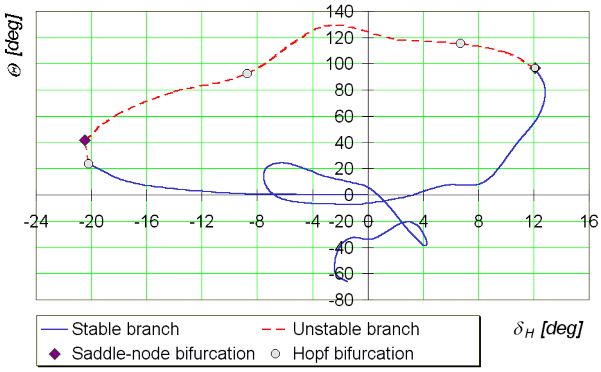

Fig. 4 Steady states for longitudinal manoeuvres

Hopf bifurcations can lead to periodic motions, so it is possible that for elevator deflections between -20.1^{o} and -8.8^{o}, 7.1^{o} and 12.1^{o} the aircraft will undergo periodic motion (wing-rock instability). The saddle-node bifurcations (occurred at the elevator deflections: -20.5^{o} and 12.1^{o}) signify that aircraft loss their longitudinal stability.

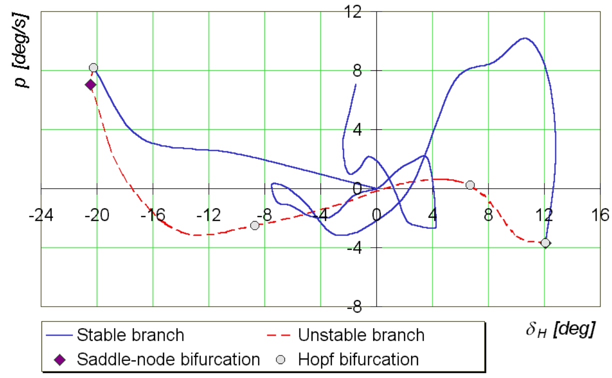

Fig. 5 Steady states for longitudinal manoeuvres

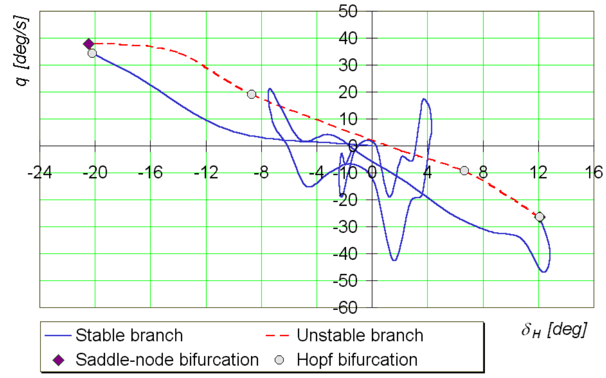

Fig. 6 Steady states for longitudinal manoeuvres

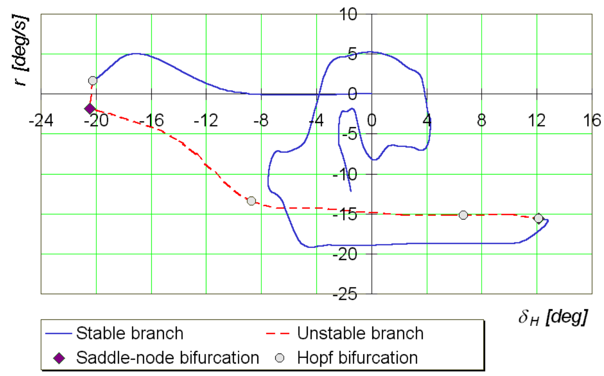

Fig. 7 Steady states for longitudinal manoeuvres

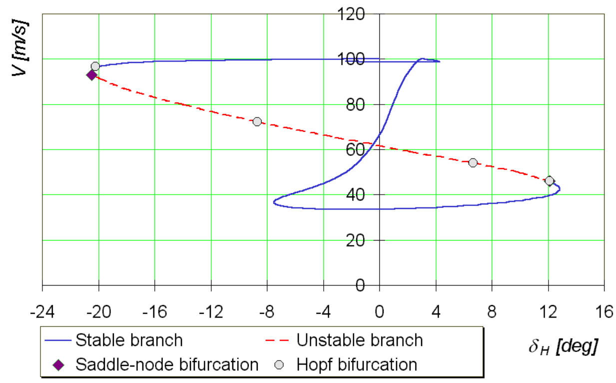

Fig. 8 Steady states for longitudinal manoeuvres

It can by explain by loosing of effectiveness of the control surface. The lifting force generated on the tail control surface depends on the local angle of attack and is perpendicular to the local flow direction. As the first approximation it can be assumed that control pitching moment is generated by the normal component of the aerodynamic tail force only. In the range of AoA up to 35^{o} the normal component increases approximately linearly, then stabilises and practically the tail surface losses its effectiveness. After the further displacement of the tail control surface the pitching moment is proportional to the cosine of the displacement angle. This way, in the first approximation, one can assume that control surface is able to work effectively in range approximately ±40^{o}. Those occurred at high angles of attack longitudinal stability loosing make it possible to perform the Cobra manoeuvre. This new manoeuvre had been shown by W. Pugatchov early in 1994 at the Abu Dabi Air Show and was called "hook". The hook manoeuvre (performed in horizontal plane) processes real combat significance. Basing on flight tests (cf. O. Samoylovich [24]) it was concluded that the dynamic entrance into the high angles of attack flight could be divided into four phases as follows:

- first phase, characterised by full deflection of horizontal tail (for pull-up of the aircraft) with a maximum speed of the control stick. The main goal in this stage is to create a big, positive pitching moment as soon as possible;
- second phase, when aircraft still increases angle of attack due its inertia and at last reaches the maximum angle of attack (at the end of this phase the pitching moment is near of its maximum value for diving, pitch rate is near 0);
- third phase, (recovery from the manoeuvre) characterised by full deflection of the horizontal tail for diving with the increasing, negative pitch rate (at the end of this phase the pitch rate for diving reaches its maximum, negative value. The angle of attack approaches its value of the steady flight, but aircraft still rotates and further decreases the angle of attack due to its inertia);
- fourth phase, when angles of attack reach values lower then that of in the steady flight and the process of balancing at low angles of attack is going on.

During the third and fourth phases it is necessary to control the aircraft motion in its recovery from high angles of attack. Slow recovery creates the conditions for supplementary loss of flight speed and increases the probability of stall. Quick recovery from high angles of attack can be reason to go at angles of attack lower than 0. Thorough analysis of Cobra manoeuvre dynamics was presented in paper [25]. Exemplary results of those investigations are shown in Fig. 9.

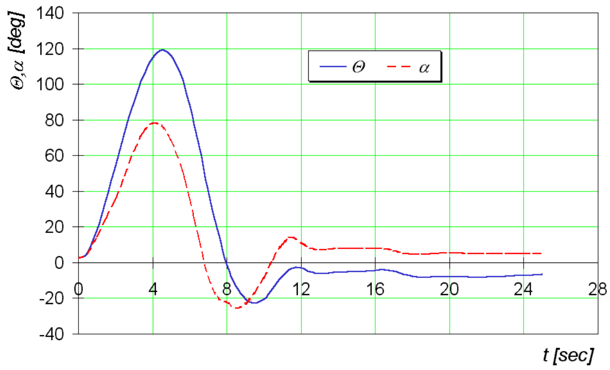

Fig. 9 Curves of angle of attack, and pitch angle during Cobra manoeuvre (cf. [25])

Based upon investigations carried out in works [24], [25] one can conclude that the necessary conditions to perform Cobra manoeuvre are:

- aircraft should be statically unstable in the range of angles of attack near critical;
- balance should be attainable in the range of angles of attack greater than critical;
- pitching moment for diving has to have a margin in the range of angles of attack $\alpha \in (30^{o}, 60^{o})$;
- natural pitching moment for diving should be big enough in the range of angles of attack grater than 60^{o};
- limiter of the extreme flying parameters should be turned off.

Moreover, the aircraft should be highly insensitive to the spin tendency, especially because of lack of lateral stability at high angles of attack. Investigations of Cobra manoeuvre carried out in work [25] were under assumption, that the aircraft is highly insensitive to the spin tendency and wing rock instability. Unfortunately, at higher angles of attack real aircraft will undergo wing-rock oscillations, and has strong tendency to bank. It is evident, when one considered steady states at middle and high angles of attack. Occurrences of Hopf and saddle-node bifurcations signify radical changes in aircraft response after those bifurcations.

Figs. 10-21 show an attempted Cobra manoeuvre entry using the elevator deflection. During the Cobra manoeuvre, all flight parameters increase their values. In terms of continuation methods, the Cobra is unstable because of Hopf bifurcation, that occurred at δ_H=–20.1^o and saddle-node bifurcation, that occurred at δ_H=-20.5^o. The Poincare maps of selected state parameters are shown in Figs. 10-12. It can be stated that taking into consideration unsteady aerodynamic model and hysteresis of aerodynamic coefficients, one counters significant irregularities in solution of equations of motion, that are characteristic for chaotic motion.

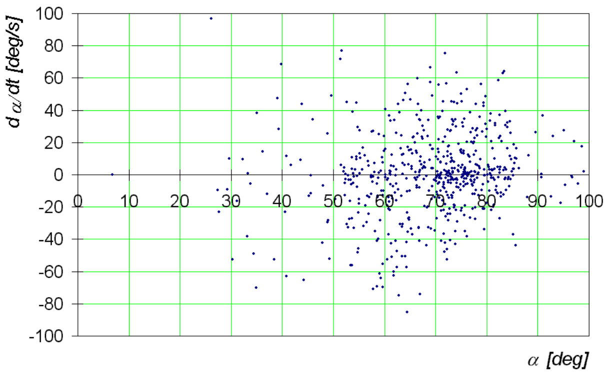

Fig. 10 Longitudinal manoeuvre performed at high angles of attack. Poincare map

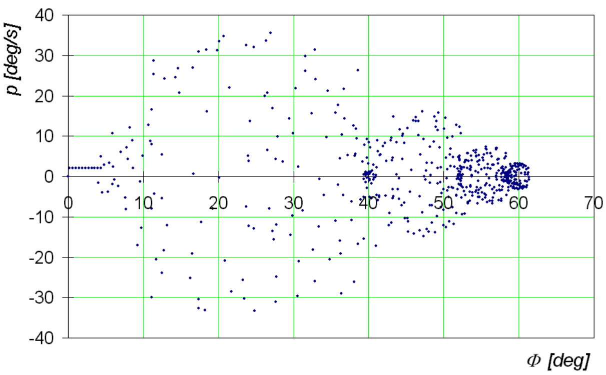

Fig. 11 Longitudinal manoeuvre performed at high angles of attack. Poincare map

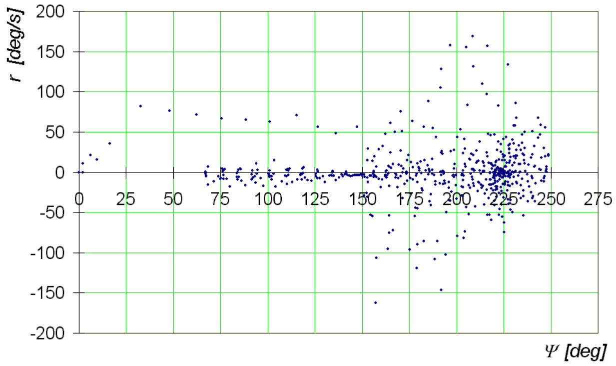

Fig. 12 Longitudinal manoeuvre performed at high angles of attack. Poincare map

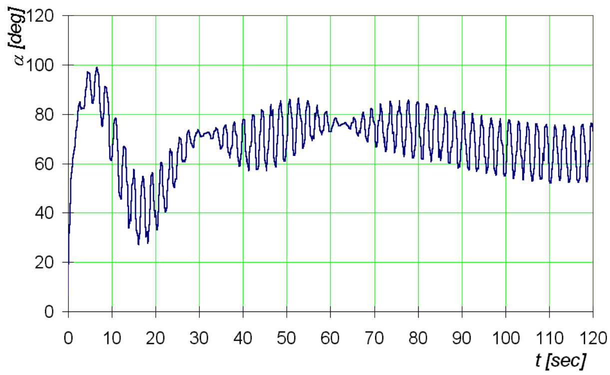

Fig. 13 Course of angle of attack

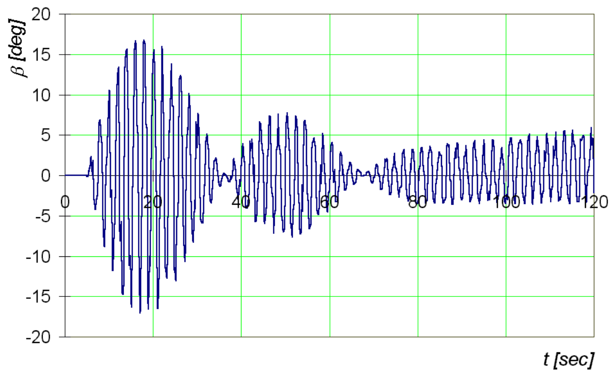

Fig. 15 Course of slip angle

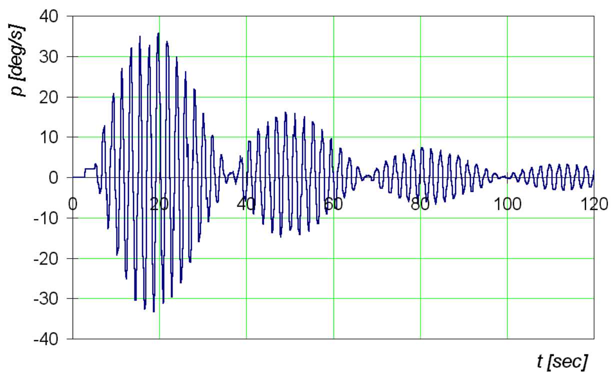

Fig. 16 Course of roll rate

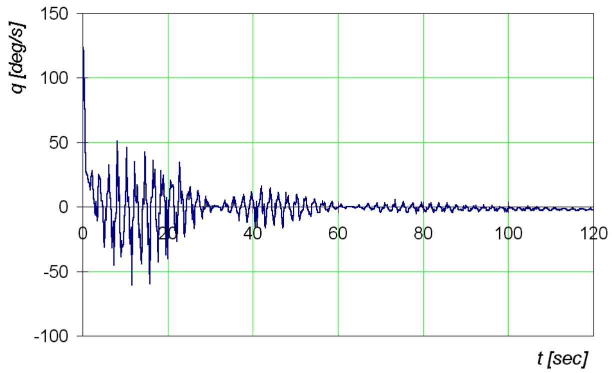

Fig. 17 Course of pitch rate

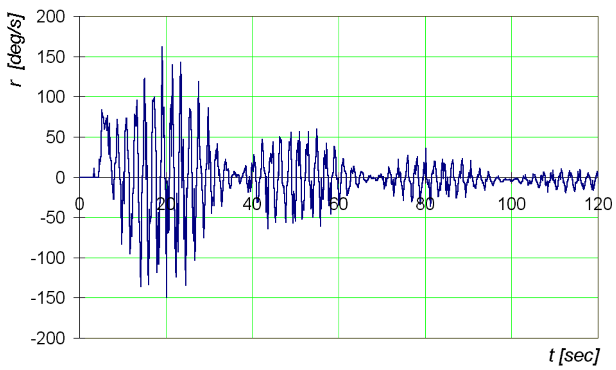

Fig. 18 Course of yaw rate

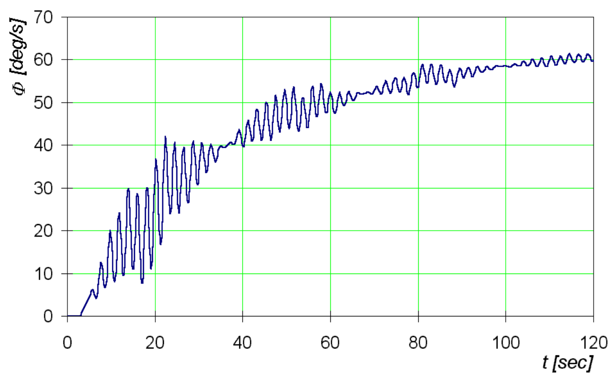

Fig. 19 Course of roll angle

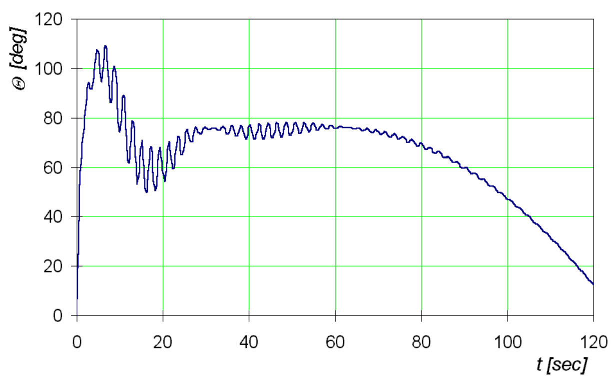

Fig. 20 Course of pitch angle

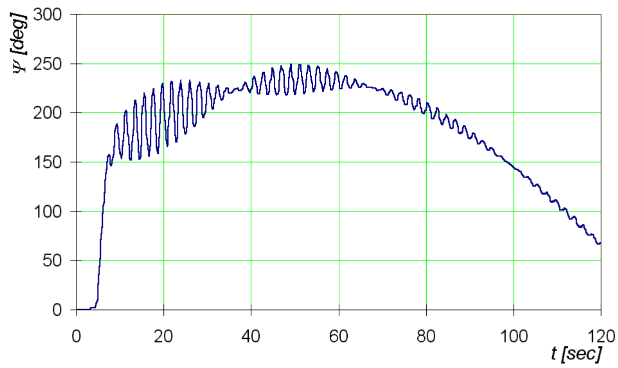

Fig. 21 Course of yaw angle

To perform successfully the Cobra manoeuvre, one should rapidly pull-up the control stick at the commencement of the manoeuvre, putting the aircraft in region of unstable steady states. During those simulations one assumed, that aircraft has the spin tendency and wing rock instability, because of lack of lateral stability at high angles of attack. Occurrences of Hopf and saddle-node bifurcations signify radical changes in aircraft response after those bifurcations At higher angles of attack the MiG-29 fighter aircraft is undergoing wing-rock instability. Figs. 13-21 show time simulation of Cobra manoeuvre. The course of change of the angle of attack α is shown in Fig. 13, the angle of pitch Θ - in Fig.20 and the roll and yaw angles - in Figs.19, and 21. It is seen from these figures that during that manoeuvre the angles of attack and pitch increase suddenly. Maximum values of α, Θ reach 100^o and 110^o respectively. Figs. 13-21 show, additionally developing wing-rock oscillations witch a period approximately 3.5s. Note that magnitude and frequency of those oscillations are irregular and have chaotic character [7], [10]. It can be stated, that during the Cobra manoeuvre the pilot must fix his attention on aircraft control, first of all keeping the aircraft precisely in the vertical plane. It is not simple, because the aircraft during this manoeuvre has strong tendency to bank. Additionally it is observed appearance of wing rock oscillations.

Conclusions

The main aim of the study was to apply modern methods for investigation of non-linear problems in flight dynamics. Based on the investigation described above, the following conclusions can be drawn:

1. The present results show the value of using continuation and bifurcation methods for analysing the equations of aircraft motion;
2. The efficiency of the methods makes it possible to analyse complicated aerodynamic models using the complete equations of motions for the whole range of control surface deflections;
3. Knowledge of such deflections, which cause bifurcation allows us to select the most probable scenario of occurrences before the accident, and to escape from risky motions;
4. The need for a precise description of aerodynamic loads is a fundamental cause of difficulties;
5. The presented approach can be applied to the prediction of space behaviour of aircraft. Therefore, it can be also applied to modification of aircraft dynamic characteristics.

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