OPTIMISATION OF COMPOSITE AIRCRAFT PANELS USING EVOLUTIONARY COMPUTATION METHODS

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Abstract

The optimisation of laminated composite structures is a highly nonlinear problem with both continuous and discrete design variables. For this kind of optimisation problem particularly evolutionary algorithms are much better suited than traditional techniques, based on mathematical programming principles. In this paper an approach is presented that uses evolutionary strategies for the minimum weight design of composite structures. In order to reduce the computer running time, the inherent parallel structure of evolution strategies has been fully utilized in the computer code, making the program well suited for multi-processor environments. Since the finite element method is employed for the structural analysis task, the optimisation technique can be applied to any kind of structure. As an example the weight of a stiffened composite panel is minimised. The obtained results show that the developed evolutionary optimisation procedure is a powerful tool for designing complex aircraft structures made of composite materials.

1 Nomenclature

Roman Symbols:

f	objection function
$\boldsymbol{g(x)}$	vector of inequality constraints
$\boldsymbol{h(x)}$	vector of equality constraints
h	stiffener height
n_skin	ply stack number in one symmetric half of skin
n_stiff	ply stack number in one symmetric half of stiffeners
r	penalty factor
type	type of stiffeners
sn	stiffener number
$\boldsymbol{x}$	design vector
N_x	compression load
P	penalty function
W	weight
$\boldsymbol{Z}$	vector of random values

Greek Symbols:

ε	strain
λ	number of offspring individuals
μ	number of parent individuals
$\boldsymbol{\sigma}$	vector of standard deviation
ϕ_skin_i	fibre orientations in skin ply stacks
ϕ_stiff_i	fibre orientations in stiffener ply stacks
Λ	buckling load factor
Φ	penalised objective function

Indices:

max	maximum value
offspring	offspring individual
parent	parent individual
ult	ultimate value

2 Introduction

Primary objectives concerning the development of next generation commercial transport aircraft are to cut severely the manufacturing costs and to achieve considerable structural weight reductions. A promising way to attain these aims is to extend the application of fibre-reinforced composite materials which is still lagging behind the usage in primary structures of military and general aviation aircraft. The use of laminated composites leads to a considerable increase in design parameters. Hence, minimum weight solutions for more complex structural components such as stiffened fuselage panels and wing skins can be obtained only by using numerical optimisation tools, often combined with finite element analysis.

The optimisation of stiffened panels made of laminated composites is characterised by a mix of continuous and discrete design variables. Typical examples for discrete parameters are the ply thickness and the ply orientation angle. The plies of the skin and stiffener laminates have fixed thicknesses and usually only a limited set of ply orientation angles such as 0, 90 and ±45 degrees is used in real structures. Other discrete parameters are the number and type of stiffeners, whereas the stiffener height is a continuous design variable. This discontinuous design space complicates the optimisation process. In particular, the traditional continuous optimisation techniques based on mathematical programming principles are not very well suited to this task. Therefore, numerical methods have to be applied which are able to deal with both continuous as well as discrete design variables. In recent years particularly techniques based on natural evolution principles have been developed to solve this kind of optimisation problem. The best known of these evolutionary algorithms are evolution strategies (ES) and genetic algorithms (GA). A general survey of these techniques has been given by Schwefel [1].

Applications of evolutionary algorithms to the optimum design of composite structures can be found in several recent studies [2 - 9]. Haftka et al. [2 - 4] solved the stacking sequence design problem of flat laminated plates subject to buckling and strength constraints by using genetic algorithms. In [5 - 8] similar techniques have been applied to minimise the weight of stiffened panels made of fibre-reinforced laminates.

Crossley et al. [8] optimised the energy absorption capability of stiffened composite panels being part of the floor structure of a crashworthy helicopter airframe. In addition to the laminate parameters the number and shape of the stiffeners had been included as design variables in the optimisation process. Since no strictly analytic techniques were available to evaluate the crushing strength of the panels, a semi-empirical analysis method was used. The need for empirical data limited the range of design variables addressed in this work to those panel configurations that had been crush tested.

In principal, evolutionary algorithms need a very large number of analyses to evaluate the effect of design changes. This requires fast structural analysis tools to keep the computer running time within practical limits. Therefore, usually closed form solution methods and approximations are chosen to determine structural strength, stiffness and buckling loads. One of the few exceptions is given in [9]. Todoroki et al. used GAs to optimise the strength of laminated composite plates with stress-concentrated open holes. In their approach a finite element analysis had been chosen to determine stress and strain distributions around the hole. Since the geometric shape of the plates remained unchanged during the optimisation process, only fixed meshes were used.

In the present paper an optimisation method for the minimum weight design of composite structures is presented that is based on evolution strategies. This technique has been chosen, because it is better suited to deal with continuous design variables than the genetic algorithms. In the present approach evolution strategies have been combined with the finite element method as the basic structural analysis tool. Since parametric finite element models are used, even different design configurations can be considered in the optimisation process. In order to increase the computational speed parallel computing techniques have been applied. The effectiveness of the method is shown by optimising a stiffened panel made of graphite-epoxy laminates.

3 Theoretical background

The approach to improve or optimise a structure implicitly presupposes changes of the structure. The potential for change is expressed in terms of permissible ranges of a group of design variables which form the design vector ***x***:

$$x = [x_1, x_2, x_3 \ldots x_n]. \qquad 3.1$$

The design variables x_i are parameters controlling the geometry and material properties of the structure. Depending on the type of parameter, design variables can take either continuous or discrete values.

The aim of the optimisation process is to find among all feasible design vectors $\boldsymbol{x}$ the one that minimises or maximises an objective or fitness function f. This function characterises the efficiency of the solution. In structural applications often the weight W is used as objective function which has to be minimised in this case:

$$W = f(x_1, x_2, x_3 ... x_n) \rightarrow \min . \qquad 3.2$$

Commonly, the design space spanned through the design variables is limited, thus leading to a constrained optimisation problem. Other constraints which have to be considered are structural limits such as stress and strain allowables, maximum deformations and buckling loads. The transformation of the constrained problem into an unconstrained one is done by associating penalties with all constraint violations and including them in the fitness function evaluation:

$$\Phi = f(\boldsymbol{x}) + r \cdot P[\boldsymbol{h}(\boldsymbol{x}), \boldsymbol{g}(\boldsymbol{x})]. \qquad 3.3$$

Generally, the penalty function P consists of equality $\boldsymbol{h}(\boldsymbol{x})$ as well as inequality constraints $\boldsymbol{g}(\boldsymbol{x})$. The penalty factor r weighs the penalty in case the limits or restrictions are violated.

Like other evolutionary algorithms evolution strategies are heuristic optimisation search methods suitable for solving complex problems. They are derived from the biological evolution process and are based on the principle of Darwin's theory of survival of the fittest. These algorithms maintain a population of individuals for each iteration step. Each individual represents a potential solution in the design space. Like a population of living creatures in biology a population of structures is considered. These structures are evolved over generation or iteration cycles and the characteristics of the parent structures are passed to the offspring structure generation. Structural individuals with better fitness values regarding the objective function have better chances to survive.

Each individual structure is represented by a pair of vectors $[\boldsymbol{x}, \boldsymbol{\sigma}]$. The first vector $\boldsymbol{x}$ contains the design variables characterising the structure and defines a point in the search space. The second vector $\boldsymbol{\sigma}$ stores standard deviations representing the step size in the optimisation process. The following example gives the stiffener configuration of a stiffened panel. The design variable x_1 describes the number of stiffeners, x_2 the height and x_3 the thickness of a single stiffener. Without restricting generality all standard deviations σ_i have been set to one. This vector pair represents a panel with four stiffeners being 24.6mm high and 3.2mm thick:

$$[\boldsymbol{x}\ \sigma] = \left[\begin{pmatrix} 4 \\ 24.6 \\ 3.2 \end{pmatrix} \begin{pmatrix} 1 \\ 1.0 \\ 1.0 \end{pmatrix} \right]. \qquad 3.4$$

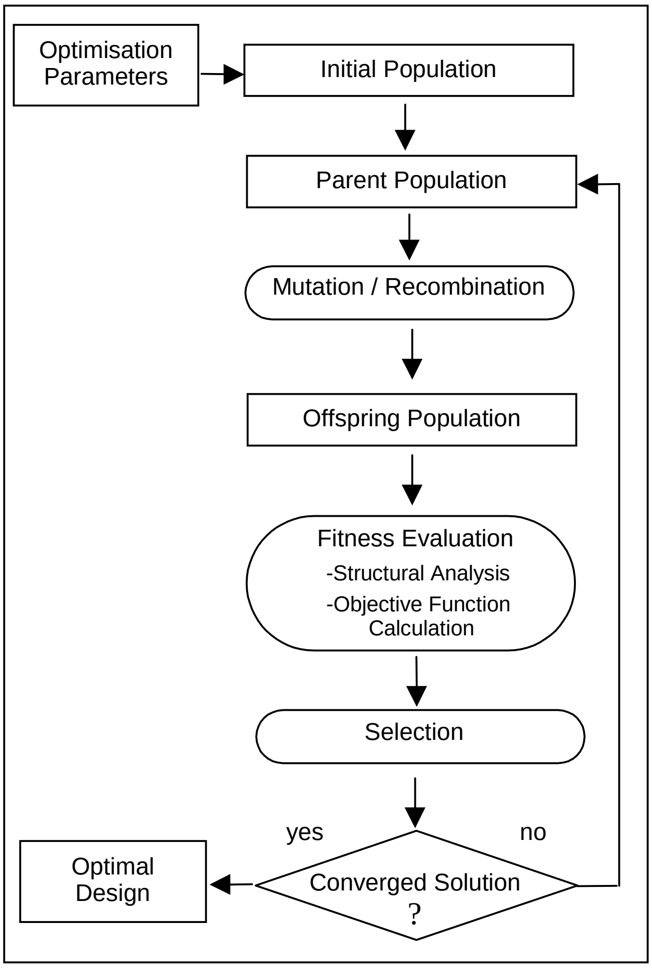

Fig. 3.1: Flow chart of evolution strategies

The optimisation procedure starts with an initial population of solutions which is improved through an iteration process. The structure of this process is given in Fig. 3.1.

The main steps of the procedure are:

Step 0: *Initialisation*

An initial population is chosen randomly consisting of μ parent individuals.

Step 1: *Mutation / Recombination*

Starting from the design points characterised by the parent individuals new design points are generated. Each parent individual creates λ/μ offspring individuals on average, so that a total of λ new offspring individuals are available. Mutations are realised using the following equation:

$$\boldsymbol{x}_{offspring} = \boldsymbol{x}_{parent} + \boldsymbol{Z}(\boldsymbol{0},\sigma). \qquad 3.5$$

The values of the added vector $\boldsymbol{Z}$ are independent random Gaussian numbers with a mean of zero and the standard deviations σ_i of the design variables. This is in accordance with observations in biological evolution that smaller changes occur more often than larger ones.

The recombination operator exchanges the genetic material (the values of the vectors $\boldsymbol{x}$ and $\boldsymbol{\sigma}$) between two parents. Mainly, there are two types of recombination operators, the discrete and the intermediate one. Using the discrete operator each component of the offspring design vector comes from the first or the second pre-selected parent vector. In case of intermediate recombination the mean value of both parent design variables is the new offspring value. The recombination operator can be applied also in a global mode, where a new pair of parents is selected for each component of the offspring vector.

Step 2: *Fitness evaluation*

The fitness of each structural individual is evaluated by using the objective function.

Step 3: *Selection*

The offspring individuals are accepted as new population members if they have better fitness values and satisfied constraints than the parent individuals. The μ best of the existing individuals become parents of the following generation. To choose μ new parent individuals from the pool of individuals is the task of the selection operator. There are two major selection methods developed for evolution strategies. These are the (μ+λ)- and (μ,λ)-strategy that differ in the individuals placed at the disposal. Fig. 3.2 shows the principle of these selection methods. The (μ+λ)-strategy use both the parent and the offspring individuals of the special iteration step for the selection process. On the other hand in the (μ,λ)-strategy the new population is selected from the set of offspring individuals only. By doing this the life time of each individual is limited to one generation. In all cases the existing population of individuals is reduced by the selection process again to μ new parent individuals for the next iteration step.

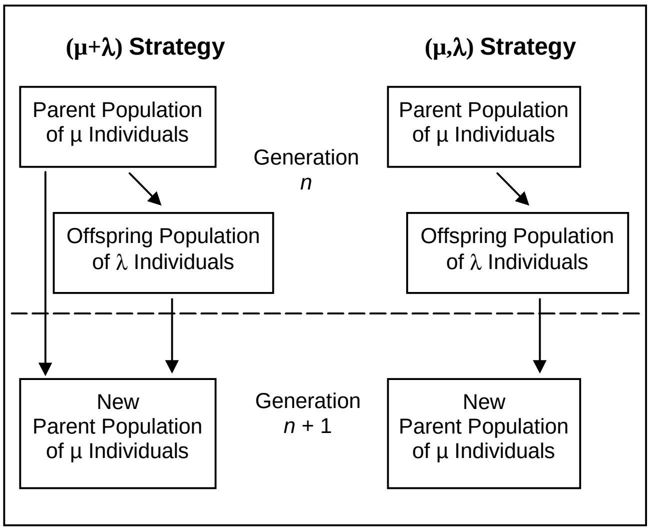

Fig. 3.2: Selection operators

Instead of a single design point search, evolution strategies evaluate simultaneously a population of points in the design space. Therefore, this method requires a larger number of fitness evaluations compared to continuous deterministic mathematical programming methods. Consequently, the resulting computational costs are higher. This drawback is even more significant, if numerical methods such as finite elements are used to analyse the structural behaviour. Fortunately, evolution strategies have an inherent parallel structure. Particularly, the fitness evaluation of the newly

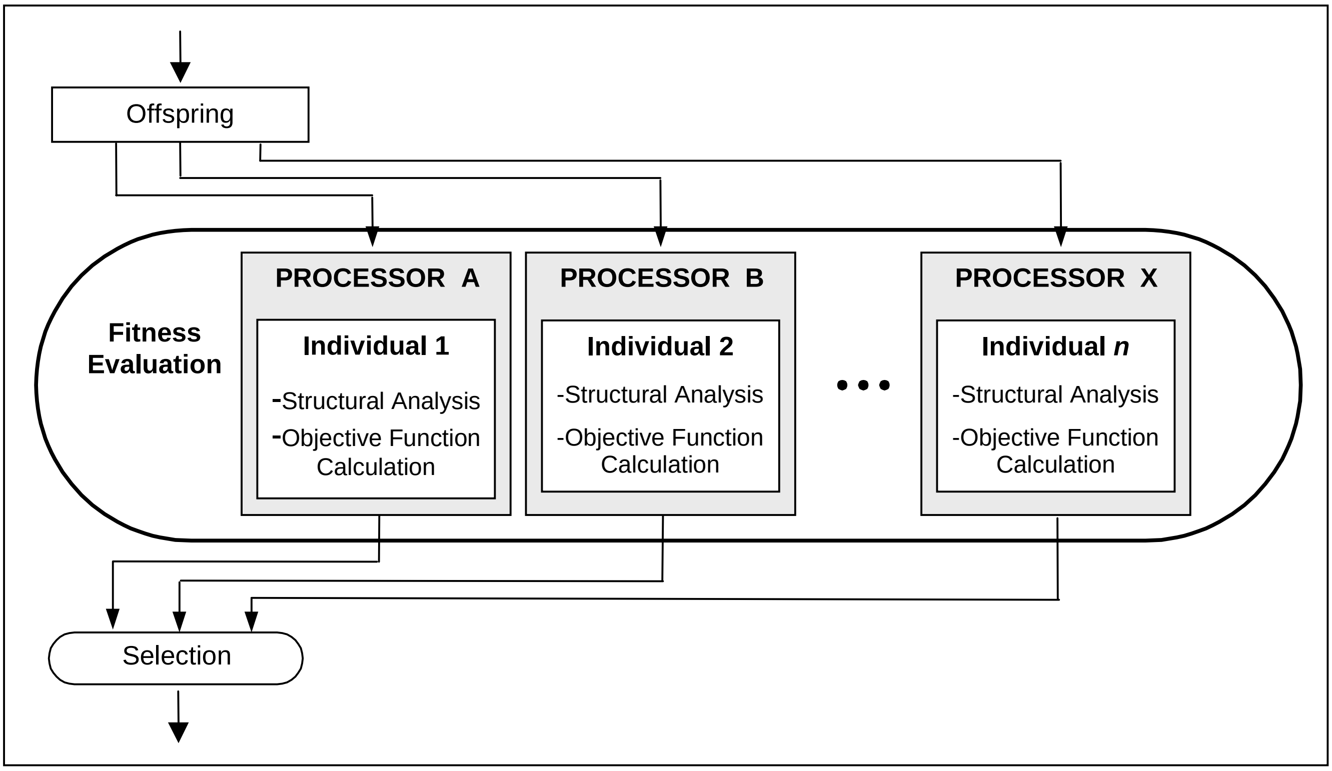

Fig. 3.3: Fitness evaluation in a parallel computing architecture

created offsprings can be done independently for each individual. In principle, this permits to perform the time consuming structural analyses concurrently for all individuals. In case parallel computing systems are available, the numerical simulation of the structural behaviour can be distributed on different processors (Fig. 3.3). Thus, the total running time of the optimisation procedure can be reduced considerably. The present evolutionary computation approach utilizes this capability. The developed program is based on the parallel environment architecture of a code written by Axmann et al. [10] for the aerodynamic design of airplane wings. This architecture permits parallel processing on computer clusters as well as on multiprocessor systems.

4 Problem description

4.1 The structural model

In this study a stiffened flat composite panel is investigated. The panel is 500mm long and 260mm wide and has equally spaced stringers (Fig. 4.1).

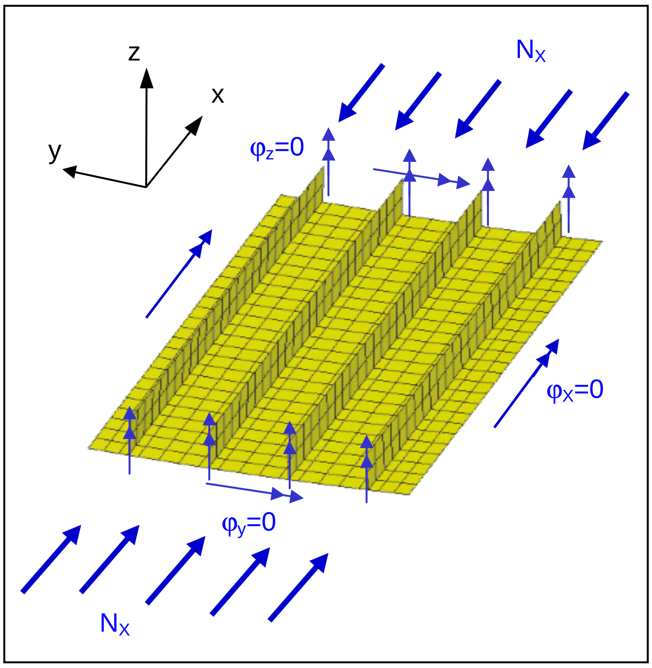

Fig. 4.1: Stiffened panel with boundary conditions

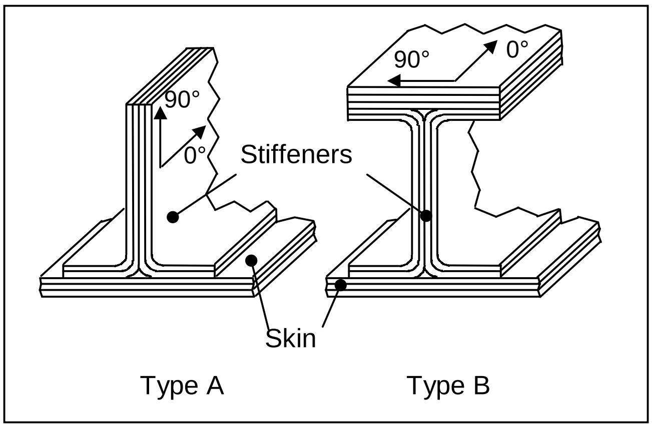

Fig. 4.2: Stiffener cross-sections

Both the number as well as the type of stiffeners are design variables which are varied during the optimisation process. The two stringer types shown in Fig. 4.2 are considered.

Skin and stiffeners of the panel are composed of 0.125mm thick layers made of unidirectional graphite/epoxy prepregs. The properties of the material are summarised in Table 4.1. Blade and flange laminates of the stiffeners are identical. The type B stiffeners have additional plies on top of the upper flanges (see Fig. 4.2). These strips have the same lay-up scheme as the skin. Thus, only two different laminates are used which are both balanced and symmetric. The fibre orientations of the plies are restricted to 0, ±45 and 90 degrees.

Young's modulus (longitudinal)	E_{11}	136000N/mm²
Young's modulus (transversal)	E_{22}	9755N/mm²
Shear modulus	G_{12}	4985N/mm²
Poisson's ratio	ν_{12}	0.3
Density	ρ	1500kg/m³

Table 4.1: Material properties

The panel is subjected to a compressive load N_x=750N/mm in the x direction. All edges of the skin as well as the stiffener ends are clamped as shown in Fig. 4.1.

4.2 The optimisation problem

The aim of the optimisation is to find lay-up schemes for the skin and stiffener laminates as well as a stiffener configuration that result in a minimum weight:

$$W(x_1, x_2, x_3 \ldots x_n) \rightarrow \min . \qquad 4.1$$

For the present example 45 design variables were defined which are summarised in Table 4.2.

Constraints of the problem are that neither local nor global buckling is allowed below the given limit load and that the maximum laminate strains must not exceed the strain allowables of the material at ultimate load. As already mentioned the constrained optimisation problem must be transformed into an unconstrained problem by introducing penalty factors in combination with penalty functions. For the

Design Variables		Explanation	Design Space
x_1	*type*	stiffener type (see Fig. 4.2)	$type \in \{A,B\}$
x_2	*sn*	stiffener number	$sn \in N$, $1 \leq sn \leq 8$
x_3	*h*	stiffener height	$h \in R$, $10mm \leq h \leq 80mm$
x_4	*n_skin*	ply stack number in one symmetric half of skin	$n\_skin \in N$, $1 \leq n\_skin \leq 20$
x_5	*n_stiff*	ply stack number in one symmetric half of stiffeners	$n\_stiff \in N$, $1 \leq n\_stiff \leq 20$
$x_6 \ldots x_{25}$	$\phi\_skin_6 \ldots \phi\_skin_{25}$	fibre orientations in skin ply stacks	$\phi\_skin_i \in \{0°, \pm45°, 90°\}$
$x_{26} \ldots x_{45}$	$\phi\_stiff_{26} \ldots \phi\_stiff_{45}$	fibre orientations in stiffener ply stacks	$\phi\_stiff_i \in \{0°, \pm45°, 90°\}$

Table 4.2: Design variables

present problem following modified objective functions Φ were defined:

- for $|\varepsilon_{max}| \leq |\varepsilon_{ult}|$ and $\Lambda \geq 1$
$$\Phi = W(x_1 \ldots x_n), \qquad 4.2$$
- for $|\varepsilon_{max}| > |\varepsilon_{ult}|$
$$\Phi = W(x_1 \ldots x_n)\{900000\,(|\varepsilon_{max}| - |\varepsilon_{ult}|) + 3\},$$
- for $|\varepsilon_{max}| \leq |\varepsilon_{ult}|$ and $\Lambda < 1$
$$\Phi = W(x_1 \ldots x_n)\left\{1 + 12\left(\frac{1}{\Lambda} - 1\right)\right\},$$

where Λ is the buckling load factor, ε_{max} the maximum laminate strain and ε_{ult} the ultimate strain. In cases where the strain constraint is violated no buckling analysis is performed, and a higher penalty factor is used than in case of premature buckling. The penalty parameters employed are the result of extensive numerical testing.

Since all laminates are symmetric, only half of the plies were taken as design variables. In order to get a further reduction of variables the laminates were composed of ply stacks similar to [7]. The constraint of balanced laminates was enforced by associating a +45° with a –45° ply into a ±45° stack of two plies. The number of the fibre orientations ϕ_skin_i and ϕ_stiff_i depends on the number of ply stacks in skin and stiffeners (*n_skin* and *n_stiff*).

For the optimisation a (5,25)-strategy was chosen. In one iteration cycle 25 offspring individuals are created by changing 5 parent structures. The parent structures for the next iteration cycle are selected only from the population of the previous offspring individuals. The optimisation is terminated after 100 iteration cycles or generations. If the solution converges the iteration is interrupted prematurely.

Both strength as well as buckling analyses were performed using the finite element code ANSYS [11]. The panels were modelled by 8-node quadrilateral shell elements, being capable of modelling multi-layered composite materials. The number of elements used varied depending on both the number and the type of stiffeners. The element meshes were chosen so as to yield reliable results for the buckling loads.

5 Results

5.1 Panel with type A stiffeners

Firstly, the optimisation method is employed to find minimum weight solutions considering only type A stiffeners. As starting design a feasible quasi-isotropic panel with 3 stiffeners is used, which satisfies both the strain as well as the buckling constraint.

	Initial Quasi-Isotropic Panel	Optimised Panels			
		Panel A-1	Panel A-2	Panel A-3	Panel A-4
w	1.23 kg	0.62 kg	0.63 kg	0.77 kg	0.78 kg
$\lvert\varepsilon_{max}\rvert / \lvert\varepsilon_{ult}\rvert$	0.996	0.986	0.996	1.000	0.996
Λ	3.30	1.01	1.00	1.21	1.11
sn	3	4	4	3	3
h	15	16 mm	72 mm	24 mm	21 mm
n_skin	12	5	3	6	6
n_stiff	12	5	3	5	6
Skin Laminate	$[[0_2/\pm45_2/90_2]_3]_S$	$[90_2/0_8]_S$	$[0_6]_S$	$[90_2/0_4/90_2/0_4]_S$	$[0_6/\pm45/90_2/0_2]_S$
Stiffener Laminate	$[[0_2/\pm45_2/90_2]_3]_S$	$[\pm45/0_4/90_2/0_2]_S$	$[\pm45/0_4]_S$	$[\pm45/90_2/0_2/\pm45/0_2]_S$	$[\pm45/0_4/\pm45/90_2/0_2]_S$

Table 5.1: Panel designs with stiffeners of type A

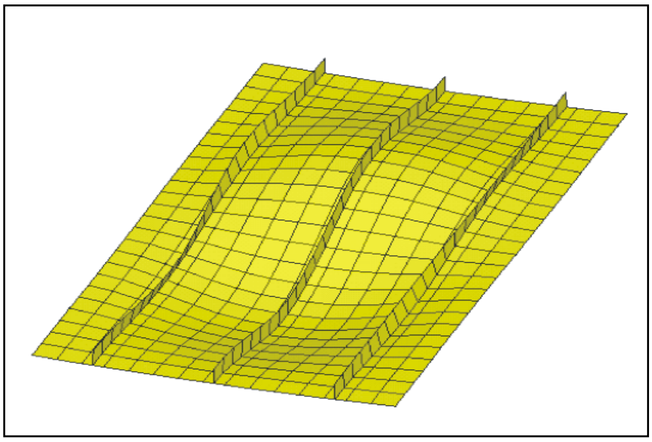

Fig. 5.1a: Buckling mode of the initial quasi-isotropic panel

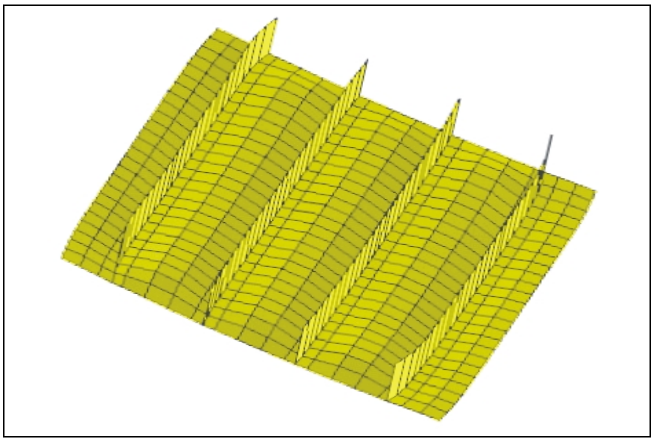

Fig. 5.1b: Buckling mode of panel A-1

The best designs obtained are summarised in Table 5.1. The configuration and the buckling modes of the initial panel and the minimum weight panel A-1 are shown in Fig. 5.1a and Fig. 5.1b. Compared to the initial design a weight saving of 50% has been achieved. This is obtained by reducing the laminate plies and by increasing the number of stiffeners. The optimisation yields two solutions that have nearly the same weight but differ considerably regarding their configuration. Whereas the panel A-1 consists of laminates with 5 ply stacks and low stiffeners, the panel A-2 has thinner laminates but higher stiffener blades. This result clearly demonstrates the ability of evolutionary methods to find not only the global optima but also several near optimal solutions. The advantage of this feature is, that the design engineer will have the final choice among several solutions, enabling him to consider additional aspects such as manufacturing effort and costs.

5.2 Panel with type A and type B stiffeners

In the next step both stiffener types (see Fig. 4.2) are included in the optimisation process. Load and starting solution are the same as in chapter 5.1.

The results obtained are listed in Table 5.2. Both the stiffener configuration as well as the buckling modes of the two lightest panels are

	Initial Quasi-Isotropic Panel	Optimised Panels		
		Panel B-1	Panel A-1	Panel B-2
w	1.23 kg	0.58 kg	0.62 kg	0.68 kg
$\lvert\varepsilon_{max}\rvert / \lvert\varepsilon_{ult}\rvert$	0.996	0.981	0.986	0.910
Λ	3.30	1.03	1.01	1.12
type	A	B	A	B
sn	3	3	4	4
h	15	10 mm	16 mm	11 mm
n_skin	12	4	5	4
n_stiff	12	4	5	5
Skin Laminate	$[[0_2/\pm45_2/90_2]_3]_S$	$[0_8]_S$	$[90_2/0_8]_S$	$[90_2/0_6]_S$
Stiffener Laminate	$[[0_2/\pm45_2/90_2]_3]_S$	$[\pm45/0_6]_S$	$[\pm45/0_4/90_2/0_2]_S$	$[\pm45/0_8]_S$

Table 5.2: Panel designs with stiffeners of both types

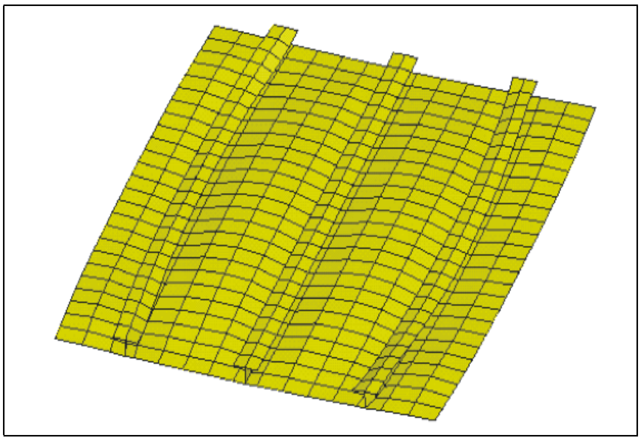

Fig. 5.2a: Buckling mode of panel B-1

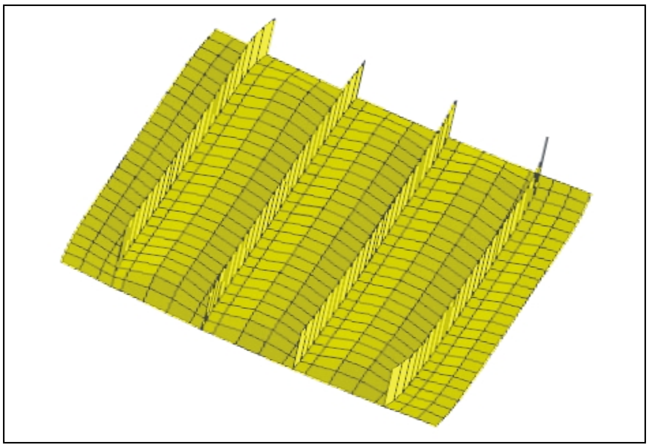

Fig. 5.2b: Buckling mode of panel A-1

compared in Fig. 5.2a and 5.2b.

Again the optimisation yields several optimal solutions that satisfy the strength and buckling constraints. The extension of the design space by an additional stiffener type has enabled the optimiser to find a better solution than before. Compared to the best solution with a type A stiffener the B-1 panel is 7% lighter. This reduction has been achieved partly by omitting the 90° plies in the laminates, resulting in a skin laminate with only 0° plies left. Since such a lay-up scheme is not feasible for practical applications, the designer has to chose a better suited solution which comes close to the optimal panel. This example emphasizes the advantage of evolution strategies to provide several near optimal panels that differ in laminate stacking sequence and geometry.

5.3 Parallel computing

In order to investigate the performance of the parallel computing features of the optimisation technique benchmark tests were carried out. For these tests up to four processors were used to do part of the fitness evaluation concurrently. The number of processors was confined by the limited number of ANSYS software licenses being available.

In Fig. 5.3 the measured computational speed is compared to the theoretical speed-up. The theoretical gain is based on the assumption that the increase in speed is proportional to the number of processors. The graph clearly shows the speed-up effect of parallel computing. As expected the actual gain in speed is lagging behind the theoretical assumption. This is due to the communication and administrative effort that increases with the number of processors.

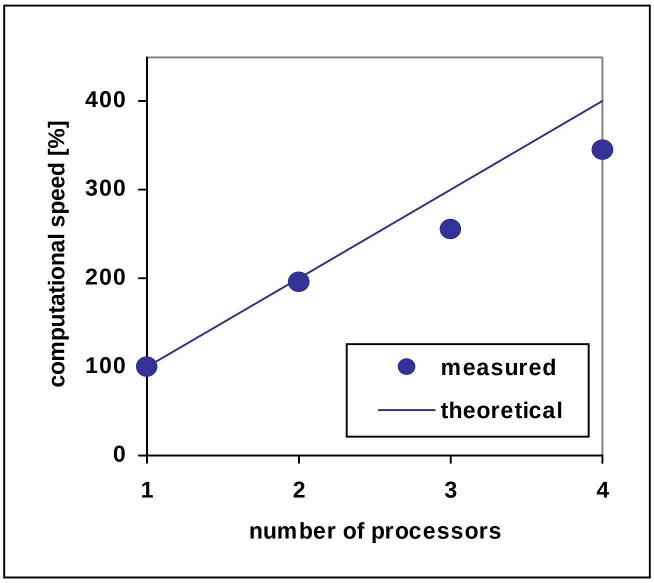

Fig. 5.3: Effect of processor number on speed up

6 Conclusions

An evolutionary computation method for the optimisation of composite structures has been developed. The approach is based on evolution strategies which are well suited for solving mixed-discrete optimisation problems such as the stacking sequence design of laminates. In order to reduce the execution time of the optimisation process parallel computing techniques have been applied, making use of the parallel structure of evolutionary algorithms. This permits to base the structural analysis task fully on numerical tools such as the finite element method.

The applicability of the present approach to complex structures was demonstrated by minimising the weight of a stiffened panel made of laminated composite material. Additional to the laminate lay-ups also the stiffener configuration had been considered as a design variable in this example. It could be shown that the optimisation method is able to find several solutions with optimal or near optimal performance, leaving the ultimate choice between various feasible designs to the structural engineer.

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