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STRUCTURAL DYNAMIC ANALYSIS OF NONLINEAR MULTIBODY SYSTEMS BY A TIME-DISCONTINUOUS GALERKIN FINITE ELEMENT FORMULATION

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Abstract

This paper is concerned with an adaptive time-stepping algorithm to solve the equations of motion of nonlinear constrained multibody systems discretized using the finite element method. A time-discontinuous Galerkin scheme is used as the basis for the formulation. The resulting scheme presents unconditional stability, third order accuracy and high frequency numerical damping. The Lagrange multiplier technique is used to enforce the kinematic constraints among the bodies. The formulation uses Cartesian coordinates to represent the position of each body with respect to an inertial system. The adaptive time-stepping algorithm selects time step sizes that reduce computational cost and maintain the accuracy of the solution. The results presented confirm the high order accuracy of the scheme and the significant reduction in computational cost of the solutions. Two examples are successfully analyzed with the algorithm: a nonlinear oscillator and a simple nonlinear constrained multibody system. In light of the results the algorithm seems to be very promising to treat complex nonlinear elastic multibody systems.

Introduction

Multibody systems usually present general and complex topologies whose accurate modeling implies on carefully addressing factors such as the coordinate systems to be used, the formulation of elastic members, the enforcement of kinematic constraints and the parametrization of finite rotations. Different methodologies of analysis result from the way such factors are addressed and the best methodology to analyze a certain type of multibody system may vary widely.

This work concerns the analysis of nonlinear constrained multibody systems discretized using the finite element method.The formulation uses Cartesian coordinates to represent the position of each body with respect to an inertial frame. The Lagrange multiplier technique is used to enforce the kinematic constraints among the various bodies. Although this approach does not involve the minimum set of coordinates[1], it allows a modular development of finite elements to represent a variety of kinematic constraints, so that general multibody configurations can effectively be modeled. The resulting systems of equations are differential-algebraic in nature. Such systems are stiff due to the presence of high frequencies in the elastic members, and also the infinite frequencies associated with the kinematic constraints. In reality, no mass is associated with the Lagrange multipliers degrees of freedom resulting algebraic equations coupled to the differential equations of the system. This study applies an adaptive time-stepping algorithm to solve the equations of motion of nonlinear constrained multibody systems.

Using the natural framework of the second-order hyperbolic equations, instead of relying on converting the equations to a first-order symmetric hyperbolic form, Hulbert[2] developed time and space-time discontinuous Galerkin finite element methods to solve structural dynamics and elastodynamics problems. The resulting systems of equations are larger than the original ones, and fully coupled, which increases the computational cost of the solution. However, the scheme may be solved in a predictor-multicorrector form, which al-

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leviates the high computational cost of the fully coupled system, maintains the high order of accuracy and unconditional stability of the original scheme, and improves its characteristics of high frequency numerical damping.

An adaptive time-stepping procedure, based on a time-discontinuous Galerkin scheme, for selecting the proper time step size is presented by Li and Wiberg[3]. They use a two-field formulation, namely the P1-P1 formulation[2], which interpolates displacements and velocities as piecewise linear funtions. In the study herein developed a single-field formulation is used with displacements approximated as a quadratic function. The resulting system of equations is smaller than the one resulting from a two-field formulation with the advantage of improved accuracy characteristics[2]. A single degree of freedom nonlinear undamped oscillator is used to assess the applicability of the time adaptive algorithm to nonlinear problems. Then, a nonlinear simple pendulum is analyzed as a two degree of freedom problem and to enforce its length to remain constant a constraint equation is added to the Lagrangian of the system. Due to its inherent high frequency numerical dissipation the time-discontinuous Galerkin scheme does not present the weak instability observed in the Newmark scheme[6] and the high frequency numerical oscillations of the solution are completely damped out at the first time step of the analysis.

A Time-Discontinuous Galerkin Finite Element Method

Consider[2] a partition of the time domain $I = (0, T)$ in the form $0 = t_0 < t_1 < \cdots < t_N = T$ with corresponding time steps $\Delta t_n = t_n - t_{n-1}$ and intervals $I_n = (t_{n-1}, t_n)$. The finite element interpolation functions for the trial displacements are

$$S^h = \{\underline{u}^h \in \bigcup_{n=1}^{N} (P^2(I_n))^{n_{eq}}\} \tag{1}$$

where P^2 stands for second-order polynomial and each member of S^h is a vector consisting of n_{eq} quadratic functions on each time interval I_n. By construction, the interpolation functions are continuous within each time interval and may be discontinuous across time slabs. To enforce the continuity across time intervals a temporal jump operator

$$[\underline{w}(t_n)] = \underline{w}(t_n^+) - \underline{w}(t_n^-) \tag{2}$$

where

$$\underline{w}(t_n^\pm) = \lim_{\epsilon \to 0^\pm} \underline{w}(t_n + \epsilon) \tag{3}$$

is used. The displacement weighting function space is identical to the trial displacement's.

The statement[2] of the time-discontinuous Galerkin finite element method for the single-field formulation, applied to the ordinary differential equations associated with the semidiscrete form of linear elastodynamics is:

Find $\underline{u}^h \in S^h$ such that for all $\underline{w}^h \in S^h$

$$\begin{aligned} \int_{t_{n-1}^+}^{t_n^-} [\underline{\dot{w}}^h \cdot (M\underline{\ddot{u}}^h + C\underline{\dot{u}}^h + K\underline{u}^h - \underline{F})]dt \quad + \\ \underline{\dot{w}}^h(t_{n-1}^+) \cdot M[\underline{\dot{u}}^h(t_{n-1}^+) - \underline{\dot{u}}^h(t_{n-1}^-)] \quad + \\ \underline{w}^h(t_{n-1}^+) \cdot K[\underline{u}^h(t_{n-1}^+) - \underline{u}^h(t_{n-1}^-)] \quad = \quad 0. \end{aligned} \tag{4}$$

In equation (4) $n = 1, 2, \cdots, N$, where N is the number of time intervals. Variables $\underline{u}^h$ and $\underline{w}^h$ are, respectively, displacements and weighting functions; $\underline{\dot{u}}^h$ and $\underline{\ddot{u}}^h$ are, respectively, velocities and accelerations. The last two terms on the left-hand side weakly enforce the initicial conditions for each time interval. These jump terms are stabilizing operators that have the effect of "upwinding" information with respect to time[2]. Also, M, C and K are the mass, damping and stiffness matrices, respectively, and F is the force vector. Since the displacements are interpolated as quadratic functions, the resulting system of equations is 3 times larger than the ones solved by commonly used semidiscrete methods. To circumvent the high computational cost of solving the fully coupled equations, the system is cast in a predictor-multicorrector form, which maintains the characteristics of high order accuracy and unconditional stability of the scheme, besides improving its high frequency numerical damping capability.

The Adaptive Time-Stepping Procedure

In modern structural dynamic analysis it is convenient that an integration scheme allows automatic time step size control. It will reduce computational cost while a required accuracy in the solution is maintained. Based on a two-field formulation of the time-discontinuous Galerkin method Li and Weiberg[3] applied an adaptive time-stepping algorithm to solve linear structural dynamics problems. In their formulation displacements and velocities are interpolated as piecewise linear funtions, resulting in systems of equations 4 times larger than those emanating from commonly used semidiscrete methods.

The work described in this article is concerned with the solution of nonlinear constrained structural dynamics problems. Hulbert[2] has shown the stability and accuracy characteristics of the time-discontinuous Galerkin scheme when applied to linear structural dynamics. However, there is no proof of stability when

the method is used to solve nonlinear problems. A natural measure of stability for the differential equations associated with the semidiscrete form of structural dynamics is the total energy of the system. Indeed, a scheme that eliminates the energy associated with vibratory motions at high frequencies, which are of a purely numerical origin, implies the presence of numerical dissipation and guarantees the stability of the solution. Bauchau[4] has shown a shortcoming of using a method whose stability has only been established for linear systems: energy can be created in the numerical solution process, when applied to a nonlinear system, leading to potential instabilities. Due to the lack of proof of stability for the time-discontinuous Galerkin method when applied to nonlinear constrained systems, the total energy is used to monitor the stability characteristics of the scheme. Also, the use of an enegy-like quantity as convergence criterion does not necessarily imply true energy convergence when dealing with complex constrained nonlinear structural problems[4]. Hence, the total energy of the system is used to estimate an error to control the time step size. The time adaptive algorithm presented by Li and Weiberg[3] is herein applied to the problems of nonlinear constrained structural dynamics. For the automatic time step size control the relative error at a time t_n is defined as

$$\epsilon_n = \left| \frac{E(t_n) - E(\bar{t})}{E(\bar{t})} \right| \tag{5}$$

where E is the total energy, $E(\cdot) = K(\cdot) + V(\cdot)$, i.e, the sum of kinetic and potential energies, and $E(\bar{t})$ is a reference energy of the system. It is expected that the relative errors satisfy the condition

$$\epsilon_n \leq \epsilon^{tol} \quad , \tag{6}$$

where ϵ^{tol} is a specified error tolerance. If requirement (6) is not satisfied, a time step refinement is performed: the corresponding solution is rejected and, given that the convergence rate for the algorithm is $O(\Delta t^3)$, the new time step size that will satisfy the error tolerance criterion is calculated[3, 5] as

$$\Delta t_n^{tol} = \left(\frac{\theta_t \, \epsilon^{tol}}{\epsilon_n} \right)^{1/3} \Delta t_n \quad , \tag{7}$$

where $\theta_t \leq 1.0$ is a reducing factor used to avoid the new predicted time step size being rejected. On the contrary, if the calculated error is much smaller than the tolerance, i.e.,

$$\epsilon_n < \gamma \epsilon^{tol} \tag{8}$$

the solution is accepted but the time step size may be increased according to equation (7) when the criterion (8) is satisfied for a certain successive number of time steps. In equation (8) γ is a number much smaller than 1.0.

Numerical Examples

In this section, two numerical examples are presented to assess the applicability of the fore mentioned algorithm to nonlinear problems of structural dynamics. The first example deals with a single degree of freedom nonlinear undamped oscillator. The second example envolves a kinematic constraint, which increases the complexity of the problem.

Nonlinear Oscillator

A nonlinear undamped oscillator represented by

$$m\ddot{\underline{u}} + k\underline{u}^3 = 0 \tag{9}$$

is studied. The system has mass $m = 1.0kg$ and stiffness $k = 25.0N/m$. Initial conditions are $\underline{u}(0) = 0.0$ and $\dot{\underline{u}}(0) = -1.5m/s$. The solution is computed for 30 seconds with Δt = 1.0E-03, i.e., 30000 time steps, without adaptivity. The total energy is $E = 1.125$ and at the last time step it shows an inaccuracy of the order of 1.0E-12. The total CPU time is 337 seconds. The adaptive algorithm with $\theta_t = 1.0$, $\gamma = 1.0$ and ϵ^{tol} = 1.0E-06 reduces the computation time to 20 seconds and 1234 time steps, i.e., less than 6% of the CPU time needed for the solution without adaptivity. The total energy is preserved to the accuracy of 1.0E-07. Results for displacements, velocities and accelerations are in excellent agreement with the exact solutions. Figure (1) presents accelerations obtained with the time-discontinuous Galerkin adaptive algorithm (TDG) in perfect agreement with the exact solution. Figure (2) shows the results for the time step

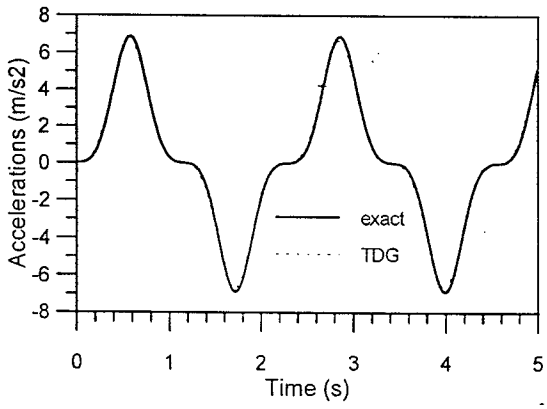

Figure 1: Accelerations for the nonlinear oscillator.

sizes throughout the calculation period. As expected, the algorithm increases, and reduces, the time step size according to the calculated error at each time step.

It is important to remark that since this is a very simple problem the trend observed in energy error for

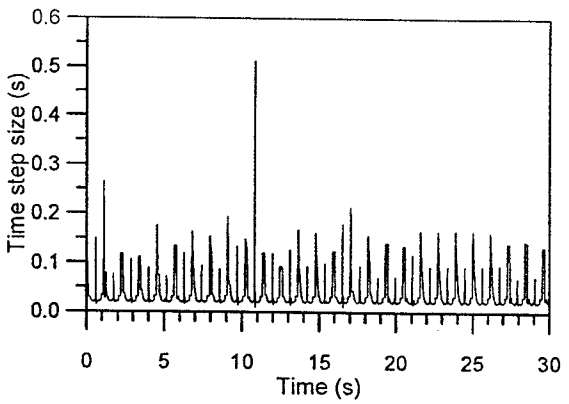

Figure 2: Time step size for the nonlinear oscillator.

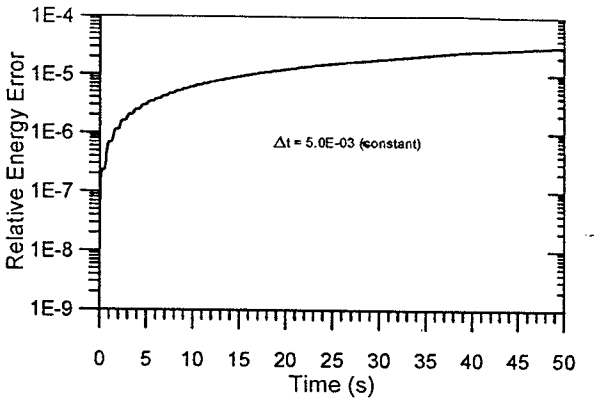

Figure 3: Relative energy error for the nolinear pendulum with constant time step.

this example will certainly not be the same in case of complex nonlinear problems. The following example will allow to outline some conclusions regarding such a matter.

Simple Pendulum

Consider a simple pendulum problem modeled as a point mass m and a massless rigid link of length l. Two degrees of freedom u_x and u_y, respectively, vertical and horizontal displacements, describe the position of the mass m. Only gravity acts upon the system. Nonlinearity is due to the initial conditions. Through an augmented Lagrange multiplier technique a constraint equation that guarantees the constant length l of the pendulum is added to the system. Such a constraint introduces an infinite frequency into the system of equations, which then becomes prone to numerical instabilities and oscillations of a purely numerical origin. Cardona and Geradin[6] have shown the impossiblity of solving this problem with the Newmark time integration scheme. Nevertheless, the time-discontinuous Galerkin scheme efficiently solves this nonlinear constrained problem as shown by Damilano[7].

Despite the reduced number of degrees of freedom and its rigid body nature, this problem is studied aiming the possibility of applying the technique herein described to more representative multibody systems, e.g., problems with large number of degrees of freedom, several kinematic constraints and nonlinear elastic members. For the present study $m = 1.0kg$, $l = 0.5m$, $g = 9.81m/s^2$, and the initial conditions are $u_x = 0.5m$, $u_y = \dot{u}_x = 0.0$, $\dot{u}_y = -1.695m/s$. The solution is calculated for 50 seconds and the numerical results are in excellent agreement with their analytical counterparts. The reduced loss in total energy observed in the first example is not prresent in this case. Initially solved without adaptivity, i.e., with a constant Δt, a large increase in relative energy error is observed during the first 10 seconds of the solution, as shown in Figure (3). In fact, it increases about 3 orders of magnitude. This error keeps increasing during the computations of the solution, however, at a low gradient in the instants that follow the initial 10 seconds. As previously pointed out, the presence of a kinematic constraint makes this problem prone to numerical instabilities and oscillations. Algorithms that are unconditionally stable but present no numerical dissipation, such as the Newmark method, are not able to solve this type of problem. The inherent high frequency numerical dissipation of the time-discontinuous Galerkin method completely eliminates the undesired high frequency numerical oscillations. This numerical dissipation guarantees the stability of the scheme. However, it also implies the energy decaying results observed. A conventional analysis of the scheme based on the characteristics of the amplification matrix[8], for linear systems, results the period elongation $\Delta T/T = \omega^4\Delta t^4/270 + O(\omega^6\Delta t^6)$, and the algorithmic damping $\zeta = \omega^3\Delta t^3/72 + O(\omega^5\Delta t^5)$, where $\omega^2 = k/m$. Thus the scheme is third-order accurate. However, there is no guarantee the same accuracy will be observed with nonlinear constrained systems. To assess the order of accuracy of the scheme applied to the nonlinear pendulum, the results at time $t = 10$ seconds were used to calculate the errors of the solution as functions of the time step size. The results are presented in Figure (4), where u, v, and E stand for displacements, velocities and total energy, respectively. In fact, the scheme is third-order accurate. However, there is no guarantee that the same high order of accuracy will be maintained by the scheme in the solution of different nonlinear problems.

The time adaptive algorithm predicts the solution for two different error tolerances, 1.0E-05 and 1.0E-04, with $\theta_t = 0.95$ and $\gamma = 0.6$ for both cases. Figure

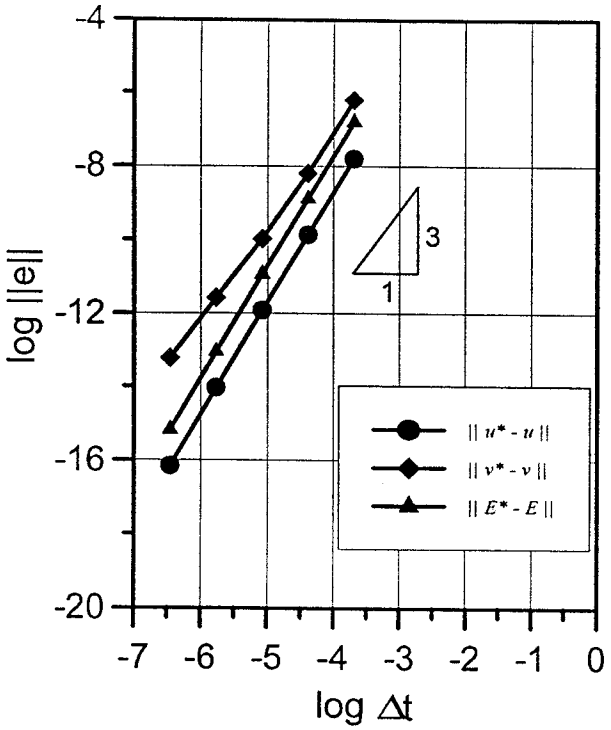

Figure 4: Rate of convergence at time $t = 10.0$ seconds for the nonlinear pendulum.

(5) shows the smooth results for horizontal accelerations at the first second of calculations. There are no high frequency numerical oscillations present in the response. The solution obtained without time adaptivity falls on top of its analytical counterpart. The adaptive time-stepping routine allows time step changes right at the very first time step of integration, which explains the minor difference around $t = 0.1$ seconds. However, the scheme is capable of recovering from that slight error and brings the solution to coincide with the exact one. Accelerations at the last two seconds of computation are shown in figure (6). Notice the error in period elongation as large time step sizes are used. However, it seems that the changes in time step size carried out by the adaptive algorithm did not affect the amplitude of the response. The estimated errors in energy for the initial 10 seconds of computation are presented in figure (7). It is important to remember that the numerical dissipation inherently present in the scheme results a total energy decay. Evidently, it turns out to be impossible for the algorithm to use the same reference energy $E(\bar{t})$ in equation (5), throughout the entire computation of the soltuion. Thus, the calculation of the responce starts using $E(\bar{t})$ equal to the initial total energy of the system. Then, if at a given time step the convergence criterion (6) is not satisfied the time step size will be reduced. At each time step that Δt has to be reduced, this process goes on up to

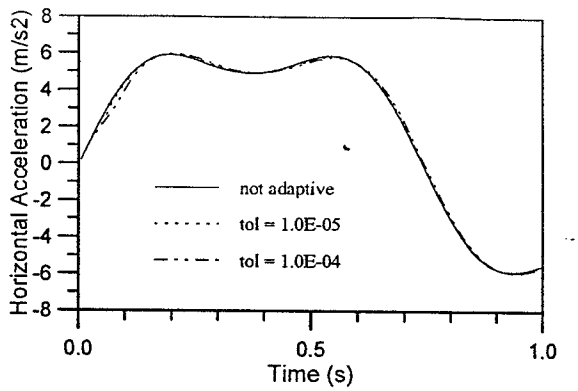

Figure 5: Horizontal accelerations for the simple pendulum.

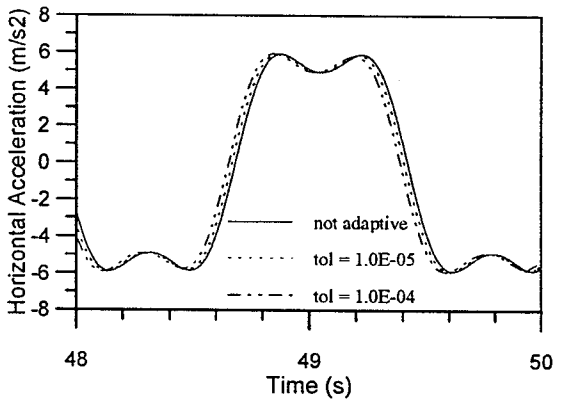

Figure 6: Horizontal accelerations for the simple pendulum.

a maximum number of repetitions, or until Δt reaches a specified minimum value. In either case, if the error is not larger than $1.1\epsilon^{tol}$, the solution is accepted but the new energy of reference is the average between the reference energy at the previous time step and the energy computed at the present time step. Results for both analyses show the energy errors bounded by their respective limits. In both cases, the maximum value allowed for the time step size is Δt = 5.0E-02. A spectral analysis of the solution for constant Δt showed that Δt = 3.7E-02 is the largest time step size that could accurately integrate the equations of motion. Figure (8) shows that the algorithm keeps the time step size within the necessary limit for accuracy. During the entire period of the computations the kinematic constraint, which guarantees the constant length of the pendulum, is satisfied at the order of machine accuracy, as shown if figure (9). The solution without adaptivity used 223 seconds of CPU time

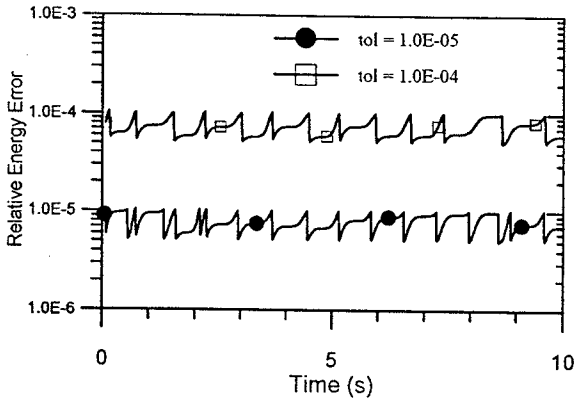

Figure 7: Error in energy for the simple pendulum.

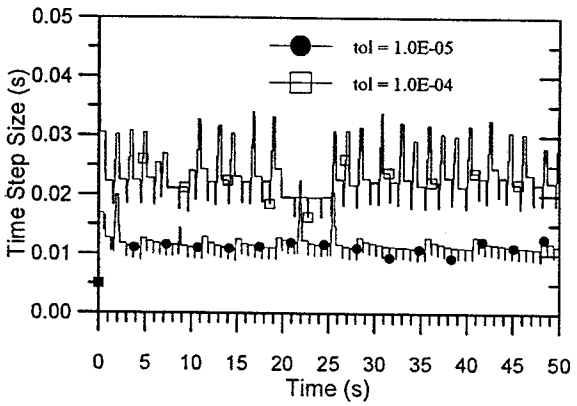

Figure 8: Time step size variation for the simple pendulum.

and 10000 time steps. The adptive algorithm with the error tolerance 1.0E-05 used 57% of the computation time above and reduced the number of time steps to 4341. Relaxing the error tolerance to 1.0E-04 results a reduction in time of computation to 36% of the computation time for a constant Δt, further reducing the number of time steps to 2195.

Conclusion

An adaptive time-stepping algorithm is applied to a time-discontinuous Galerkin scheme on the solution of a nonliear constrained multibody system. The resulting systems of differential-algebraic equations are larger than the original ones, fully coupled, and prone to high frequency oscillations and instabilities of a purely numerical origin. The inherent high frequency

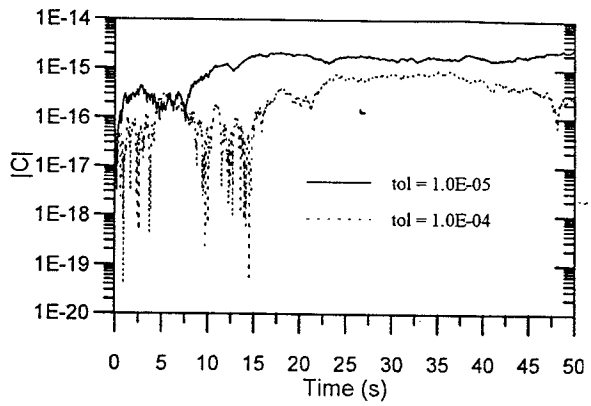

Figure 9: The kinematic constraint for the simple pendulum.

numerical dissipation of the scheme completely eliminates the undesired instabilities. A single field formulation of the time-discontinuous Galerkin scheme using quadratic functions produces coupled systems that are 3 times larger than the original system of equations. The high computational cost of solving the fully coupled systems of equations is reduced when a predictor-multicorrector form is used in the computation of the response. The time adaptive algorithm further reduces the computation time by enlarging the time step sizes in certain instants of the solution. The total energy of the system, which monitors the stability characteristics of the response, is used as error measure. The time step sizes are adjusted such that these errors are within a specified tolerance. The results presented showed excellent agreement with their analytical counterparts. The kinematic constraint present in the second example is satisfied at the order of machine accuracy, which confirms the high order of precision of the scheme and the efficiency of the algorithm as well. The results presented in this study encourages the use of the algorithm in the solution of complex multibody systems, i.e., systems with large number of nonlinear elastic members and several kinematic constraints.

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