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EFFECTS OF STRETCHING-BENDING COUPLINGS ON THE BUCKLING AND THERMAL BUCKLING BEHAVIOR OF UNSYMMETRIC LAMINATES

GuangMin Cheng, Engineer
The First Research Institute
Beijing 9203-11
Beijing 100076,P.R.China

David Tsao, Professor
Department of Aircraft Engineering
North-western Polytechnical University
Xi'an 710072,P.R.China

Abstract Implicit in the linear analysis of buckling behavior of isotropic plates .homogenous anisotropic plates,or symmetrically laminated composite plates ,are the three assumptions that. " no additional inplane deformations take place in the process of buckling " , " the external edged loads remain constant in the process of buckling " , and " for a perfect plate,the plate remains plane before buckling onset " .

In the authors' priori work,it was pointed out that the applicability of the above three mentioned assumptions are worth to be questioned to a buckling problem of unsymmetric laminates.To unsymmetrically laminated composite plates ,due to the stretching-bending couplings,before buckling,inplane loads may cause a out-of-plane prebuckling deflection,while in the process of buckling,the bending of the plate may cause inplane deformations as well as the altering of edged loads. This phenomena is much similar to the way of buckling of shell panels

In the present paper,based on a higher order displacement theory including three midplane displacements ,one stretching of normal,two rotations of normals about the midplane.and two warps of the normals,by using the variation principle,a formulation for both buckling and thermal buckling was derived from the system's total potential energy.The formulation accounts for all of the effects of inplane deformations,edged loads altering,and prebuckling deflection on the critical buckling loads

The above three mentioned effects either as an individual or as an arbitrary combination on the buckling and thermal buckling loads are studied herein. An eight degree of freedom finite element model is used to compute the buckling and thermal buckling loads and model shapes of a number of unsymmetric laminates with simply supported and clamped boundary conditions.the results show that the importance of the three factors take an order of first the prebuckling deflection,then the inplane deformations,finally the edged loads altering.espically to antisymmetric laminates,the effects are too important to be neglected.

1.Introduction

Buckling and postbuckling behaviors of laminated composite plate and shell panels have gained great attentions in the past few decades. Leissa[1,2] made a comprehensive review of the current states of this art. From his review papers, it can be seen evidently that most of these works focuss on symmetric orthotropic composite plates. Some works focuss on symmetric anisotropic composite plates. Only a few works dealt with unsymmetrically laminated composite plates but confined in conditions of orthotropic or angle-ply plates.Works dealt with unsymmetric arbitrary anisotropic composite laminates are much more limted. This condition may be resulted by the following two factors:

First, The stretching-shearing couplings of anisotropic composite plates result in a full terms governing differential equations of buckling. Second, The stretching-bending of unsymmetric composite plates make the bending and inplane deformations coupled , There fore consequently result in a more higher-order set of governing diferential equitions of buckling. Combination of the above mentioned two factors, make the governing equition of buckling extremly complex.When using the Reissner- mindlin theory, the order of the system of governing deferential equitions is ten ,the number of boundary conditions is five for each edge, and as a result, the buckling problems of unsymmetric anisotropic composite plates

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are much more difficult to solve analytically.

Reissner and Stavsky[3,4] developed the first satisfactory theory which including stretching-bending effects.

Ashton[5] developed an approximate theory in which the bending stiffness were replaced by the "reduced bending stiffness".But results show that the approximate theory is not accurate for all problems. Jones et al[6] studied the influence of stretching-bending on buckling loads. The results show that stretching-bending couplings reduce the stiffness of composite laminates, and consequently, reduce the critical loads.Chia[7] and Prabhakara[8] analysised the buckling and postbuckling of unsymmetric laminated plates, by using an nonliear methods. Whitney[9] and Hui[10]dealt with the buckling of unsymmetri orthotropic plates. Hurris[11] analysis the buckling of unsymmetric orthotropic plates under biaxial compression. Jensen and Lagace[12,13] investigated the influence of stretching- bending couplings on the buckling and postbuckling behavior by using a Rayleigh- Ritz method and experiments. Rao[14, 15] studied the buckling behavior of symmetric or unsymmetric anisotropic sandwich plates with laminated faces using the Rayleigh-Ritz methods, where the deflection and two shearing components are assumed to satisfy the boundary conditions.

Jeng-Shian Chang[16] analyzed the buckling and thermal buckling behavior of antisymmetric angle-ply laminates by using FEM according to a higher order shear theory. But none of the above mentioned works have taken into account for the prebuckling deflection, the enplane deformations and external edged forces altering when buckling onset. The present paper will aim to develop a formulation which take into account for the all above mentioned factors.Formulation methodology

Generally, when linearly analysis the buckling behavior of isotropic or homogenous anisotropic plates, following two assumptions are introduced in or implicited [17]:

First, no additional inplane deformations take place at buckling onset.

$$\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2=0$$
$$\frac{\partial v}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2=0 \qquad (1)$$
$$\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}=0$$

In other words, the transverse deformations and in-plane deformations are independent at buckling onset.

Second, the external edged loads remain constant at buckling onset. As had pointed out in the authors' prior work[18],these two assumptions may be correct for isotropic or homogeneous anisotropic plates. But, they are worth to be questioned when linearly analysis unsymmetric anisotropic laminates. Because, due to the stretching-bending couplings in unsymmetric laminates, the inplane state is coupled with the bend state. At buckling onset, there is out-of-plane displacements take place, this may unaviodably cause inplane displacements and edged loads altering. This phenomena is much similar to that of shell buckling. there fore, in the present paper, the above two assumptions are neglected.

2. Formulation

2.1 potential energy of the pre-buckling state

According to the fundamental theory of composite material mechanics, the stress-strain relations in arbitrary one layer are:

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \sigma_{yz}\\ \sigma_{xz}\\ \sigma_{xy}\end{Bmatrix}=\begin{bmatrix}Q_{11}&Q_{12}&Q_{13}&0&0&Q_{16}\\ Q_{12}&Q_{22}&Q_{23}&0&0&Q_{26}\\ Q_{13}&Q_{23}&Q_{33}&0&0&Q_{36}\\ 0&0&0&Q_{44}&Q_{45}&0\\ 0&0&0&Q_{45}&Q_{55}&0\\ Q_{16}&Q_{26}&Q_{36}&0&0&Q_{66}\end{bmatrix}_k\begin{Bmatrix}\varepsilon_x-\int_{T_0}^{T_1}\alpha_x dT\\ \varepsilon_y-\int_{T_0}^{T_1}\alpha_y dT\\ \varepsilon_z-\int_{T_0}^{T_1}\alpha_z dT\\ \gamma_{yz}\\ \gamma_{xz}\\ \gamma_{xy}-\int_{T_0}^{T_1}\alpha_{xy} dT\end{Bmatrix}$$

Displacement field just before buckling onset is assumed to be u、v、w, they have the following forms which includes transverse shear and normal stretch:

$$u(x,y,z)=u_0(x,y)+z\,\psi_x(x,y)+z^3\,\phi_x(x,y)$$
$$v(x,y,z)=v_0(x,y)+z\,\psi_y(x,y)+z^3\,\phi_y(x,y) \qquad (2)$$
$$w(x,y,z)=w_0(x,y)+z^2\,\phi_z(x,y)$$

Where, u_0, v_0 and w_0 denote the displacements at the midplane(z=0), $\psi_x, \psi_y, \phi_x, \phi_y$ are generalized rotations of the normals to the midplane about the x, y axes respectively , ϕ_z is the stretching of the nomal .

The general strains can now be derived using the assumed displacement field, in equation(2).

$$\varepsilon_x = \frac{\partial u}{\partial x} \quad \varepsilon_y = \frac{\partial v}{\partial y} \quad \varepsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \quad \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \quad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

Then in-plane and through-thickness strains are:

$$<\varepsilon> = <\varepsilon_x \ \varepsilon_y \ \gamma_{xy} \ \varepsilon_z> = <\varepsilon_0> + z<k> + z^3<\eta>$$

Where

$$<\varepsilon_0> = <\varepsilon_{0x} \ \varepsilon_{0y} \ \gamma_{0xy} \ 0>$$
$$= (\frac{\partial u_0}{\partial x} \quad \frac{\partial v_0}{\partial y} \quad \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \quad 2\phi_z)$$

$$<k> = <k_x \ k_y \ k_{xy} \ k_z>$$
$$= (\frac{\partial \psi_x}{\partial x} \quad \frac{\partial \psi_y}{\partial y} \quad \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \quad 2\phi_z)$$

$$<\eta> = <\eta_x \ \eta_y \ \eta_{xy} \ 0>$$
$$= (\frac{\partial \phi_x}{\partial x} \quad \frac{\partial \phi_y}{\partial y} \quad \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \quad 0)$$

The inter-laminar transverse strains are:

$$<\gamma> = <\gamma_0> + z^2 <\xi>$$

Where

$$<\gamma_0> = <\gamma_{0xy} \ \gamma_{0yz}> = (\frac{\partial \omega_0}{\partial x} + \psi_x \quad \frac{\partial \omega_0}{\partial y} + \psi_y)$$

$$<\xi> = <\xi_{xz} \ \xi_{yz}> = (\frac{\partial \phi_z}{\partial x} + 3\phi_x \quad \frac{\partial \phi_z}{\partial y} + 3\phi_y)$$

The total potential energy of the pre-buckling state is

$$\Pi = \frac{1}{2}\iiint_V (<\sigma_x \sigma_y \tau_{xy} \sigma_z> \left[\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\\ \varepsilon_z\end{Bmatrix} - \int_{T0}^{T1}\begin{Bmatrix}\alpha_x\\ \alpha_y\\ \alpha_{xy}\\ \alpha_z\end{Bmatrix}dT\right] + <\tau_{xz} \ \tau_{yz}>\begin{Bmatrix}\gamma_{xz}\\ \gamma_{yz}\end{Bmatrix})dV$$
$$-\iint_R\left[\overline{N_x}\frac{\partial u_0}{\partial x} + \overline{N_y}\frac{\partial v_0}{\partial y} + \overline{N_{xy}}\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right)\right]dR$$

$$= \frac{1}{2}\iint_R[\langle\varepsilon_0\rangle_0[A]\{\varepsilon_0\} + \langle k\rangle[D]\{k\} + \langle\eta\rangle[H]\{\eta\}$$
$$+2\langle\varepsilon_0\rangle[B]\{k\} + 2\langle\varepsilon_0\rangle[E]\{\eta\} + 2\langle k\rangle[F]\{\eta\}$$
$$+\frac{1}{2}\iiint_V(\int_{T_0}^{T_1} <\alpha> [Q]\{\alpha\})dTdV$$
$$-\iint_R(\langle\varepsilon_0\rangle[D_{aa}] + \langle k\rangle[D_{ab}] + \langle\eta\rangle[Dae])dR$$
$$-\iint_R\left[\overline{N_x}\frac{\partial u_0}{\partial x} + \overline{N_y}\frac{\partial v_0}{\partial y} + \overline{N_{xy}}\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right)\right]dR \qquad (3)$$

$$\{\cdots\} = \langle\cdots\rangle^T$$

Where

$$([A]\,[B]\,[C]\,[E]\,[F]\,[H]) = \int_{-h/2}^{h/2}[Q](1 \ z \ z^2 \ z^2 \ z^4 \ z^6)dz$$

$$[Q] = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} & Q_{13} \\ Q_{12} & Q_{22} & Q_{26} & Q_{23} \\ Q_{16} & Q_{26} & Q_{66} & Q_{36} \\ Q_{13} & Q_{23} & Q_{36} & Q_{33} \end{bmatrix}$$

$$[D_s] = \int_{-h/2}^{h/2}\begin{bmatrix}[Q_s] & z^2[Q_s] \\ z^2[Q_s] & z^4[Q_s]\end{bmatrix}dz$$

$$[D_{aa}] = \int_{-h/2}^{h/2}[Q]\int_{T0}^{T1}\begin{Bmatrix}a_x\\ a_y\\ 2a_{xy}\\ a_z\end{Bmatrix}dTdz$$

$$[D_{ab}] = \int_{-h/2}^{h/2}z[Q]\int_{T0}^{T1}\begin{Bmatrix}a_x\\ a_y\\ 2a_{xy}\\ a_z\end{Bmatrix}dTdz$$

$$[D_{ae}] = \int_{-h/2}^{h/2}z^3[Q]\int_{T0}^{T1}\begin{Bmatrix}a_x\\ a_y\\ 2a_{xy}\\ a_z\end{Bmatrix}dTdz$$

Equation(3) is equivalent to equation(29) reference [16].

Because it is a equilibrium state before buckling onset, according to the total potential energy principle, the first order variation of the total potential energy should be zero.

$$\delta\Pi = 0 \qquad (4)$$

2.2 formulation of critical buckling load

Buckling shape is assumed to be similar to displacement filed of prebuckling. they take forms as following:

$$\overline{u}(x,y,z) = \overline{u_0}(x,y) + z\overline{\psi_x}(x,y) + z^3\overline{\phi_x}(x,y)$$
$$\overline{v}(x,y,z) = \overline{v_0}(x,y) + z\overline{\psi_y}(x,y) + z^3\overline{\phi_y}(x,y) \qquad (5)$$
$$\overline{w}(x,y,z) = \overline{w_0}(x,y) + z^2\overline{\phi_z}(x,y)$$

Since the buckling is basically nonlinear phenomena, the Von-Karman's large deflection nonlinear strain-displacement relations should be adopted here. So, the general strains are

$$\langle \varepsilon_{NL} \rangle = \langle \bar{\varepsilon}_x \ \bar{\varepsilon}_y \ \bar{\gamma}_{xy} \rangle$$
$$= \left\langle \frac{\partial \bar{u}}{\partial x} + \frac{1}{2}\left(\frac{\partial \bar{w}}{\partial x}\right)^2 \quad \frac{\partial \bar{v}}{\partial y} + \frac{1}{2}\left(\frac{\partial \bar{w}}{\partial y}\right)^2 \quad \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} + \frac{\partial \bar{w}}{\partial x} \bullet \frac{\partial \bar{w}}{\partial y} \quad 0 \right\rangle$$
$$= \langle \bar{\varepsilon}_L \rangle + \langle \bar{\varepsilon}_w \rangle + \langle \bar{\varepsilon}_{w\phi} \rangle + \langle \bar{\varepsilon}_\phi \rangle \tag{6}$$

Where

$$\langle \bar{\varepsilon}_L \rangle = \langle \bar{\varepsilon}_0 \rangle + z\langle \bar{k} \rangle + z^2 \langle \bar{\eta} \rangle \tag{7}$$

$$\langle \bar{\varepsilon} \rangle = \langle \bar{\varepsilon}_x \ \bar{\varepsilon}_y \ \bar{\gamma}_{xy} \ 0 \rangle = \left\langle \frac{\partial \bar{u}_0}{\partial x} \quad \frac{\partial \bar{v}_0}{\partial y} \quad \frac{\partial \bar{u}_0}{\partial y} + \frac{\partial \bar{v}_0}{\partial x} \quad 0 \right\rangle$$

$$\langle \bar{k} \rangle = \langle \bar{k}_x \ \bar{k}_y \ \bar{k}_{xy} \ k_z \rangle = \left\langle \frac{\partial \bar{\psi}_x}{\partial x} \quad \frac{\partial \bar{\psi}_y}{\partial y} \quad \frac{\partial \bar{\psi}_x}{\partial y} + \frac{\partial \bar{\psi}_y}{\partial x} \quad 2\phi_z \right\rangle$$

$$\langle \bar{\eta} \rangle = \langle \bar{\eta}_x \ \bar{\eta}_y \ \bar{\eta}_{xy} \ 0 \rangle = \left\langle \frac{\partial \bar{\phi}_x}{\partial x} \quad \frac{\partial \bar{\phi}_y}{\partial y} \quad \frac{\partial \bar{\phi}_x}{\partial y} + \frac{\partial \bar{\phi}_y}{\partial x} \quad 0 \right\rangle$$

$$\langle \bar{\gamma}_0 \rangle = \langle \bar{\gamma}_{xz} \ \bar{\gamma}_{yz} \rangle = \left\langle \frac{\partial \bar{w}_0}{\partial x} + \bar{\psi}_x \quad \frac{\partial \bar{w}_0}{\partial y} + \bar{\psi}_y \right\rangle$$

$$\langle \bar{\xi} \rangle = \langle \bar{\xi}_{xz} \ \bar{\xi}_{yz} \rangle = \left\langle \frac{\partial \bar{\phi}_z}{\partial x} + 3\bar{\phi}_x \quad \frac{\partial \bar{\phi}_z}{\partial y} + 3\bar{\phi}_y \right\rangle$$

$$\langle \bar{\varepsilon}_w \rangle = \left\langle \frac{1}{2}\left(\frac{\partial \bar{w}_0}{\partial x}\right)^2 \quad \frac{1}{2}\left(\frac{\partial \bar{w}_0}{\partial y}\right)^2 \quad \frac{\partial \bar{w}_0}{\partial y} \bullet \frac{\partial \bar{w}_0}{\partial y} \quad 0 \right\rangle$$

$$\langle \bar{\varepsilon}_{w\phi} \rangle = \left\langle \frac{\partial \bar{w}_0}{\partial x} \bullet \frac{\partial \bar{\phi}_z}{\partial x} \quad \frac{\partial \bar{w}_0}{\partial y} \bullet \frac{\partial \bar{\phi}_z}{\partial y} \quad \frac{\partial \bar{w}_0}{\partial x} \bullet \frac{\partial \bar{\phi}_z}{\partial y} + \frac{\partial \bar{w}_0}{\partial y} \bullet \frac{\partial \bar{\phi}_z}{\partial x} \quad 0 \right\rangle$$

$$\langle \bar{\varepsilon}_\phi \rangle = \left\langle \frac{1}{2}\left(\frac{\partial \bar{\phi}_z}{\partial x}\right)^2 \quad \frac{1}{2}\left(\frac{\partial \bar{\phi}_z}{\partial y}\right)^2 \quad \frac{\partial \bar{\phi}_z}{\partial x} \bullet \frac{\partial \bar{\phi}_z}{\partial y} \quad 0 \right\rangle$$

$$\{\cdots\} = \langle \cdots \rangle^T$$

According to equation(3),the the total potential energy at buckling onset yields.

$$\Pi = \Pi_1 + \Pi_2 \tag{8}$$

Where

$$\Pi_1 = \frac{1}{2}\iint_R [\langle \bar{\varepsilon}_0 \rangle [A] \{\bar{\varepsilon}_0\} + \langle \bar{k} \rangle [D] \{\bar{k}\} + \langle \bar{\eta} \rangle [H] \{\bar{\eta}\} + 2\langle \bar{\varepsilon}_0 \rangle [B] \{\bar{k}\} + 2\langle \bar{\varepsilon}_0 \rangle [E] \{\bar{\eta}\}$$
$$+ 2\langle \bar{k} \rangle [F] \{\bar{\eta}\} + \langle \langle \bar{\gamma}_0 \rangle \langle \bar{\xi} \rangle \rangle [D_s] \{\bar{\gamma}_0 \ \bar{\xi}\}]dR + \frac{1}{2}\iiint \left(\int_{T0}^{T1} \langle \alpha \rangle dT [Q] \int_{T0}^{T1} \{\alpha\} dT \right) dV$$
$$- \iint_R (\langle \bar{\varepsilon}_0 \rangle [D_{ar}] + \langle \bar{k} \rangle [D_{ab}] + \langle \bar{\eta} \rangle [D_{ac}]) dR - \iint_R [N_x \frac{\partial \bar{u}_0}{\partial x} + N_y \frac{\partial \bar{v}_0}{\partial y}$$
$$+ N_{xy}\left(\frac{\partial \bar{u}_0}{\partial y} + \frac{\partial \bar{v}_0}{\partial x} \right)]dR \tag{9}$$

$$\Pi_2 = \iint_R [\langle \bar{\varepsilon}_0 \rangle [A] \{\bar{\varepsilon}_w\} + \langle \bar{\varepsilon}_0 \rangle [D] \{\bar{\varepsilon}_{w\phi}\} + \langle \bar{\varepsilon}_0 \rangle [F] \{\bar{\varepsilon}_\phi\} + \langle \bar{\varepsilon}_w \rangle [B] \{\bar{k}\} + \langle \bar{\varepsilon}_{w\phi} \rangle [E] \{\bar{k}\}$$
$$+ \langle \bar{\varepsilon}_\phi \rangle [G] \{\bar{k}\} + \langle \varepsilon_{w\phi} \rangle [G] \{\bar{\eta}\} + \langle \varepsilon_\phi \rangle [I] \{\bar{\eta}\}] dR$$
$$- \iint_R (\langle \bar{\varepsilon}_w \rangle [D_{aa}] + \langle \bar{\varepsilon}_{w\phi} \rangle [D_{ad}] + \langle \bar{\varepsilon}_\phi \rangle [D_{af}]) dR \tag{10}$$

Where

$$(G_{ij} \quad I_{ij}) = \int_{-h/2}^{h/2} Q_{ij} (z^5 \quad z^7) dz$$

$$[D_{ad}] = \int_{-h/2}^{h/2} z^2 [Q] \int_{T0}^{T1} \begin{Bmatrix} a_x \\ a_y \\ 2a_{xy} \\ a_z \end{Bmatrix} dT dz$$

$$[D_{af}] = \int_{-h/2}^{h/2} z^4 [Q] \int_{T0}^{T1} \begin{Bmatrix} a_x \\ a_y \\ 2a_{xy} \\ a_z \end{Bmatrix} dT dz$$

It is worth to be noted that the meaning of general forces $\bar{N}_x$, $\bar{N}_y$ and $\bar{N}_{xy}$ in equation (3) is different form that of the general force in equation(9).The former ones are the edged boundary forces on an equilibrium state, the later ones are the internal forces of the system. In an equilibrium state,N_x, N_y, N_{xy} are equal to $\bar{N}_x$, $\bar{N}_y$ and $\bar{N}_{xy}$ respectively on the boundary. But in a state of buckling onset, due to the stretching-bending couplings, N_x, N_y, N_{xy} may alter from their initial values, and take values evaluated on the boundary.

The first order variation of general strains can be obtained form equation(6) (For semplicity we denote u＝δ u， v＝δ v， w＝δ w， ψ x＝δ ψ x， ψ y＝δ ψ y， φ x＝δ φ x， φ y＝δ φ y， φ z＝δ φ z）：

$$\langle \delta\varepsilon_0 \rangle = \langle \varepsilon_0 \rangle = \left\langle \frac{\partial u_0}{\partial x} \quad \frac{\partial v_0}{\partial y} \quad \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \quad 0 \right\rangle$$

$$\langle \delta k \rangle = \langle k \rangle = \left\langle \frac{\partial \psi_x}{\partial x} \quad \frac{\partial \psi_y}{\partial y} \quad \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \quad 2\phi_z \right\rangle$$

$$\langle \delta\eta \rangle = \langle \eta \rangle = \left\langle \frac{\partial \phi_x}{\partial x} \quad \frac{\partial \phi_y}{\partial y} \quad \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \quad 0 \right\rangle$$

$$\langle \delta\gamma \rangle = \langle \gamma \rangle = \left\langle \frac{\partial w_0}{\partial x} + \psi_x \quad \frac{\partial w_0}{\partial y} + \psi_y \right\rangle$$

$$\langle \delta\xi \rangle = \langle \xi \rangle = \left\langle \frac{\partial \phi_z}{\partial x} + 3\phi_x \quad \frac{\partial \phi_z}{\partial y} + 3\phi_y \right\rangle$$

$$\langle \delta\varepsilon_w \rangle = \langle \varepsilon_w \rangle = \left\langle \frac{\partial w_0}{\partial x} \bullet \frac{\partial \bar{w}_0}{\partial x} \quad \frac{\partial w_0}{\partial y} \bullet \frac{\partial \bar{w}_0}{\partial y} \cdot \frac{\partial w_0}{\partial x} + \frac{\partial \bar{w}_0}{\partial y} + \frac{\partial w_0}{\partial y} \bullet \frac{\partial \bar{w}_0}{\partial x} \quad 0 \right\rangle$$

$$\langle \delta\varepsilon_{w\phi} \rangle = \langle \varepsilon_{w\phi} \rangle = \langle \frac{\partial w_0}{\partial x} \bullet \frac{\partial \bar{\phi}_z}{\partial x} + \frac{\partial \bar{w}_0}{\partial x} \bullet \frac{\partial \phi_z}{\partial x} \quad \frac{\partial w_0}{\partial y} \bullet \frac{\partial \bar{\phi}_z}{\partial y} + \frac{\partial \bar{w}_0}{\partial y} \bullet \frac{\partial \phi_z}{\partial y} \quad \frac{\partial w_0}{\partial x} \bullet \frac{\partial \bar{\phi}_z}{\partial y}$$
$$+ \frac{\partial \bar{w}_0}{\partial x} \bullet \frac{\partial \phi_z}{\partial y} \quad \frac{\partial w_0}{\partial y} \bullet \frac{\partial \bar{\phi}_z}{\partial x} + \frac{\partial \bar{w}_0}{\partial y} \bullet \frac{\partial \phi_z}{\partial x} \rangle$$

$$\langle \delta\varepsilon_{\phi} \rangle = \langle \varepsilon_{\phi} \rangle = \left\langle \frac{\partial\bar{\phi}_z}{\partial x}\bullet\frac{\partial\bar{\phi}_z}{\partial x} \quad \frac{\partial\bar{\phi}_z}{\partial y}\bullet\frac{\partial\bar{\phi}_z}{\partial y} \quad \frac{\partial\bar{\phi}_z}{\partial x}\bullet\frac{\partial\bar{\phi}_z}{\partial y}+\frac{\partial\bar{\phi}_z}{\partial y}\bullet\frac{\partial\bar{\phi}_z}{\partial x} \quad 0 \right\rangle$$

The second order variation of general strains are as following.

$$\langle \delta^2 \varepsilon_0 \rangle = \langle \delta\varepsilon_0 \rangle = \langle 0 \quad 0 \quad 0 \quad 0 \rangle$$

$$\langle \delta^2 k \rangle = \langle \delta k \rangle = \langle 0 \quad 0 \quad 0 \quad 0 \rangle$$

$$\langle \delta^2 \eta \rangle = \langle \delta \eta \rangle = \langle 0 \quad 0 \quad 0 \quad 0 \rangle$$

$$\langle \delta^2 \gamma \rangle = \langle \delta \gamma \rangle = \langle 0 \quad 0 \rangle$$

$$\langle \delta^2 \xi \rangle = \langle \delta \xi \rangle = \langle 0 \quad 0 \rangle$$

$$\langle \delta^2 \varepsilon_w \rangle = \langle \delta \varepsilon_w \rangle = \left\langle \left(\frac{\partial \bar{w}_0}{\partial x}\right)^2 \quad \left(\frac{\partial \bar{w}_0}{\partial y}\right)^2 \quad 2\frac{\partial \bar{w}_0}{\partial x}\bullet\frac{\partial \bar{w}_0}{\partial y} \quad 0 \right\rangle$$

$$\langle \delta^2 \varepsilon_{w\phi} \rangle = \langle \delta \varepsilon_{w\phi} \rangle = \left\langle \frac{\partial \bar{w}_0}{\partial x}\bullet\frac{\partial\bar{\phi}_z}{\partial x} \quad \frac{\partial \bar{w}_0}{\partial y}\bullet\frac{\partial\bar{\phi}_z}{\partial y} \quad \frac{\partial \bar{w}_0}{\partial x}\bullet\frac{\partial\bar{\phi}_z}{\partial y}+\frac{\partial \bar{w}_0}{\partial y}\bullet\frac{\partial\bar{\phi}_z}{\partial x} \quad 0 \right\rangle$$

$$\langle \delta^2 \varepsilon_{\phi} \rangle = \langle \delta \varepsilon_{\phi} \rangle = \left\langle \left(\frac{\partial \bar{\phi}_z}{\partial x}\right)^2 \quad \left(\frac{\partial \bar{\phi}_z}{\partial y}\right)^2 \quad 2\frac{\partial \bar{\phi}_z}{\partial x}\bullet\frac{\partial \bar{\phi}_z}{\partial y} \quad 0 \right\rangle$$

Since $\bar{w}$ is small values at buckling onset, the higher order terms of $\bar{w}_0$、$\bar{\phi}_z$ can be neglected. Then the second order variation of the total potential energy are:

$$\delta^2\Pi = \delta^2\Pi_1 + \delta^2\Pi_2 + \delta^2\Pi_3 \qquad (11)$$

Where

$$\delta^2\Pi_1 = \frac{1}{2}\iint_R \Big[\langle\varepsilon_0\rangle[A]\{\varepsilon_0\} + \langle k\rangle[D]\{k\} + 2\langle\varepsilon_0\rangle[B]\{\varepsilon_0\} + 2\langle\varepsilon_0\rangle[E]\{\eta\} + 2\langle k\rangle[F]\{\eta\}$$
$$+ \langle\langle\gamma_0\rangle\langle\xi\rangle\rangle[D_s]\{\langle\gamma_0\rangle\langle\xi\rangle\} \Big] dR \qquad (12)$$

$$\delta^2\Pi_2 = -\iint_R \left[\delta^2 N_x \frac{\partial \bar{u}_0}{\partial x} + \delta^2 N_y \frac{\partial \bar{v}_0}{\partial y} + \delta^2 N_{xy}\left(\frac{\partial \bar{u}_0}{\partial y} + \frac{\partial \bar{v}_0}{\partial x}\right) \right] dR$$
$$-2\iint_R \left[\delta N_x \frac{\partial u_0}{\partial x} + \delta N_y \frac{\partial v_0}{\partial y} + \delta N_{xy}\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) \right] dR$$

$$\delta^2\Pi_3 = \iint_R \Big[\langle\bar{\varepsilon}_0\rangle[A]\{\delta\varepsilon_w\} + \langle\varepsilon_0\rangle[D]\{\delta\varepsilon_{w\phi}\} + \langle\bar{\varepsilon}_0\rangle[F]\{\delta\varepsilon_{\phi}\} + \langle\delta\varepsilon_w\rangle[B]\{\bar{k}\}$$
$$+ \langle\delta\varepsilon_{w\phi}\rangle[E]\{\bar{k}\} + \langle\delta\varepsilon_{\phi}\rangle[G]\{\bar{k}\} + \langle\delta\varepsilon_w\rangle[E]\{\bar{\eta}\} + \langle\delta\varepsilon_{w\phi}\rangle[G]\{\bar{\eta}\}$$
$$+ \langle\delta\varepsilon_{\phi}\rangle[I]\{\bar{\eta}\} \Big] dR + \iint_R \Big[\langle\varepsilon_0\rangle[A]\{\varepsilon_w\} + \langle\varepsilon_0\rangle[D]\{\varepsilon_{w\phi}\} + \langle\varepsilon_0\rangle[F]\{\varepsilon_{\phi}\}$$
$$+ \langle\varepsilon_w\rangle[B]\{k\} + \langle\varepsilon_{w\phi}\rangle[E]\{k\} + \langle\varepsilon_{\phi}\rangle[G]\{k\} + \langle\varepsilon_w\rangle[E]\{\eta\} + \langle\varepsilon_{w\phi}\rangle[G]\{\eta\}$$
$$+ \langle\varepsilon_{\phi}\rangle[I]\{\eta\} \Big] dR - \iint_R \Big[\langle\delta\varepsilon_w\rangle[D_{aa}] + \langle\delta\varepsilon_{w\phi}\rangle[D_{ad}] + \langle\delta\varepsilon_{\phi}\rangle[D_{af}] \Big] dR$$
$$= \delta^2\Pi_4 + \delta^2\Pi_5 \qquad (14)$$

Where

$$\delta^2\Pi_4 = 2\iint_R \Big[\langle\varepsilon_w\rangle([A]\{\varepsilon_0\} + [B]\{k\} + [E]\{\eta\}) + \langle\varepsilon_{w\phi}\rangle([D]\{\varepsilon_0\} + [E]\{k\} + [G]\{\eta\})$$
$$+ \langle\varepsilon_{\phi}\rangle([F]\{\varepsilon_0\} + [G]\{k\} + [I]\{\eta\}) \Big] dR \qquad (15)$$

$$\delta^2\Pi_5 = \iint_R \left[\left\langle \frac{\partial w_0}{\partial x} \quad \frac{\partial w_0}{\partial y} \right\rangle [N] \begin{Bmatrix} \frac{\partial w_0}{\partial x} \\ \frac{\partial w_0}{\partial y} \end{Bmatrix} + \left\langle \frac{\partial w_0}{\partial x} \quad \frac{\partial w_0}{\partial y} \right\rangle [R] \begin{Bmatrix} \frac{\partial \phi_z}{\partial x} \\ \frac{\partial \phi_z}{\partial y} \end{Bmatrix} \right.$$
$$\left. + \left\langle \frac{\partial \phi_z}{\partial x} \quad \frac{\partial \phi_z}{\partial y} \right\rangle [F] \begin{Bmatrix} \frac{\partial \phi_z}{\partial x} \\ \frac{\partial \phi_z}{\partial y} \end{Bmatrix} \right] dR \qquad (16)$$

$$[N] = \begin{bmatrix} N_x & N_{xy} \\ N_{xy} & N_y \end{bmatrix} \qquad [R] = \begin{bmatrix} R_x & R_{xy} \\ R_{xy} & R_y \end{bmatrix} \qquad [F] = \begin{bmatrix} F_x & F_{xy} \\ F_{xy} & F_y \end{bmatrix}$$

$$\langle N_x \quad N_y \quad N_{xy} \rangle^T = [A]\{\bar{\varepsilon}_0\} + [B]\{\bar{k}\} + [E]\{\bar{\eta}\} - [D_{aa}]$$

$$\langle R_x \quad R_y \quad R_{xy} \rangle^T = [D]\{\bar{\varepsilon}_0\} + [E]\{\bar{k}\} + [G]\{\bar{\eta}\} - [D_{ad}]$$

$$\langle F_x \quad F_y \quad F_{xy} \rangle^T = [F]\{\bar{\varepsilon}_0\} + [G]\{\bar{k}\} + [I]\{\bar{\eta}\} - [D_{af}]$$

From equation(13),it can be obviously seen that, if the external edged boundary forces remain constant at buckling onset, $\delta^2\Pi_2$ will take a value of zero.

To isotropic or homogeneous anisotropic plates, because no stretching-bending couplings, general internal forces N_x, N_y, N_{xy} will equal to the external edged loads at the equilibrium state, equition(13) will be exactly same in form as equation(19) shown in reference[17]. If we don't take into account for prebuckling deflection and inplane deformation, then $\delta^2\Pi_3$ equils zero,equation(11) is equivalent to equation(39) in reference[16]. Here, we take into account for the altering of external edged loads and inplane deformations at buckling onset, and also take into account for the prebuckling deflection caused by stretching-bending couplings. Because only the state at buckling onset is interested in, then $\bar{w}$ is a infinitesimal perturbation, and the prebuckling deflection is generally a small value too, higher order terms of them can be neglected . The prebuckling state is much close to the state of buckling onset, so the deflection at buckling onset can be replaced by the prebuckling deflection infinite close to buckling. therefore, result in linearization of equation(11).In equation(11), $\delta^2\Pi_2$ represent the energy increments induced by the boundary loads altering, and $\delta^2\Pi_4$, represent energy increment induced by pre-buckling deflection. If assume boundary loads remain constant in buckling, and neglect the pre-buckling deflection, then equate ion (11) equivalent to equation(39) in reference[16].

According the work of Washizu[19],the second order variation of a system's total potential energy is always positive definited. When the system reaches its critical state, the second order variation of the system's total potential energy take a minimum value of zero. That is

$$\delta^2\Pi = 0$$
$$\overline{\delta}\left(\delta^2\Pi\right) = 0 \qquad (17)$$

3.Finite element methodology

Here,a 8 node isoparamater element is used to obtain numeracal result.If we denote N i(i=1,2,⋯,8) shape functions,then the general strains and their variations can be written as

$$\{\varepsilon_0\} = [B_\varepsilon]\{\Delta^e\} \qquad (18)$$

Where

$$[B_\varepsilon] = [[B_{\varepsilon 1}]\ \ [B_{\varepsilon 2}]\ \ \cdots\ \ [B_{\varepsilon 8}]]$$
$$[B_{\varepsilon i}] = [L_1]N_i \qquad (i=1,2,\cdots,8)$$
$$\{\Delta^e\} = \left\{\{\delta_1^e\}^T\ \ \{\delta_2^e\}^T\ \cdots\ \{\delta_8^e\}^T\right\}^T$$
$$[L_1] = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\{k\} = [B_k]\{\Delta^e\} \qquad (19)$$

Where

$$[B_k] = [[B_{k1}]\ \ [B_{k2}]\ \ \cdots\ \ [B_{k8}]]$$
$$[B_{ki}] = [L_2]N_i \qquad (i=1,2,\cdots,8)$$
$$[L_2] = \begin{bmatrix} 0 & 0 & \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial y} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\{\eta\} = [B_\eta]\{\Delta^e\} \qquad (20)$$

Where

$$[B_\eta] = [[B_{\eta 1}]\ \ [B_{\eta 2}]\ \ \cdots\ \ [B_{\eta 8}]]$$
$$[B_{\eta i}] = [L_3]N_i \qquad (i=1,2,\cdots,8)$$
$$[L_3] = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{\partial}{\partial x} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial y} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{Bmatrix} \{\gamma_0\} \\ \{\xi\} \end{Bmatrix} = [B_\xi]\ \{\Delta^e\} \qquad (21)$$

Where

$$[B_\xi] = [[B_{\xi 1}]\ \ [B_{\xi 2}]\ \ \cdots\ \ [B_{\xi 8}]]$$
$$[B_{\xi i}] = [L_4]N_i \qquad (i=1,2,\cdots,8)$$
$$[L_4] = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & \frac{\partial}{\partial x} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & \frac{\partial}{\partial x} \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & \frac{\partial}{\partial y} \end{bmatrix}$$

$$\begin{Bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{Bmatrix} = [B_w]\ \{\Delta^e\} \qquad (22)$$

Where

$$[B_w] = [[B_{w1}]\ \ [B_{w2}]\ \ \cdots\ \ [B_{w8}]]$$
$$[B_{wi}] = [L_5]N_i \qquad (i=1,2,\cdots,8)$$
$$[L_5] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial x} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial y} & 0 \end{bmatrix}$$

$$\begin{Bmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \end{Bmatrix} = [B_\phi]\ \{\Delta^e\} \qquad (23)$$

Where

$$[B_\phi] = [[B_{\phi 1}]\ \ [B_{\phi 2}]\ \ \cdots\ \ [B_{\phi 8}]]$$
$$[B_{\phi i}] = [L_6]N_i \qquad (i=1,2,\cdots,8)$$
$$[L_6] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial x} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial y} \end{bmatrix}$$

$$\langle \varepsilon_w \rangle = \left\langle \frac{\partial w_0}{\partial x}\ \ \frac{\partial w_0}{\partial y} \right\rangle \begin{bmatrix} \frac{\partial \overline{w_0}}{\partial x} & 0 & \frac{\partial \overline{w_0}}{\partial y} & 0 \\ 0 & \frac{\partial \overline{w_0}}{\partial y} & \frac{\partial \overline{w_0}}{\partial x} & 0 \end{bmatrix}$$

$$\langle \varepsilon_{w\phi} \rangle = \left\langle \frac{\partial w_0}{\partial x} \quad \frac{\partial w_0}{\partial y} \right\rangle \begin{bmatrix} \frac{\partial \bar{c}}{\partial x} & 0 & \frac{\partial \bar{\phi}_z}{\partial y} & 0 \\ 0 & \frac{\partial \bar{\phi}_z}{\partial y} & \frac{\partial \bar{\phi}_z}{\partial x} & 0 \end{bmatrix}$$

$$+ \left\langle \frac{\partial \phi_z}{\partial x} \quad \frac{\partial \phi_z}{\partial y} \right\rangle \begin{bmatrix} \frac{\partial \bar{w}_0}{\partial x} & 0 & \frac{\partial \bar{w}_0}{\partial y} & 0 \\ 0 & \frac{\partial \bar{w}_0}{\partial y} & \frac{\partial \bar{w}_0}{\partial x} & 0 \end{bmatrix}$$

$$\langle \varepsilon_{\phi} \rangle = \left\langle \frac{\partial \phi_z}{\partial x} \quad \frac{\partial \phi_z}{\partial y} \right\rangle \begin{bmatrix} \frac{\partial \phi_z}{\partial x} & 0 & \frac{\partial \phi_z}{\partial y} & 0 \\ 0 & \frac{\partial \phi_z}{\partial y} & \frac{\partial \phi_z}{\partial x} & 0 \end{bmatrix}$$

$$\langle \varepsilon_w \rangle = \langle \Delta^e \rangle [B_w]^T ([B_0] \bullet w_0)$$

$$\langle \varepsilon_{w\phi} \rangle = \langle \Delta^e \rangle [B_w]^T ([B_0] \bullet \phi_z) + \langle \Delta^e \rangle [B_\phi]^T ([B_0] \bullet w_0)$$

$$\langle \varepsilon_{\phi} \rangle = \langle \Delta^e \rangle [B_\phi]^T ([B_0] \bullet \phi_z)$$

$$[B_0] = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} & 0 \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix}$$

Let

$$[B_{0w}]^T = [B_w]^T ([B_0] \bullet w_0)$$

$$[B_{0T}] = [B_\phi]^T ([B_0] \bullet \phi_z)$$

Then

$$\langle \varepsilon_w \rangle = \langle \Delta^e \rangle [B_{0w}]^T \tag{24}$$

$$\langle \varepsilon_{w\phi} \rangle = \langle \Delta^e \rangle [B_{0w\phi}]^T \tag{25}$$

$$\langle \varepsilon_{\phi} \rangle = \langle \Delta^e \rangle [B_{0\phi}]^T \tag{26}$$

If no edged loads,then total potential energy of element at prebuckling state is :

$$\Pi^e = \frac{1}{2} \iint_R \langle \Delta^e \rangle \Big([B_\varepsilon]^T [A][B_\varepsilon] + [B_k]^T [D][B_k] + [B_\eta]^T [H][B_\eta] + 2[B_\varepsilon]^T [B][B_k] + 2[B_\varepsilon]^T [E][B_\eta] + 2[B_k]^T [F][B_\eta] + [B_\xi]^T [D_s][B_\xi] \Big) \{\Delta^e\} dR + \frac{1}{2} \iiint \langle a\Delta T \rangle [Q] \{a\Delta T\} dV - \iint_R \langle \Delta^e \rangle \Big([B_\varepsilon][D_{aa}] + [B_k][D_{ab}] + [B_\eta][D_{ae}] \Big) dR \tag{27}$$

The system's total potential energy is a sume of all elements' potential energy,that is

$$\Pi = \sum_{e=1}^{n} \Pi^e \tag{28}$$

According to $\delta \Pi = 0$,then matrix of steffness and loads expectively are:

$$[K^e] = \iint_R \Big([B_\varepsilon]^T [A][B_\varepsilon] + [B_k]^T [D][B_k] + [B_\eta]^T [H][B_\eta] + 2[B_\varepsilon]^T [B][B_k] + 2[B_\varepsilon]^T [E][B_\eta] + 2[B_k]^T [F][B_\eta] + [B_\xi]^T [D_s][B_\xi] \Big) dR \tag{29}$$

$$\{F^e\} = \iint_R \Big([B_\varepsilon][D_{aa}] + [B_k][D_{ab}] + [B_\eta][D_{ae}] \Big) dR \tag{30}$$

Similarly,the system's total potential energy at buckling onset , is a sume of all elements' total potential energy,that is

$$\delta^2 \Pi = \sum_{e=1}^{n} \delta^2 \Pi^e \tag{31}$$

The second variation of element's potential energy is:

$$\delta^2 \Pi^e = \iint_R \langle \Delta^e \rangle \Big([B_\varepsilon]^T [A][B_\varepsilon] + [B_k]^T [D][B_k] + [B_\eta]^T [H][B_\eta] + 2[B_\varepsilon]^T [B][B_k] + 2[B_\varepsilon]^T [E][B_\eta] + 2[B_k]^T [F][B_\eta] + [B_\xi]^T [D_s][B_\xi] \Big) \{\Delta^e\} dR$$
$$+ \iint_R \langle \Delta^e \rangle \Big([B_w]^T [N][B_w] + 2[B_w]^T [R][B_\phi] + [B_\phi]^T [F][B_\phi] \Big) \{\Delta^e\} dR$$
$$+ 2\iint_R \langle \Delta^e \rangle \Big([B_{0w}]^T [A][B_\varepsilon] + [B_{0w}]^T [B][B_k] + [B_{0w}]^T [E][B_\eta] + [B_{0w\phi}]^T [D][B_\varepsilon] + [B_{0w\phi}]^T [E][B_k] + [B_{0w\phi}]^T [G][B_\eta] + [B_{0\phi}]^T [F][B_\varepsilon] + [B_{0\phi}]^T [G][B_k] + [B_{0\phi}]^T [I][B_\eta] \Big) \{\Delta^e\} dR$$
$$= \langle \Delta^e \rangle \Big([K^e] + [K^e_w] + [K^e_g] \Big) \{\Delta^e\} \tag{32}$$

Where

$$[K^e_w] = 2\iint_R \langle \Delta^e \rangle \Big([B_{0w}]^T [A][B_\varepsilon] + [B_{0w}]^T [B][B_k] + [B_{0w}]^T [E][B_\eta] + [B_{0w\phi}]^T [D][B_\varepsilon] + [B_{0w\phi}]^T [E][B_k] + [B_{0w\phi}]^T [G][B_\eta] + [B_{0\phi}]^T [F][B_\varepsilon] + [B_{0\phi}]^T [G][B_k] + [B_{0\phi}]^T [I][B_\eta] \Big) \{\Delta^e\} dR \tag{33}$$

$$[K^e_g] = \iint_R \langle \Delta^e \rangle \Big([B_w]^T [N][B_w] + 2[B_w]^T [R][B_\phi] + [B_\phi]^T [F][B_\phi] \Big) \{\Delta^e\} dR \tag{34}$$

Then we obtain

$$\delta^2 \Pi = \sum_{e=1}^{n} \delta^2 \Pi^e = \langle \Delta \rangle \big([K] + [K_w] + [K_g] \big) \{\Delta\} \tag{35}$$

where in equation(35)

$[K]$, $[K_w]$, $[K_g]$ are steffness matrix. According to $\bar{\delta}(\delta^2 \Pi) = 0$,then obtain

$$\big([K] + [K_w] + [K_g] \big) \{\Delta\} = 0 \tag{36}$$

Equation(36) is a nomal engenvalue problem.Compared to equation(60) of reference[16], there is a matrix $[K_w]$.

According to equation(33)and equation(34), We know $[K_w]$ and $[K_g]$ is function of prebuckling deflection and general internal loads respectively.Because the critical loads and critical thermal temperature are not known,we can not immediately obtain $[K_w]$ and $[K_g]$.but we can computing the diplacements and general loads caused by unit edged loads and temperature,Because it is linear before buckling,we obtain

$$w_0(\Delta T) = \Delta T \bullet w_0(1)$$
$$\phi_z(\Delta T) = \Delta T \bullet \phi_z(1)$$
$$[N(\Delta T)] = \Delta T \bullet [N(1)]$$
$$[R(\Delta T)] = \Delta T \bullet [R(1)]$$
$$[F(\Delta T)] = \Delta T \bullet [F(1)]$$

According to equation(33) and equation(34),we obtain.

$$[K_w(w_0(\Delta T), \phi_z(\Delta T))] = \Delta T[K_w(w_0(1), \phi_z(1))]$$
$$[K_g([N](\Delta T),[R](\Delta T),[F](\Delta T))] = \Delta T[K_g([N(1)],[R(1)],[F(1)])]$$

Then equation(36) can be written as the following engenvalue equation

$$[[K] + \Delta T([K'_{w0}] + [K'_g])]\{\Delta\} = 0 \qquad (37)$$

4.Numerical Exemple and Discussion

The exemple model is same as that in reference[12],the material properties are.

$E_1 = 139.3\text{GPa}, E_2 = 11.1\text{GPa}, G_{12} = 4.9\text{GPa},$
$G_{13} = 1.9\text{GPa}, \nu_{12} = 0.3. \quad \nu_{13} = 0.21 \quad \nu_{23} = 0.33$
$\alpha_1 = -0.21E-6$ mm/(mm • F)
$\alpha_2 = \alpha_3 = 1.0E-6$ mm/(mm • F)

thickness of each layer :t=0.125mm.,plate size is: 254 × 254(mm^2).

In case of inplane mechanical buckling,boundary condition of two side simply-supported and two side free is considered:

at x=0,a $\quad u_0 = \psi_x = \phi_x = w_0 = 0$

at y=0,a free

In case of thermal buckling,

at x=0,a and y=0,a

$$u_0 = v_0 = \psi_x = \psi_y = \phi_x = \phi_y = w_0 = 0$$

The numerical results of critical loads are given in table.1,and thermal buckling critical temperature In table.2

Table 1 Critical loads account for ecfects of the three factors

stackling sequence	refer[12]	present	A	B	C	D	E	F	G
$[0_3/90_3]_S$	7.2	7.18	6.9	7.0	7.16	7.19	7.17	7.10	7.07
$[0_3//90_3//90_3//0_3]_T$	8.3	8.27	8.25	8.26	8.23	8.27	8.24	8.27	8.25
$[0_3//90_3//0_3//90_3]_T$	5.3	5.25	4.87	4.87	5.23	5.27	5.25	4.89	4.89
$[0_2//45_2//0_2//45_2//0_2]_T$	5.2	5.18	4.85	4.84	5.11	5.18	5.14	4.87	4.86
$[0_2//45_2//0_2//-45_2//0_2]_T$	4.9	4.72	4.31	4.40	4.63	4.71	4.69	4.43	4.46

Table.2 Critical loads account for ecfects of the three factors

stackling sequence	present	A	B	C	D	E	F	G
$[0_3/90_3]_S$	194	190	193	191	194	191	193	192
$[0_3//90_3//90_3//0_3]_T$	215	209	211	213	214	214	212	210
$[0_3//90_3//0_3//90_3]_T$	187	180	183	184	188	186	184	182
$[0_2//45_2//0_2//45_2//0_2]_T$	164	155	158	161	163	164	159	157
$[0_2//45_2//0_2//-45_2//0_2]_T$	159	149	152	154	158	156	151	150

where in the tables:

A—Acount for all three factors.

B---Acount for prebuckling deflection only

C---Acount for inplane deformation only

D---Acount for edged loads only .

E---neglect prebuckling deflection only

F---neglect inplane deformation only

G---Neglect edged loads only

From table 1,it can be obviously seen that the the effects on buckling loads is mainly contributed by prebuckling deflection ,secondarly the inplane deformation ,finally the edged load altering.especialy to antisymmetric laminates ,the effects is too great to be neglected.

From table 2,it is shown that both prebuckling inplane deformation and.prebuckling deflection effect on the buckling critical temperature ,but not as great as in the mechnical buckling .In all cases,the.effects of the prebuckling deflection is greater than inplane deformation.

5. Conclusion

In this paper a formulation ,which take into account the effects of prebuckling deflection,inplane deformation and edged loads altering,is developed by using the variation princeple. A eight freedom finite element is used to numerically calculate the buckling inplane loads and critical temperature.the result show

deflection have an important

s of unsymetric laminates.

ric laminates,the effects is too

d.

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