NUMERICAL CALCULATION OF OPTIMAL 3D-TRAJECTORIES FOR THE MANEUVERABLE AIRCRAFT

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Keywords: *Maneuverable Aircraft, 3D-Trajectories, Optimal Control Problem*

Abstract

A problem of how to move the center of gravity of the maneuverable aircraft from one given point in the three-dimensional space to another point as quickly as possible with the fixed vectors of the respective velocities is considered. The example is given for the case when the velocity vector equals the initial one at the end of motion.

1 Introduction

A great number of works published both in Russia and abroad deal with aircraft flight control. Most elaborated are the problems related to automatic control methods used to stabilize the motion parameters set by the pilot (see, for instance, [1, 2]). The involved problems of choosing the optimal flight trajectory using classical variational calculus methods, the maximum principle and direct numerical optimization methods are mostly [3, 4] connected with searching for the best parameters of cruising flight regimes, increasing the flight range, reducing the fuel consumption, finding optimal flight profiles, 4-D navigation, etc. All these issues are first of all critical for performance optimization of non-maneuverable aircraft [5].

Much fewer works deal with optimization of characteristics of a maneuverable aircraft, basically limiting to a rather narrow problem of optimizing the angular velocity of the bank in the horizontal plane with no restrictions on the coordinates of the aircraft at the end of the maneuver (the problem with the free right end). A sufficiently complete solution to this problem was obtained in [6–8]. Trajectories that are lying in the vertical plane and cannot be reduced to conventional civil aviation trajectories (change of the flight height, approach, etc.) were principally studied for the space industry (see, for instance, [9]).

Current aircraft development and improvement needs require solving problems to improve aircraft characteristics for a wider class of trajectories both plane and 3-D. The latter is for the class of maneuverable aircraft that are to perform, according to their specialization, various maneuvers [10], with optimality of the latter being significantly important, for instance, in air combat [11–14]. This, in turn, makes it significantly more difficult to state the optimal control problem. It becomes 3-D while the positions of coordinates of the aircraft and its flight direction at the end of the maneuver should satisfy the particular, given conditions (the problem with fixed right end). The results of the respective studies are of absolute interest since they help find whether the optimal solution is efficient for typical maneuvers such as loop, oblique loop, split S, half-roll, up line, etc. as compared to conventional ways of their execution and estimate whether the optimal control can be implemented in practice.

One of the principal research objectives in this direction is to form control algorithms for the aircraft when it is maneuvering based on solutions to the optimization problem that can be used in the onboard intellectual pilot support system. The latter must provide the pilot with the possibility of optimal control of the aircraft when it performs a maneuver. At present, such systems are becoming increasingly common in aviation since onboard hardware and software of modern aircraft allow implementing sufficiently complicated control algorithms. The experience of the latest conflicts [15] shows that during air fights “enemies approached at almost double sound speed. The increased pace of combat still had the requirement that, if not met, made success impossible, viz. forestalling the enemy as they attack. The pilot should have thought and acted much faster. The rapid nature of fight contradicted the abruptly increased scope of work the pilot did in a combat flight.” In-cabin actions needed to be combined with piloting, air situation evaluation, and tracking the enemy. As a result, the pilot reached the limits of his psychophysiologic abilities. The conflict between thinking and the rough work [15] required “developing expertise on-board systems to assist the pilot in fight planning and decision-making in the challenging air situation.” In particular [15], the surprise attack of the enemy “makes him instinctively, without thinking it over” perform defensive maneuvers practiced during trainings. By [15], in this case, the best are the “loop” or slanted turn (“oblique loop”), and one can make next steps in the fight “not before they drastically evade the enemy’s attack”.

In this work, we calculate the minimum time loop maneuver based on the Pontryagin maximum principle [16]. This problem for two-dimensional case was studied in [17] (see also [18] and references in [19]). Another thing that makes this work different from [17] is that all calculations in [17] were performed given the assumption that the speed value at the end of the maneuver can be arbitrary. As a result, this value was almost always significantly lower than the initial one since the kinetic energy storage was used to complete the motion as soon as possible. However, it is critical not to lose speed both when evading the enemy and when tracking the target [20].

2 Motion Equations

Since the form of the motion equations of the aircraft we use in this work is not very common, we briefly describe the way they are derived following [21, 22]. We also introduce the values of constants needed in calculations.

Neglecting the wind, the Earth rotation and its surface curvature, we consider the center of gravity of the aircraft moving in the three-dimensional space with respect to a terrestrial fixed right Cartesian rectangular inertial coordinate system $OXYZ$, with its axes OX and OZ lying in the horizontal plane. We have

$$\begin{aligned} &m\ddot{r} = G + R + P, \quad m = \text{const} \\ &\dot{r} = v(t) \in R^3, \quad |v| = V \geq \text{const} > 0 \end{aligned} \tag{1}$$

Here, m is the mass of the aircraft, which we consider constant, r is the radius vector of the center mass of the aircraft in the system $OXYZ$ with the components x, y and z, v is the velocity vector, G is the gravitation force, R is the principal vector of aerodynamic forces, and P is the tractive force of the engines. By the above said, we consider the value V to be always sufficiently high and one can neglect the restriction on it.

We put the gravitation field to be homogeneous so that

$$\begin{gathered} G = \text{col}(0, -mg, 0), \quad |G| = mg \\ g = 9.81 \text{ m/s}^2 \end{gathered} \tag{2}$$

where g is the acceleration of gravity and col stands for the column vector. We consider the atmosphere to be isothermal with the following approximation dependence of the mass density of the air ρ on the height y

$$\begin{aligned} &\rho = \rho_0 \exp(-y/h) \\ &\rho_0 = 9.81 \cdot 0.125\,\text{kg/m}^3 = 1.22625\,\text{kg/m}^3 \\ &h = 10^4\,\text{m} \end{aligned} \tag{3}$$

The vector R in (1) is generally considered as the sum of three vectors

$$R = R_x + R_y + R_z \tag{4}$$

where R_x is the drag force, R_y is the lifting force, and R_z is the lateral force. The vector R_x is opposite in direction to the velocity vector v, the vector R_y lies in the vertical symmetry plane of the aircraft and is orthogonal to R_x, and the vector R_z is orthogonal to R_x and R_y. We consider the aircraft to move with no slip, i.e., v is always in the same plane as R_y. Then, we can put $R_z \equiv 0$.

We assume that the lifting force R_y can be changed instantaneously both in its value and direction, and $|R_y|$ can be expressed via a dimensionless scalar lift coefficient C_y

$$|R_y| = C_y qS, \quad 0 \le C_y <= C_y^{\max} = 1.5$$
$$q = \rho v^2/2 \tag{5}$$

where q is the dynamic pressure and S is the wing area of the aircraft. We explain the physical sense of the variable $C_y^{\max}$. During the flight, the pilot can change the variable R_y by varying the angle of attack α, i.e., the angle between the longitudinal axis of the aircraft and the velocity vector projected onto its vertical symmetry plane. One knows that the coefficient C_y is linear with respect to α for subsonic speeds and $|\alpha| \lesssim 15^\circ$. For $|\alpha| \gtrsim 15^\circ$, this dependence becomes significantly nonlinear, which is caused by flow separation on the wing that worsens stability and controllability of the aircraft with the subsequent stall. This means that the value C_y should not exceed some maximal value $C_y^{\max}$ during the aircraft operation.

The module of the drag force can be written using the dimensionless scalar drag coefficient C_x as

$$|R_x| = C_x qS, \quad C_x = C_{x0} + C_{xi} \tag{6}$$

The scalar C_{x0} is the zero-lift drag coefficient. Its value depends on air viscosity and compressibility and corresponds to passive drag, i.e., when it is independent of the lift. Hence, the part of the drag force that depends on the lift is described using C_{xi} , which is the lift-induced drag coefficient. For subsonic speeds and symmetric aircraft configuration and linear dependence of C_y on α the function $C_x(C_y)$ called a polar line is a parabola. We put

$$C_x = C_{x0} + AC_y^2,$$
$$C_{x0} = 0.025, \quad A = 0.14 \tag{7}$$

where A is the airplane efficiency factor. In other words, by (6), we put $C_{xi} = AC_y^2$. Note that the quadratic dependence can be applied in engineering practice for non-symmetric configurations of the aircraft for transonic speeds.

We assume that the thrust vector P is always directed along the velocity vector v. The variable $|P|$ is bounded by the minimal and maximal admissible power of the engines

$$P_{\min} \le |P| \le P_{\max}, \quad \frac{P_{\min}}{|G|} = \frac{1}{2}$$
$$\frac{P_{\max}}{|G|} = -B_1 y + B_2 V + B_3 \tag{8}$$
$$B_1 = 10^{-4}\,\mathrm{m}^{-1}, \quad B_2 = 0.002\,\mathrm{s/m}, \quad B_3 = 1$$

Here, we use the linear approximation of the dependence $P_{\max}$ on the altitude y and the value of the speed V. Note that $P_{\min}$ is chosen to equal about half of $P_{\max}$. We introduce the scalar u by the formulas

$$U_p = \frac{g}{2|G|}\left(P_{\max} + P_{\min}\right)$$
$$U_m = \frac{g}{2|G|}\left(P_{\max} - P_{\min}\right) \tag{9}$$
$$\frac{|P|}{m} = U_p + uU_m, \quad |u| \le 1$$

This transformation allows us to move from the variable $|P|$ with bounds that depend on the phase variables to the variable u whose bounds do not depend on the phase variables.

The ratio $|G|/S$ is called the unit wing load. We consider it constant $|G|/S = 9.81 \times 300\,\mathrm{N/m^2}$. We divide both parts of the differential equation for $\dot{r}$ in (1) by m. Using expressions (2) and (4)–(9), we have

$$\dot{r} = v$$
$$\dot{v} = \frac{G}{m} + \left(U_p + uU_m - \left(C_{x0} + AC_y^2\right) qk\right)\tau + C_y qk\nu$$
$$\tau, \nu \in R^3, \ \tau = \frac{v}{V}, \ \nu = \frac{R_y}{|R_y|}, \ (\nu, \tau) = 0 \tag{10}$$
$$k = \frac{S}{m} = \frac{1}{300}\,\mathrm{m^2/kg}$$

where $(\cdot,\cdot)$ stands for the scalar product of two vectors.

We introduce the load factor

$$n_y = \frac{|R_y|}{|G|} = \frac{C_y qS}{gm} = \frac{C_y qk}{g} \qquad (11)$$
$$0 \le n_y \le n_y^{\max} = 8$$

It is bounded by the constructive features of the aircraft and physiological abilities of the pilot. Note that the latter are quite significant (see, for instance, [23]) therefore the condition $n_y^{\max} = 8$ actually reflects the limit capabilities of the aircraft. Similarly, we introduce the tangential load factor

$$n_x = \frac{|P| - |R_x|}{|G|} \qquad (12)$$

Using (11) and (12), we can write differential equations (10) as

$$\dot{r} = v, \quad \dot{v} = f_g + g n_x \tau + g n_y \nu$$
$$f_g = \frac{G}{m}, \quad |f_g| = g \qquad (13)$$

3 Optimal Control Problem in the Three Dimensional Case

Taking into account (5), (8), (11) and (13), we can re-write (10) as

$$\dot{r} = v$$
$$\dot{v} = \left(U_p + uU_m - qkC_{x0} - qk|c_l|^2 A\right)\tau + qkc_l + f_g$$
$$|u| \le 1, \quad c_l = C_y \nu \qquad (14)$$
$$|c_l| = C_y \le C_y^{\max}, \quad |c_l| qk \le g n_y^{\max}$$

Choosing the vector c_l and the scalar u, we need to spend minimal time T to move the center of gravity of the aircraft from its initial state

$$r(0) = \mathrm{col}(x_0, y_0, z_0), \quad v(0) = \mathrm{col}(v_{0x}, v_{0y}, v_{0z}) \qquad (15)$$

to its final state

$$r(T) = \mathrm{col}(x_T, y_T, z_T), \quad v(T) = \mathrm{col}(v_{Tx}, v_{Ty}, v_{Tz}) \qquad (16)$$

Without loss of generality, we can always take $x_0 = z_0 = 0$.

We introduce the vectors ψ_v and ψ_r conjugate to the vectors v and r, respectively. We decompose ψ_v into the components collinear to τ and ν. We have

$$\psi_{v\tau} = (\psi_v, \tau), \quad \psi_{v\nu} = \psi_v - \psi_{v\tau}\tau \qquad (17)$$

We compose the Hamiltonian

$$H = \left(U_p + uU_m - qkC_{x0} - qk|c_l|^2 A\right)\psi_{v\tau} +$$
$$+ qk(\psi_{v\nu}, c_l) + (\psi_v, f_g) + (\psi_r, v) +$$
$$+ \lambda\left(|c_l| qk - g n_y^{\max}\right) \qquad (18)$$

where λ is the additional scalar undetermined Lagrange multiplier introduced due to the phase constraint on the control in (14). By the Pontryagin maximum principle [16], we find the maximum of (18) with respect to u and c_l. Since later in this section we use only the functions that satisfy the necessary optimality conditions, we can preserve the designations for them that were used above. In particular, the controls found as a result of searching for the extremum will be still denoted by u and c_l. Substituting the respective expressions into the phase constraint $|c_l| qk \le g n_y^{\max}$, we have

$$\lambda = \frac{2Ag n_y^{\max}}{qk}\psi_{v\tau} - |\psi_{v\nu}| \qquad (19)$$

The optimal values c_l should be calculated by the following algorithm

if $\lambda < 0$ and $\frac{g n_y^{\max}}{qk} \le C_y^{\max}$, then $|c_l| = \frac{g n_y^{\max}}{qk}$

else

$$\chi = \frac{|\psi_{v\nu}|}{2A\psi_{v\tau}}$$

if $\chi < 0$ or $\chi > C_y^{\max}$, then $|c_l| = C_y^{\max}$

else $|c_l| = \chi$

$$c_l = |c_l| \frac{\psi_{v\nu}}{|\psi_{v\nu}|} \qquad (20)$$

We have for the optimal u

$$u = \mathrm{sign}(\psi_v, v) \qquad (21)$$

Since the differential equations for ψ_v and ψ_r are very cumbersome, we cannot give them here. The respective C++ algorithm was obtained using the MAPLE system by the known formulas [16]. The exception is

$$\dot\psi_{rx} = 0, \qquad \dot\psi_{rz} = 0 \tag{22}$$

since by (18) the coordinates of the center of gravity x and z are not included in the function H explicitly.

We performed numerical integration of differential equations for phase and conjugate variables in the neighborhood of the regular points by the typical fourth-order accurate Runge-Kutta method with the variable step and the check term in the England form [24]. We applied the first-order Euler method with the step 10^{-13} s on the time intervals of about 10^{-13} s during which the first derivative of the functions to be calculated jumped. For additional accuracy control, we used the values of the Hamiltonian H that, by [16], should preserve its initial value. As a result, we managed to ensure conditions (16) are met up to the accuracy of less than 0.1 m with respect to spatial variables and less than 0.1 m/s with respect to projections of the velocity vector for the relative integration accuracy of about 10^{-8} and absolute accuracies with respect to time and components of the vectors r and v of about 10^{-13} s, 10^{-13} m, and 10^{-13} m/s, respectively.

Two-dimensional results can be found in [25].

4 Example of Numerical Calculation

As an example, we consider the case with the following initial and final phase points (15) and (16)

$$\begin{aligned} r(0) &= \mathrm{col}(0, 5000\,\mathrm{m}, 0) \\ v(0) &= \mathrm{col}(250\,\mathrm{m/s}, 0, 0) \\ r(T) &= \mathrm{col}(0, 5000\,\mathrm{m}, 500\,\mathrm{m}) \\ v(T) &= \mathrm{col}(250\,\mathrm{m/s}, 0, 0) \end{aligned} \tag{23}$$

The results of calculations are shown in the figures.

5 Conclusion

We studied the two-point three-dimensional optimal performance problem for (14). We found

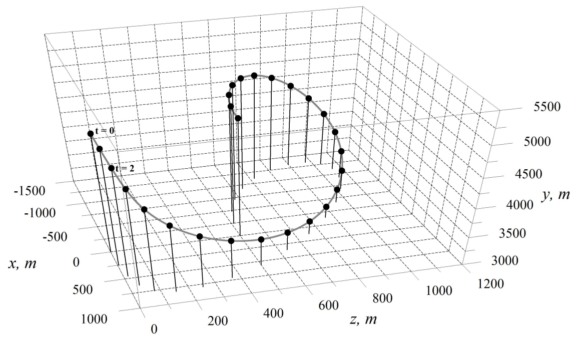

Fig. 1 The obtained 3D-trajectory. The optimal time for considered case is T = 24.98 s

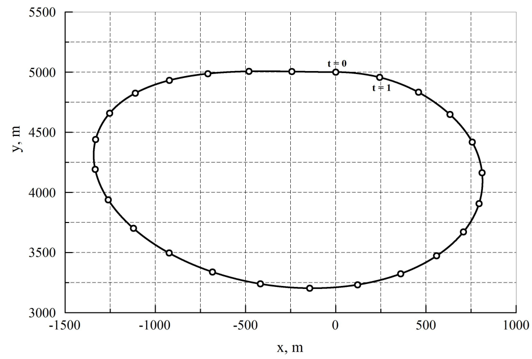

Fig. 2 The projection of the trajectory to the vertical plane OXY

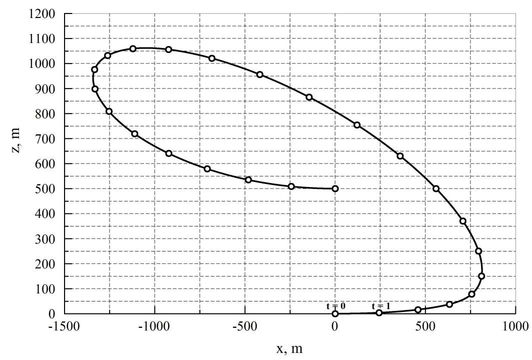

Fig. 3 The projection of the trajectory to the horizontal plane OXZ

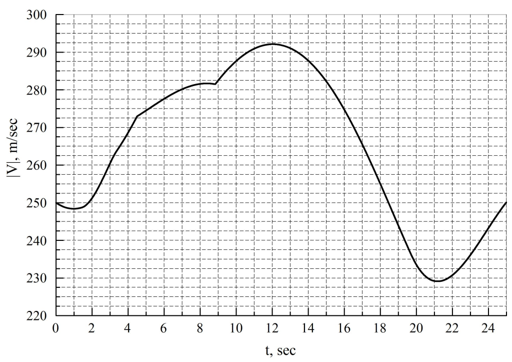

Fig. 4 The time dependence of the module of the speed of the aircraft center of gravity V

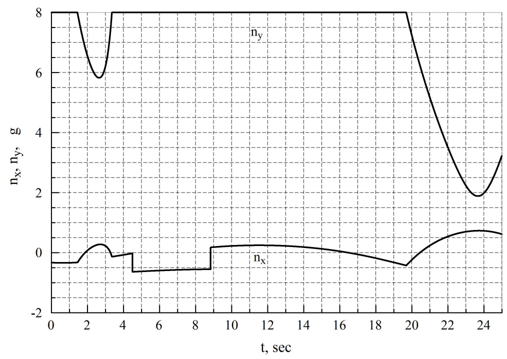

Fig. 5 The tangential load factor n_x and load factor n_y depending on time

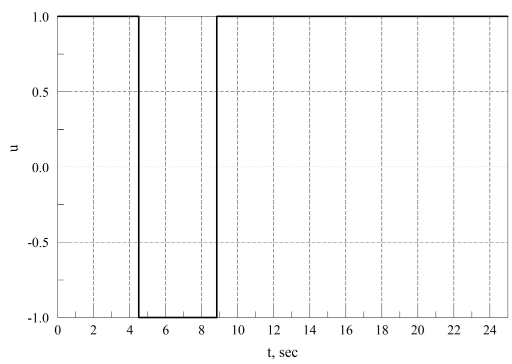

Fig. 6 The scalar control u depending on time

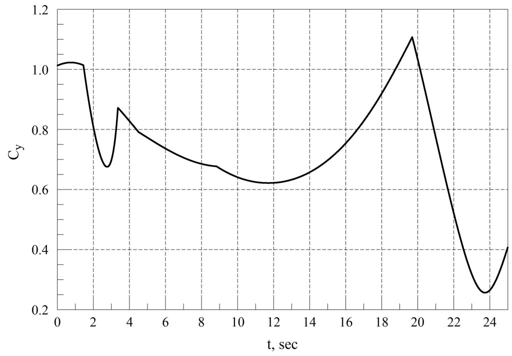

Fig. 7 The lift coefficient C_y depending on time

the respective optimal controls (20) and (21). We constructed the numerical solutions for boundary conditions (23).

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Acknowledgments

This study was financially supported by the Russian Foundation for Basic Research (projects no. 13-01-00180, 14-08-00606, and 14-01-00476) and Program for Support of Leading Scientific Schools of the Russian Federation (NSh-369.2012.1).

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